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Variable Separation and Exact Separable Solutions for Equations of Type $u_{xt} = A(u, u_x)u_{xx} + B(u, u_x)$*

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Abstract *The generalized conditional symmetry is developed to study the variable separation for equations of type $u_{xt} = A(u, u_x)u_{xx} + B(u, u_x)$. Complete classification of those equations which admit derivative-dependent functional separable solutions is obtained and some of their exact separable solutions are constructed.*

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1 Introduction

The study on the variable separation for nonlinear partial differential equations (PDEs) arising from various fields has been made much progress in the past decades. A few variable separation approaches to deal with evolution equations have been developed. These include the Lie-point symmetry method,[1,2] the differential Stäckel matrix approach,[3] the ansatz-based method,[4] the geometrical method,[5] the formal variable separation approach,[6] the multi-linear variable separation approach (MLVSA),[7] the functional variable separation approach (FVSA),[8−10] the derivative-dependent functional variable separation approach (DDFVSA),[11] and so on.

Among these methods, the FVSA and the DDFVSA based on the generalized conditional symmetry (GCS)[12] have been playing important roles in the study of PDEs. These two methods were successfully applied to generalized nonlinear diffusion equations, extended wave equations, and KdV-type equations.[8−11] The DDFVSA newly introduced is an extension of the FVSA. However, on the variable separation for equations of type

$$u_{xt} = F(u, u_x, u_{xx}, \ldots)\,, \tag{1}$$

the DDFVSA did not seem to work until lately. This is the very question we shall address in this paper.

Recently, we defined a concept of the derivative-dependent functional separable solutions (DDFSSs),[11]

$$f(u, u_x) = a(x) + b(t)\,, \tag{2}$$

to (1+1)-dimensional evolution equations. The functional separable solution (FSS)

$$f(u) = a(x) + b(t) \tag{3}$$

is a special case of the DDFSS as $f_{u_x} \equiv 0$. If $f(u) = \ln u$, it is just the product separable solution, while if $f(u) = u$, it is the additive separable solution. The FVSA on nonlinear evolution equations has been involved by many authors.[8−10]

Equations of type (1) enjoy a wide range of applications in mathematical physics. This type includes the sine-Gordon, sinh-Gordon, Liouville equations, etc. Calogero[13] studied an evolution equation in integral form that can be rewritten in the purely differential form (via substitutions),

$$u_{xt} = -uu_{xx} + g(u_x)\,, \tag{4}$$

where g is an arbitrary differentiable function. The Vakhnenko equation[14]

$$(u_t + uu_x)_x + u = 0\,, \tag{5}$$

which governs the propagation of waves in a relaxing medium, shares the remarkable properties inherent to the KdV equation. In Ref. [15], Rabelo and Tenenblat characterized equations of type

$$u_{xt} = F\Big(u, u_x, u_{xx}, \ldots, \frac{\partial^k u}{\partial x^k}\Big)\,, \quad k \geq 2\,, \tag{6}$$

which describe pseudo-spherical surfaces. Konno, Kameyama, and Sanuki[16] obtained the following equation of motion for nonlinear lattice under a weak dislocation potential:

$$u_{xt} = -u_{xxxx} - \frac{3}{2}u_x^2 u_{xx} + \alpha \sin u = 0\,, \tag{7}$$

where α is a real nonzero constant.

Motivated mainly by the existence of these important examples, in this paper, as a special case of Eq. (1), we

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intend to discuss the derivative-dependent functional variable separation for evolution equations of the type,

$$u_{xt} = A(u, u_x)u_{xx} + B(u, u_x)\,, \tag{8}$$

and the functional variable separation for the system of equations,

$$v_t = A(u, v)v_x + B(u, v)\,, \quad u_x = v\,. \tag{9}$$

The layout of the paper is as follows. In Sec. 2, we present the basic theory on DDFSSs and GCSs for evolution equations. In Sec. 3, we classify systems (9) which admit FSSs and classify evolution equations (8) which admit DDFSSs respectively. Some exact separable solutions to the resulting systems and equations are constructed in Sec. 4. Section 5 is summary and discussion.

2 Preliminaries

Consider the general m-th order nonlinear evolution equation

$$E \equiv E(t, x, u, u_1, u_2, \ldots, u_m, u_t, u_{tt}) = 0\,, \tag{10}$$

where $u_k = \partial^k u/\partial x^k$, $1 \le k \le m$, E is a smooth function of the indicated variables. Let

$$V = \eta\frac{\partial}{\partial u} = \eta(t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{xxx}, u_{xxt}, \ldots,)\frac{\partial}{\partial u} \tag{11}$$

be an evolutionary vector field and η its characteristic.

Definition 1 A solution $u = u(x, t)$ is said to be a derivative-dependent functional separable solution (DDFSS) to Eq. (10) if it satisfies Eq. (10) and the ansatz

$$f(u, u_x) = a(x) + b(t)\,, \tag{12}$$

where $f(u, u_x)$ is a smooth function of u and u_x, and $a(x)$ and $b(t)$ are some functions of x and t. In particular, when f does not depend on u_x, the DDFSS turns to be the FSS.

Definition 2 The evolutionary vector field (11) is said to be a generalized conditional symmetry (GCS) of Eq. (10) if and only if

$$V^{(m)}(E)|_{L\cap W} = 0\,, \tag{13}$$

where $V^{(m)}$ is the m-th prolongation of Eq. (11), L is the solution manifold of Eq. (10) and W is the set of all the total derivatives of $\eta|_W = 0$ with respect to x, namely, $D_x^i\eta|_W = 0$, $i = 0, 1, 2, \ldots$, which are invariant surface condition and its partial derivatives with respect to x by appending to Eq. (10).

It can be derived from Eq. (13) that equation (10) admits the GCS (11) if and only if

$$(E_{u_t}D_t\eta + E_{u_{tt}}D_t^2\eta)|_{L\cap W} = 0\,, \tag{14}$$

where D_t and D_t^2 denote the first- and second-order total derivatives in t, respectively.

How to determine whether a PDE possesses the DDFSSs? And if it does, how to derive its DDFSSs? To sum up the results in Ref. [11], we have the following theorem.

Theorem 1 Equation (10) possesses the DDFSS (12) if and only if it admits the GCS

$$\begin{aligned} V &= \eta\frac{\partial}{\partial u} \\ &= [f_{uu}u_xu_t + f_{uu_x}(u_xu_{xt} + u_{xx}u_t) + f_{u_xu_x}u_{xx}u_{xt} \\ &\quad + f_uu_{xt} + f_{u_x}u_{xxt}]\frac{\partial}{\partial u}\,. \end{aligned} \tag{15}$$

Unfortunately, Definition 2 does not work upon equations of type (1). In fact, in virtue of the invariant surface condition and its partial derivatives $D_x^i\eta|_W = 0$, $i = 0, 1, 2, \ldots$, we have $D_{xt}\eta|_W = 0$. If letting Eq. (13) act on Eq. (1), then considering the determining condition $D_{xt}\eta|_{L\cap W} \equiv 0$, we cannot get any useful information. Now we have to try another way. By transformation $u_x = v$, equation (1) turns to be the following system of equations

$$v_t = F(u, v, v_x, \ldots,)\,, \quad u_x = v\,. \tag{16}$$

For system (16), we have the following definition.

Definition 3 A solution $u = u(x, t)$, $v = v(x, t)$ is a functional separable solution (FSS) to system (16) if it satisfies both system (16) and the ansatz

$$f(u, v) = a(x) + b(t)\,, \tag{17}$$

where f is a smooth function of the indicated variables. By transformation $v = u_x$, Definition 3 corresponds to the following definition.

Definition 4 A solution $u = u(x, t)$ is a derivative-dependent functional separable solution (DDFSS) to Eq. (1) if it satisfies both Eq. (1) and the ansatz

$$f(u, u_x) = a(x) + b(t)\,, \tag{18}$$

where f is a smooth function of the indicated variables.

By Definitions 3 and 4, we know that seeking DDFSSs to Eq. (1) is equivalent to seeking FSSs to system (16). If we can clarify the functional variable separation for system (16), by transformation $v = u_x$, we then fulfill the derivative-dependent functional variable separation for Eq. (1) subsequently.

Theorem 2 System (16) possesses the FSS (17) if and only if it admits the GCS

$$V^{(1)} = \eta^u\frac{\partial}{\partial u} + \eta^v\frac{\partial}{\partial v} + D_x\eta^v\frac{\partial}{\partial v_x} + D_t\eta^v\frac{\partial}{\partial v_t}\,, \tag{19}$$

where

$$\begin{aligned} \eta^u \equiv{} & f_{uu}u_xu_t + f_{uu_x}(u_xu_{xt} + u_{xx}u_t) + f_{u_xu_x}u_{xx}u_{xt} \\ & + f_uu_{xt} + f_{u_x}u_{xxt}\,, \end{aligned} \tag{20}$$

$$\begin{aligned} \eta^v \equiv{} & f_{uu}vu_t + f_{uv}(vv_t + v_xu_t) + f_{vv}v_xv_t \\ & + f_uv_t + f_vv_{xt}\,, \end{aligned} \tag{21}$$

and $V^{(1)}$ is the first prolongation of $V = \eta^u\partial/\partial u + \eta^v\partial/\partial v$ of system (16).

Proof Differentiating expressions (18) and (17) with respect to x and t respectively, we have

$$\eta^u \equiv \frac{\partial^2 f(u,u_x)}{\partial x\partial t} = f_{uu}u_xu_t + f_{uu_x}(u_xu_{xt} + u_{xx}u_t) + f_{u_xu_x}u_{xx}u_{xt} + f_uu_{xt} + f_{u_x}u_{xxt} = 0\,,$$

$$\eta^v \equiv \frac{\partial^2 f(u,v)}{\partial x\partial t} = f_{uu}vu_t + f_{uv}(vv_t + v_xu_t) + f_{vv}v_xv_t + f_uv_t + f_vv_{xt} = 0\,.$$

In the same way, as verifying Theorem 1, we can complete the proof. Similar to Definition 2, we have the following definition.

Definition 5 The evolutionary vector field (19) is a GCS of system (16) if and only if

$$V^{(1)}(\tilde{E})|_{\tilde{L}\cap\tilde{W}} = 0\,, \tag{22}$$

where $\tilde{E} \equiv \{v_t - F(u, v, v_x), u_x - v\}$, $V^{(1)}$ is the first prolongation of $V = \eta^u\partial/\partial u + \eta^v\partial/\partial v$ of system (16), $\tilde{L}$ is the solution manifold of system (16) and $\tilde{W}$ is the set of all the total derivatives of $\eta^u|_{\tilde{W}} = 0$ and $\eta^v|_{\tilde{W}} = 0$ with respect to x, namely, $D_x^i\eta^u|_{\tilde{W}} = 0$, $D_x^i\eta^v|_{\tilde{W}} = 0$, $i = 0, 1, 2, \ldots$, which are invariant surface conditions and their partial derivatives with respect to x by appending to system (16).

By Definition 5, we conclude that system (16) admits the GCS (19) if and only if

$$D_t\eta^v|_{\tilde{L}\cap\tilde{W}} = 0\,, \tag{23}$$

where D_t denotes the first order total derivative in t.

3 Classification of System (9) Admits FSSs and Equation (8) Admits DDFSSs

From Eq. (23) and Theorem 2, we know that classification of system (9) which admits FSSs (17) is equivalent to seeking the GCS of the form (19) which satisfies

$$D_t\eta^v|_{\tilde{L}\cap\tilde{W}} = 0\,, \tag{24}$$

while $\tilde{E} \equiv \{v_t - F(u, v, v_x), u_x - v\}$ and

$$\eta^v \equiv f_{uu}vu_t + f_{uv}(vv_t + v_xu_t) + f_{vv}v_xv_t + f_uv_t + f_vv_{xt}\,. \tag{25}$$

We now calculate the left side of expression (24).

Making full use of expressions $D_x^i\eta^u|_{\tilde{W}} = 0$, $D_x^i\eta^v|_{\tilde{W}} = 0$, $i = 0, 1, 2, \ldots$, and system (9) and excluding the higher-order derivatives of u and v in expression (24), at last, the left-hand side of expression (24) is changed into an expression of independent derivatives of u and v, which reads

$$D_t\eta^v|_{\tilde{L}\cap\tilde{W}} = (h_0v_x + h_1)u_{tt} + (h_2v_x + h_3){u_t}^2 + (h_4{v_x}^2 + h_5v_x + h_6)u_t + h_7{v_x}^3 + h_8{v_x}^2 + h_9v_x + h_{10}\,, \tag{26}$$

where $h_i = h_i(u, v)$, $i = 0, 1, \ldots, 10$, are expressed by $f(u, v)$, $A(u, v)$, $B(u, v)$ and their derivatives with respect to u and v.

Equating expression (26) to zero leads to $f(u, v)$, $A(u, v)$ and $B(u, v)$ satisfying the following system of PDEs:

$$h_i = h_i(u, v) = 0\,, \quad i = 0, 1, \ldots, 10\,. \tag{27}$$

The detailed expressions for h_i are listed in Appendix.

After solving system (27) for $f(u, v)$, $A(u, v)$ and $B(u, v)$, besides the case $A(u, v) \equiv 0$, we obtain the following classification theorem.

Theorem 3 The system

$$v_t = A(u, v)v_x + B(u, v)\,, \quad u_x = v\,,$$

admits FSS of the form $f(u, v) = a(x) + b(t)$ if and only if it is equivalent to one of the following cases, up to translation and dilatation for u and v:

(i)

$$v_t = c_2\,\mathrm{e}^{c_1f(v)}v_x + b(v)\,, \quad u_x = v\,, \tag{28}$$

$$f(u, v) = f(v)\,, \tag{29}$$

where $b(v)$ and $f(v)$ satisfy

$$[c_1(f'(v))^2f''(v) + (f''(v))^2 - f'''(v)f'(v)]b(v) + [c_1(f'(v))^2 - f''(v)]f'(v)b'(v) - (f'(v))^2b''(v) = 0 \tag{30}$$

(ii)

$$v_t = c_1vv_x + v(u + c_2v + c_3)\,, \quad u_x = v\,, \tag{31}$$

$$f(u, v) = \ln(v)\,; \tag{32}$$

(iii) $c_2 \neq 0$,

$$v_t = c_2(u + c_3)\,\mathrm{e}^{c_1f(v)}v_x + \frac{c_4 - c_2\int^v \xi f'(\xi)\,\mathrm{e}^{c_1f(\xi)}\,\mathrm{d}\xi}{f'(v)}\,,$$

$$u_x = v\,, \tag{33}$$

$$f(u, v) = f(v)\,; \tag{34}$$

(iv) $c_4 \neq 0$,

$$v_t = c_2(u + c_1)vv_x - \frac{1}{2}v[c_2v^2 + c_4u(u + 2c_1) + c_3]\,,$$

$$u_x = v\,, \tag{35}$$

$$f(u, v) = \ln(v)\,; \tag{36}$$

(v) $c_1 \neq 0$,

$$v_t = (c_2\,\mathrm{e}^{-c_1u} + c_3)v_x + v(c_1c_2v + c_4)\,\mathrm{e}^{-c_1u}\,,$$

$$u_x = v\,, \tag{37}$$

$$f(u, v) = c_1u + \ln(v)\,; \tag{38}$$

(vi) $c_1 \neq 0$,

$$v_t = \frac{c_2\,\mathrm{e}^{-c_1u}v_x}{v} - v[c_1c_2(c_1u + \ln(v)) - c_3]\,\mathrm{e}^{-c_1u}\,,$$

$$u_x = v\,, \tag{39}$$

$$f(u,v) = c_1 u + \ln(v)\,; \tag{40}$$

(vii) $c_1 c_3 \neq 0$,

$$v_t = c_2\,\mathrm{e}^{-c_1 u} v_x + v[c_1(c_2 v + c_3 u) + c_3 \ln(v) + c_4]\,\mathrm{e}^{-c_1 u}\,,$$

$$u_x = v\,, \tag{41}$$

$$f(u,v) = c_1 u + \ln(v)\,; \tag{42}$$

(viii)

$$v_t = c_2(v + c_3)^{2/3}\,, \quad u_x = v\,, \tag{43}$$

$$f(u,v) = c_1 u + \frac{3}{4}(v + c_3)^{4/3}\,; \tag{44}$$

(ix)

$$v_t = c_2 v_x + c_3\,, \quad u_x = v\,, \tag{45}$$

$$f(u,v) = u + c_1 v\,; \tag{46}$$

where c_i, $i = 1,2,3,4$ are arbitrary constants and $f(v)$ is an arbitrary function in their definition sets when not specified.

By transformation $v = u_x$, as a simple consequence of the foregoing theorem, we have the following classification theorem for Eq. (8):

Theorem 4 The equation

$$u_{xt} = A(u,u_x)u_{xx} + B(u,u_x)\,,$$

admits DDFSS of the form $f(u,u_x) = a(x) + b(t)$ if and only if it is equivalent to one of the following cases, up to translation and dilatation for u:

(i)

$$u_{xt} = c_2\,\mathrm{e}^{c_1 f(u_x)} u_{xx} + b(u_x)\,, \tag{47}$$

$$f(u,u_x) = f(u_x)\,, \tag{48}$$

where $b(u_x)$ and $f(u_x)$ satisfy

$$[c_1(f'(u_x))^2 f''(u_x) + (f''(u_x))^2 - f'''(u_x)f'(u_x)]b(u_x) + [c_1(f'(u_x))^2 - f''(u_x)]f'(u_x)b'(u_x) - (f'(u_x))^2 b''(u_x) = 0\,; \tag{49}$$

(ii)

$$u_{xt} = c_1 u_x u_{xx} + u_x(u + c_2 u_x + c_3)\,, \tag{50}$$

$$f(u,u_x) = \ln(u_x)\,; \tag{51}$$

(iii) $c_2 \neq 0$,

$$u_{xt} = c_2(u + c_3)\,\mathrm{e}^{c_1 f(u_x)} u_{xx} + \frac{c_4 - c_2\int^{u_x} \xi f'(\xi)\,\mathrm{e}^{c_1 f(\xi)}\,\mathrm{d}\xi}{f''(u_x)}\,, \tag{52}$$

$$f(u,u_x) = f(u_x)\,; \tag{53}$$

(iv) $c_4 \neq 0$,

$$u_{xt} = c_2(u + c_1)u_x u_{xx} - \frac{1}{2}u_x[c_2 {u_x}^2 + c_4 u(u + 2c_1) + c_3]\,, \tag{54}$$

$$f(u,u_x) = \ln(u_x)\,; \tag{55}$$

(v) $c_1 \neq 0$,

$$u_{xt} = (c_2\,\mathrm{e}^{-c_1 u} + c_3)u_{xx} + u_x(c_1 c_2 u_x + c_4)\,\mathrm{e}^{-c_1 u}\,, \tag{56}$$

$$f(u,u_x) = c_1 u + \ln(u_x)\,; \tag{57}$$

(vi) $c_1 \neq 0$,

$$u_{xt} = \frac{c_2\,\mathrm{e}^{-c_1 u} u_{xx}}{u_x} - u_x[c_1 c_2(c_1 u + \ln(u_x)) - c_3]\,\mathrm{e}^{-c_1 u}\,, \tag{58}$$

$$f(u,u_x) = c_1 u + \ln(u_x)\,; \tag{59}$$

(vii) $c_1 c_3 \neq 0$,

$$u_{xt} = c_2\,\mathrm{e}^{-c_1 u} u_{xx} + u_x[c_1(c_2 u_x + c_3 u) + c_3 \ln(u_x) + c_4]\,\mathrm{e}^{-c_1 u}\,, \tag{60}$$

$$f(u,u_x) = c_1 u + \ln(u_x)\,; \tag{61}$$

(viii)

$$u_{xt} = c_2(u_x + c_3)^{2/3}\,, \tag{62}$$

$$f(u,u_x) = c_1 u + \frac{3}{4}(u_x + c_3)^{4/3}\,; \tag{63}$$

(ix)

$$u_{xt} = c_2 u_{xx} + c_3\,, \tag{64}$$

$$f(u,u_x) = u + c_1 u_x\,, \tag{65}$$

where c_i, $i = 1,2,3,4$ are arbitrary constants and $f(u_x)$ is an arbitrary function in their definition sets when not specified.

4 Some Exact Separable Solutions to Eq. (8) and System (9)

In this section, we construct some exact separable solutions to the resulting equations and systems listed in Theorems 3 and 4. To perform this, one can refer to Ref. [11]. In the following, k_i ($i = 1,2,3,4$), λ, and μ are arbitrary constants in their definition sets when not being indicated, while the primes denote derivatives.

Example 1 For case (i) in Theorems 3 and 4, we distinguish three special subcases:

(i) $f(v) = \ln(v)$ or $f(u_x) = \ln(u_x)$

Substituting $f(v) = \ln(v)$ into Eq. (30), we get

$$v^2 b''(v) - (c_1 + 1)(vb'(v) - b(v)) = 0\,,$$

whose general solution $b(v) = c_3 v + c_4 v^{c_1+1}$, then system (28) is reduced to

$$v_t = c_2 v^{c_1} v_x + c_3 v + c_4 v^{c_1+1}\,, \quad u_x = v\,, \tag{66}$$

$$v_t = c_2 v_x + (c_3 + c_4)v\,, \quad u_x = v\,, \quad c_1 = 0\,. \tag{67}$$

From Eqs. (29) and (17), we have $f(v) = \ln(v) = a(x) + b(t)$, or $v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}$, then $u = \mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + f_1(t)$ via $u_x = v$. Now we formally obtain the solution

$$v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}\,, \quad u = \mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + f_1(t)\,. \tag{68}$$

To fix the unknown functions $a(x)$, $b(t)$, and $f_1(t)$, we substitute Eq. (68) into Eq. (66) and split it into a system of ordinary differential equations (ODEs) with dependent variables $a(x)$, $b(t)$, and $f_1(t)$. The system of ODEs reads

$$c_2 a'(x) = \lambda\,\mathrm{e}^{-c_1 a(x)} - c_4\,, \quad b'(t) = \lambda\,\mathrm{e}^{c_1 b(t)} + c_3\,.$$

After solving it, we have the following.

Systems (66) and (67) possess FSSs (68), where $f_1(t)$ is an arbitrary function, $a(x)$ and $b(t)$ are respectively given by

$$a(x) = -\frac{1}{c_1}\ln\Big(\frac{c_4}{\lambda\,\mathrm{e}^{c_4 c_1 (x+k_1)/c_2} - 1}\Big) - \frac{c_4(x+k_1)}{c_2}\,, \tag{69}$$

$$b(t) = \frac{1}{c_1}\ln\Big(\frac{c_3}{1-\lambda\,\mathrm{e}^{c_3 c_1 t + k_2 c_1 c_3}}\Big) + c_3(t+k_2)\,, \quad c_1 \neq 0\,, \tag{70}$$

and

$$a(x) = \frac{(\lambda - c_4)x}{c_2} + k_1\,, \quad b(t) = (\lambda + c_3)t + k_2\,, \quad c_1 = 0\,. \tag{71}$$

As a result, by transformation $v = u_x$, case (i) in Theorem 3 corresponds to case (i) in Theorem 4. Thus, in this subcase, equation (47) is reduced to

$$u_{xt} = c_2 {u_x}^{c_1} u_{xx} + c_3 u_x + c_4 {u_x}^{c_1+1}\,, \tag{72}$$

$$u_{xt} = c_2 u_{xx} + (c_3 + c_4)u_x\,, \quad c_1 = 0\,, \tag{73}$$

whose DDFSSs are respectively given by

$$u = \mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + f_1(t)\,,$$

where $f_1(t)$ is an arbitrary function, $a(x)$ and $b(t)$ satisfy Eqs. (69) and (70) or (71).

(ii) $f(v) = v$ or $f(u_x) = u_x$

Substituting $f(v) = v$ into Eq. (30), we get

$$b''(v) - c_1 b'(v) = 0\,.$$

Solving it we have $b(v) = c_3 + c_4\,\mathrm{e}^{c_1 v}$, then system (28) is reduced to

$$v_t = c_2\,\mathrm{e}^{c_1 v} v_x + c_3 + c_4\,\mathrm{e}^{c_1 v}\,, \quad u_x = v\,, \tag{74}$$

$$v_t = c_2 v_x + c_3 + c_4\,, \quad u_x = v\,, \quad c_1 = 0\,. \tag{75}$$

From Eqs. (29) and (17), we have $f(v) = v = a(x) + b(t)$, then $u = \int a(x)\mathrm{d}x + b(t)x + f_1(t)$ via $u_x = v$. Now we formally obtain the solution

$$v = a(x) + b(t)\,, \quad u = \int a(x)\,\mathrm{d}x + b(t)x + f_1(t)\,. \tag{76}$$

To fix the functions $a(x)$, $b(t)$, and $f_1(t)$, we substitute Eq. (76) into Eq. (74) and split equation (74) into a system of ordinary differential equations (ODEs) with dependent variables $a(x)$, $b(t)$, and $f_1(t)$, which read

$$c_2 a'(x) + c_4 = \lambda\,\mathrm{e}^{-c_1 a(x)}\,, \quad b'(t) - c_3 = \lambda\,\mathrm{e}^{c_1 b(t)}\,.$$

After solving them for $a(x)$ and $b(t)$, at last, we have the following solutions.

Systems (74) and (75) possess respectively the following FSSs:

$$u = -\frac{1}{c_1}\int \ln\Big(\frac{c_4}{-1+\lambda\,\mathrm{e}^{c_4 c_1 (x+k_1)/c_2}}\Big)\mathrm{d}x - \frac{c_4 x^2}{2c_2} + \Big[\frac{1}{c_1}\ln\Big(\frac{c_3}{1-\lambda\,\mathrm{e}^{c_3 c_1 t + k_2 c_1 c_3}}\Big) + c_3 t - \frac{k_1 c_4}{c_2} + k_2 c_3\Big]x + f_1(t)\,, \tag{77}$$

$$v = u_x\,, \quad c_1 \neq 0\,; \tag{78}$$

and

$$u = \frac{(-c_4+\lambda)x^2}{2c_2} + [(c_3+\lambda)t + k_2 + k_1]x + f_1(t)\,, \tag{79}$$

$$v = u_x\,, \quad c_1 = 0\,. \tag{80}$$

Consequently, in this subcase, equation (47) is reduced to

$$u_{xt} = c_2\,\mathrm{e}^{c_1 u_x} u_{xx} + c_4\,\mathrm{e}^{c_1 u_x} + c_3\,, \quad c_1 \neq 0\,, \tag{81}$$

and

$$u_{xt} = c_2 u_{xx} + c_3 + c_4\,, \quad c_1 = 0\,, \tag{82}$$

whose DDFSSs are given by Eqs. (77) and (79) respectively, where $f_1(t)$ is an arbitrary function.

(iii) $f(v) = 1/v$ or $f(u_x) = 1/u_x$

In the same way, substituting $f(v) = 1/v$ into Eq. (30), and solving it for $b(v)$, system (28) is reduced to the following:

$$v_t = c_2\,\mathrm{e}^{c_1/v} v_x + (c_3 + c_4\,\mathrm{e}^{c_1/v})v^2\,, \quad u_x = v\,, \tag{83}$$

$$v_t = c_2 v_x + (c_3 + c_4)v^2\,, \quad u_x = v\,, \quad c_1 = 0\,. \tag{84}$$

From Eqs. (29) and (17), we know $f(v) = 1/v = a(x) + b(t)$. Hence we obtain the following FSSs to system (83) and (84)

$$v = \frac{1}{a(x)+b(t)}\,, \quad u = \int \frac{1}{a(x)+b(t)}\,\mathrm{d}x + f_1(t)\,, \tag{85}$$

where $f_1(t)$ is an arbitrary function, $a(x)$ and $b(t)$ are expressed by

$$a(x) = \frac{c_4(x+k_1)}{c_2} - \frac{1}{c_1}\ln\Big(\frac{c_4}{1-\lambda\,\mathrm{e}^{-c_4 c_1 (x+k_1)/c_2}}\Big)\,, \tag{86}$$

$$b(t) = \frac{1}{c_1}\ln\Big(\frac{c_3}{-1+\lambda\,\mathrm{e}^{-c_3 c_1 (t+k_2)}}\Big) - c_3(t+k_2)\,, \quad c_1 \neq 0 \tag{87}$$

or

$$a(x) = \frac{(c_4+\lambda)x}{c_2} + k_1\,, \quad b(t) = (-c_3+\lambda)t + k_2\,, \quad c_1 = 0\,. \tag{88}$$

Correspondingly, equation (47) turns out to be

$$u_{xt} = c_2\,\mathrm{e}^{c_1/u_x} u_{xx} + (c_3 + c_4\,\mathrm{e}^{c_1/u_x}){u_x}^2\,, \quad c_1 \neq 0\,, \tag{89}$$

and

$$u_{xt} = c_2 u_{xx} + (c_3 + c_4){u_x}^2\,, \quad c_1 = 0\,, \tag{90}$$

whose DDFSSs are shown by the second formula of Eq. (85), where $f_1(t)$, $a(x)$, and $b(t)$ are the same as the above.

Example 2 Now we seek for the separable solutions to Eq. (31) and (50) in case (ii) of Theorems 3 and 4 respectively.

From Eq. (32) and (17), we have $f(v) = \ln(v) = a(x) + b(t)$, or equivalently, $v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}$. Substituting $v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}$ into Eq. (31) and solving it for u gives the FSSs

$$u = [-c_1 a'(x) - c_2]\,\mathrm{e}^{b(t)}\,\mathrm{e}^{a(x)} + b'(t) - c_3\,,$$

$$v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}\,, \tag{91}$$

to system Eq. (31), where $b(t)$ is an arbitrary function and $a(x)$ satisfies ODE

$$c_1 a''(x) + c_1(a'(x))^2 + c_2 a'(x) + 1 = 0\,. \tag{92}$$

In fact, substituting Eq. (91) into $u_x = v$ and simplifying it, we find that $a(x)$ satisfies ODE (92), which can be reduced to the following linear ODE,

$$c_1 M''(x) + c_2 M'(x) + M(x) = 0\,, \tag{93}$$

by transformation $a'(x) = M'(x)/M(x)$. The general solutions of Eqs. (93) and (92) are listed below:

$$M(x) = \mathrm{e}^{-c_2 x/2c_1}\Big[k_1\cosh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{2c_1}\Big) + k_2\sinh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{2c_1}\Big)\Big]\,, \tag{94}$$

$$\begin{aligned} a(x) = {} & \ln\Big[k_1 + k_1\Big(\tanh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{4c_1}\Big)\Big)^2 + 2k_2\tanh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{4c_1}\Big)\Big] \\ & + \Big(\frac{c_2}{\sqrt{{c_2}^2-4c_1}} - 1\Big)\ln\Big[\tanh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{4c_1}\Big) - 1\Big] + k_3 - \Big(1 + \frac{c_2}{\sqrt{{c_2}^2-4c_1}}\Big) \\ & \times\ln\Big[1 + \tanh\Big(\frac{\sqrt{{c_2}^2-4c_1}x}{4c_1}\Big)\Big]\,, \quad {c_2}^2 - 4c_1 > 0\,; \end{aligned} \tag{95}$$

$$M(x) = \mathrm{e}^{-c_2 x/2c_1}\Big[k_1\cos\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{2c_1}\Big) + k_2\sin\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{2c_1}\Big)\Big]\,, \tag{96}$$

$$\begin{aligned} a(x) = {} & \ln\Big[-k_1\Big(\tan\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{4c_1}\Big)\Big)^2 + k_1 + 2k_2\tan\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{4c_1}\Big)\Big] \\ & - \frac{2c_2}{\sqrt{-{c_2}^2+4c_1}}\arctan\Big(\tan\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{4c_1}\Big)\Big) + k_3 \\ & - \ln\Big[1 + \Big(\tan\Big(\frac{\sqrt{-{c_2}^2+4c_1}x}{4c_1}\Big)\Big)^2\Big]\,, \quad {c_2}^2 - 4c_1 < 0\,; \end{aligned} \tag{97}$$

$$M(x) = (k_1 + k_2 x)\,\mathrm{e}^{-c_2 x/2c_1}\,, \tag{98}$$

$$a(x) = -\frac{c_2 x}{2c_1} + \ln(k_1 + k_2 x) + k_3\,, \quad {c_2}^2 - 4c_1 = 0\,. \tag{99}$$

Correspondingly, to obtain the DDFSSs to Eq. (92), considering $v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}$ and $u_x = v$, we have

$$u = \mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + g_1(t)\,. \tag{100}$$

Substituting Eq. (100) into Eq. (60) and splitting it give the following system:

$$\int \mathrm{e}^{a(x)}\,\mathrm{d}x + [c_1 a'(x) + c_2]\,\mathrm{e}^{a(x)} = \lambda\,, \quad b'(t) - g_1(t) - c_3 = \lambda\,\mathrm{e}^{b(t)}\,. \tag{101}$$

The combination of Eqs. (101) and (100) with Eq. (60) leads to the following DDFSS of (60):

$$u = [\lambda - (c_1 a'(x) + c_2)\,\mathrm{e}^{a(x)}]\,\mathrm{e}^{b(t)} + g_1(t)\,, \tag{102}$$

$$b(t) = \int g_1(t)\,\mathrm{d}t - \ln\Big[-k_1 - \lambda\int \mathrm{e}^{\int g_1(t)\,\mathrm{d}t + c_3 t}\,\mathrm{d}t\Big] + c_3 t\,, \tag{103}$$

where $g_1(t)$ is an arbitrary function and $a(x)$ satisfies ODE (92).

Example 3 For FSSs to Eq. (33) and DDFSSs to Eq. (52) in the case (iii) of Theorems 3 and 4 respectively, we just display the solutions relating to two special evaluations of $f(v) = \ln(v) = a(x) + b(t)$ and $f(v) = v = a(x) + b(t)$. The FSSs to Eq. (33) and DDFSSs to Eq. (52) in the case (iii) of Theorems 3 and 4 are given by

(i) $f(v) = \ln(v)$ or $f(u_x) = \ln(u_x)$

$$u = \mathrm{e}^{b(t)}\int \mathrm{e}^{a(x)}\,\mathrm{d}x + f_1(t)\,, \quad v = \mathrm{e}^{a(x)}\,\mathrm{e}^{b(t)}\,,$$

where $a(x)$, $b(t)$, and $f_1(t)$ are listed as follows.

Specifically, for different subcases, the FSSs $\{u, v\}$ to Eq. (33) and DDFSSs $\{u\}$ to Eq. (52) read

(i.1) $c_1 \neq 0$:

$$v = [\,\mu c_1(x + k_1)]^{1/c_1}\,\mathrm{e}^{b(t)}\,, \quad b(t) \text{ is an arbitrary function}\,,$$

$$u = \frac{1}{\mu(c_1 + 1)}[\,\mu c_1(x + k_1)]^{(c_1+1)/c_1}\,\mathrm{e}^{b(t)} - \frac{c_4 - b'(t)}{\mu c_2\,\mathrm{e}^{c_1 b(t)}} - c_3\,;$$

(i.2) $c_1 \neq -1$, $c_4 \neq 0$:

$$v = \Big[\frac{c_4}{\lambda\,\mathrm{e}^{c_4(c_1+1)(t+k_1)} - 1}\Big]^{1/(c_1+1)}\,\mathrm{e}^{a(x)+c_4(t+k_1)}\,,$$

$$u = \Big(\int \mathrm{e}^{a(x)}\,\mathrm{d}x + \mu\Big)\Big[\frac{c_4}{\lambda\,\mathrm{e}^{c_4(c_1+1)(t+k_1)} - 1}\Big]^{1/(c_1+1)}\,\mathrm{e}^{c_4(t+k_1)} - c_3\,,$$

where $a(x)$ satisfies

$$c_2\,\mathrm{e}^{c_1 a(x)}\Big[a'(x)\Big(\int \mathrm{e}^{a(x)}\,\mathrm{d}x + \mu\Big) - \frac{\mathrm{e}^{a(x)}}{c_1 + 1}\Big] + \lambda = 0\,; \tag{104}$$

(i.3) $c_1 \neq -1$, $c_4 = 0$:

$$v = \mathrm{e}^{a(x)}[\lambda(c_1 + 1)(t + k_1)]^{-1/(c_1+1)}\,, \quad u = [\lambda(c_1 + 1)(t + k_1)]^{-1/(c_1+1)}\Big(\int \mathrm{e}^{a(x)}\,\mathrm{d}x + \mu\Big) - c_3\,,$$

where $a(x)$ satisfies Eq. (104);

(i.4) $c_1 = -1$:

$$v = -\frac{\mathrm{e}^{b(t)}}{\mu(x + k_1)}\,, \quad b(t) \text{ is an arbitrary function}\,,$$

$$u = \Big[\frac{b'(t) + c_2 b(t) - c_2\ln(-\mu) - c_4}{c_2\mu} - \frac{\ln(x + k_1)}{\mu}\Big]\,\mathrm{e}^{b(t)} - c_3\,;$$

or

$$v = \mathrm{e}^{a(x)+k_1\,\mathrm{e}^{-c_2 t}-(\lambda-c_4)/c_2}\,, \quad u = \Big(\int \mathrm{e}^{a(x)}\,\mathrm{d}x + \mu\Big)\,\mathrm{e}^{k_1\,\mathrm{e}^{-c_2 t}-(\lambda-c_4)/c_2} - c_3\,;$$

where $a(x)$ satisfies

$$c_2 a'(x)\Big(\int \mathrm{e}^{a(x)}\,\mathrm{d}x + \mu\Big) + [\lambda - c_2 a(x)]\,\mathrm{e}^{a(x)} = 0\,.$$

(ii) $f(v) = v$ or $f(u_x) = u_x$:

$$v = -\frac{1}{c_1}\ln\Big(\frac{c_2}{k_1 x - \lambda c_1}\Big) + b(t)\,, \quad b(t) \text{ is an arbitrary function}\,,$$

$$u = \Big(x - \frac{\lambda c_1}{k_1}\Big)b(t) - \frac{1}{c_1 k_1}\Big[\ln\Big(\frac{c_2}{k_1 x - \lambda c_1}\Big) + 1\Big](k_1 x - \lambda c_1) - \frac{c_1(c_4 - b'(t))}{k_1\,\mathrm{e}^{c_1 b(t)}} - c_3\,.$$

Example 4 The FSSs to Eq. (35) are given by

$$v = u_x\,, \quad c_2 c_4 > 0\,,$$

$$u = \frac{1}{2}\sqrt{\frac{(-c_3 + c_4{c_1}^2)c_4}{c_4 + ({c_4}^2\lambda^2 + {c_2}^2 k_1 k_2)\,\mathrm{e}^{(-c_3+c_4{c_1}^2)(t+k_3)}}}$$

$$\times\Big(\frac{c_2 k_1}{c_4}\,\mathrm{e}^{\sqrt{c_4/c_2}x} - \frac{c_2 k_2}{c_4}\,\mathrm{e}^{-\sqrt{c_4/c_2}x} + 2\lambda\Big)\,\mathrm{e}^{(-c_3+c_4{c_1}^2)(t+k_3)/2} - c_1\,,$$

and

$$v = u_x\,, \quad c_2 c_4 < 0\,,$$

$$u = \sqrt{\frac{(-c_3 + c_4{c_1}^2)c_4}{c_4 + ({c_4}^2\lambda^2 - {c_2}^2({k_2}^2 + {k_1}^2))\,\mathrm{e}^{(-c_3+c_4{c_1}^2)(t+k_3)}}}$$

$$\times\Big[-\frac{c_2 k_1}{c_4}\cos\Big(\sqrt{-c_4/c_2}x\Big) + \frac{c_2 k_2}{c_4}\sin\Big(\sqrt{-c_4/c_2}x\Big) + \lambda\Big]\,\mathrm{e}^{(-c_3+c_4{c_1}^2)(t+k_3)/2} - c_1\,.$$

The DDFSSs to Eq. (54) are given by the expressions of u listed above.

Example 5 An FSS to Eq. (37) is given by

$$v=\frac{k_1\,\mathrm{e}^{k_1x+k_2}}{c_1(\,\mathrm{e}^{k_1x+k_2}+k_1f_1(t))}\,,\quad u=\frac{1}{c_1}\ln\Big[-\frac{(c_2k_1+c_4)(\,\mathrm{e}^{k_1x+k_2}+k_1f_1(t))}{k_1(f_1'(t)+c_3k_1f_1(t))}\Big]\,,$$

and A DDFSS to Eq. (56) is given by the above expression of u, where $f_1(t)\neq 0$ is an arbitrary function.

Example 6 An FSS to Eq. (39) is given by

$$v=\frac{k_1}{c_1(k_1x+k_2)(\ln(k_1x+k_2)+k_1f_1(t))}\,,\quad u=\frac{1}{c_1}\Big[\ln\Big(\frac{c_1(\ln(k_1x+k_2)+k_1f_1(t))}{k_1}\Big)+b(t)\Big]\,,$$

and A DDFSS to Eq. (58) is given by the above expression of u, where

$$f_1(t)=\Big\{\frac{1}{c_1}\int(c_1c_2(b(t)+1)-c_3)\exp\Big[-b(t)-c_2k_1\int\mathrm{e}^{-b(t)}\,\mathrm{d}t\Big]\,\mathrm{d}t+k_3\Big\}\exp\Big[c_2k_1\int\mathrm{e}^{-b(t)}\,\mathrm{d}t\Big]\,,$$

and $b(t)\neq 0$ is an arbitrary function.

Example 7 The FSSs to Eq. (41) are given by

$$v=\frac{\mathrm{e}^{a(x)}}{c_1(\int\mathrm{e}^{a(x)}\,\mathrm{d}x+f_1(t))}\,,\quad u=\frac{b(t)+\ln[c_1(\int\mathrm{e}^{a(x)}\,\mathrm{d}x+f_1(t))]}{c_1}\,,$$

where

$$a(x)=k_1\,\mathrm{e}^{-c_3x/c_2}-\frac{c_4+\lambda}{c_3}\,,\quad f_1(t)=\int\frac{(\lambda-c_3b(t))\,\mathrm{e}^{-b(t)}}{c_1}\,\mathrm{d}t+k_2\,,\quad c_3\neq 0\,.$$

If $c_3=0$, then

$$v=\,\mathrm{e}^{(-c_4-\lambda)x/c_2+k_1}\Big(\frac{c_1c_2}{-c_4-\lambda}\,\mathrm{e}^{(-c_4-\lambda)x/c_2+k_1}+\lambda\int\mathrm{e}^{-b(t)}\,\mathrm{d}t+c_1k_2\Big)^{-1}\,,$$

$$u=\frac{1}{c_1}\Big[b(t)+\ln\Big(\frac{c_1c_2}{-c_4-\lambda}\,\mathrm{e}^{(-c_4-\lambda)x/c_2+k_1}+\lambda\int\mathrm{e}^{-b(t)}\,\mathrm{d}t+c_1k_2\Big)\Big]\,,$$

where $b(t)\neq 0$ is an arbitrary function.

The DDFSSs to Eq. (60) are given by the above expressions of u.

Example 8 The FSSs to Eq. (43) are given by

$$v=u_x\,,\quad u=\frac{b(t)}{c_1}-\frac{1}{108}\frac{{c_2}^4}{c_1}\Big(t-\frac{c_1x}{c_2}+k_1\Big)^4-c_3x\,,\quad c_1\neq 0\,,$$

and

$$v=u_x\,,\quad u=\Big[\frac{1}{27}{c_2}^3(t+k_1)^3-c_3\Big]x+g(t)\,,\quad c_1=0\,,$$

where $b(t)$ and $g(t)$ are arbitrary functions.

The DDFSSs to Eq. (62) are given by the expressions of u above.

Example 9 An FSS to Eq. (45) is given by

$$v=-\frac{c_3x}{c_2}+k_3\,\mathrm{e}^{-(c_2t+x)/c_1}+k_1\,,\quad u=-\frac{1}{2}\frac{c_3x^2}{c_2}+k_1x-c_1k_3\,\mathrm{e}^{-(c_2t+x)/c_1}+b(t)\,;$$

A DDFSS to Eq. (64) is given by

$$u=-\frac{1}{2}\frac{c_3x^2}{c_2}+k_1x-c_1f_1\Big(\frac{c_2t+x}{c_2}\Big)+b(t)\,,$$

where $b(t)$ and $f_1((c_2t+x)/c_2)$ are arbitrary functions.

5 Summary and Discussion

In this paper, we have applied the DDFVSA to evolution equations of type (1). For example, we have classified Eqs. (8) and systems (9) which admit FSSs and DDFSSs respectively. We also have obtained some of their exact separable solutions.

There are still some interesting topics to be treated later.

(i) How can we apply the DDFVSA to other types of evolution equations or higher-dimensional evolution equations?

(ii) How can we unify other variable separation approaches by the DDFVSA? For instance, the possible group explanation of the MLVSA may be obtained via the GCS by extending the DDFSS ansatz (2), so can we recover the results from the MLVSA by the DDFVSA?

(iii) It is interesting to extend the DDFVSA and apply it to some differential-difference equations.

Appendix Expressions of h_i in Eq. (27)

In the following $f = f(u, v)$, $A = A(u, v)$ and $B = B(u, v)$, and the subscripts of f, A and B denote the partial derivatives with respect to u and v respectively.

$$h_0 = f_{uv} = 0\,, \tag{A1}$$

$$h_1 = v f_{uu} = 0\,, \tag{A2}$$

$$h_2 = f_{uuv} - \frac{f_{uv}(f_v A_u + f_{uv}A)}{f_v A} = 0\,, \tag{A3}$$

$$h_3 = v\Big[f_{uuu} - \frac{f_{uu}(f_v A_u + f_{uv}A)}{f_v A}\Big] = 0\,, \tag{A4}$$

$$h_4 = \Big(f_{uvv} - \frac{f_{uv}f_{vv}}{f_v}\Big)A - \Big(\frac{f_v A_u}{A} + 2f_{uv}\Big)A_v + f_v A_{uv} = 0\,, \tag{A5}$$

$$\begin{aligned} h_5 = &-\Big(\frac{f_{uv}B}{A} + 3vf_{uu}\Big)A_v - \frac{f_{vv}B + f_v B_v}{A}A_u + \Big(2f_{uvv} - 2\frac{f_{uv}f_{vv}}{f_v}\Big)B \\ &- \frac{vf_v {A_u}^2}{A} - \frac{f_u f_{uv} A}{f_v} + (B_{uv} + vA_{uu})f_v + f_{vv}B_u + f_{uu}A = 0\,, \end{aligned} \tag{A6}$$

$$\begin{aligned} h_6 = &\Big[v^2\Big(\frac{f_{uv}A}{f_v} - A_u\Big) - v\Big(B_v + \frac{f_{vv}B}{f_v}\Big) + 2B\Big]f_{uu} - \frac{(vf_{uv} + f_u)A_u B}{A} + \Big[f_u + v\Big(2f_{uv} - \frac{f_v A_u}{A}\Big)\Big]B_u \\ &+ \Big[2vf_{uuv} - \frac{(vf_{uv} + f_u)f_{uv}}{f_v}\Big]B + v\Big(f_v B_{uu} - \frac{f_{uu}A_v B}{A}\Big) - v^2 f_{uuu}A = 0\,, \end{aligned} \tag{A7}$$

$$h_7 = f_v(AA_{vv} - {A_v}^2) - f_{vv}AA_v = 0\,, \tag{A8}$$

$$\begin{aligned} h_8 = &\Big[\Big(f_{vvv} - \frac{{f_{vv}}^2}{f_v}\Big)B + vf_v A_{uv} + f_v A_u + f_{vv}B_v + f_v B_{vv}\Big]A + \Big(\frac{vf_{uv}f_{vv}}{f_v} - f_{uv} - vf_{uvv}\Big)A^2 \\ &- \frac{f_v {A_v}^2 B}{A} + f_v A_{vv}B - [(2vf_{uv} + 2f_u)A + f_v B_v + 2f_{vv}B + vf_v A_u]A_v = 0\,, \end{aligned} \tag{A9}$$

$$\begin{aligned} h_9 = &\Big[f_v B - v(vf_{uv} + f_u)A - \frac{vf_v A_v B}{A}\Big]A_u + \Big(vf_v B_{uv} - \frac{f_u f_{vv}B}{f_v} + f_{uv}B\Big)A \\ &- \Big[v^2 f_{uuv} + 2vf_{uu} - \frac{vf_{uv}(vf_{uv} + f_u)}{f_v}\Big]A^2 - \frac{f_{vv}(f_{vv}A + f_v A_v)B^2}{f_v A} - \frac{f_v A_v B B_v}{A} \\ &- [3(vf_{uv} + f_u)A_v - f_{vv}B_v]B + [(f_v + vf_{vv})A - vf_v A_v]B_u \\ &+ f_v(B_{vv} + vA_{uv})B + f_{vvv}B^2 = 0\,, \end{aligned} \tag{A10}$$

$$\begin{aligned} h_{10} = &f_v\Big[\Big(B - \frac{vA_v B}{A}\Big)B_u + vBB_{uv}\Big] - \frac{(vf_{uv} + f_u)A_v B^2}{A} - \frac{(vf_{uv} + f_u)(f_{vv}B - vf_{uv}A)B}{f_v} \\ &- v(vf_{uv} + f_u)A_u B + vf_{vv}BB_u - v(vf_{uuv} + 2f_{uu})AB + (2f_{uv} + vf_{uvv})B^2 = 0\,. \end{aligned} \tag{A11}$$

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