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# A Simple Method to Obtain Exact Soliton Solutions for a Nonlinear Equation in a Loss Fibre System*

YANG Xiao-Xue[1] and WU Ying[1,2]

[1]Department of Physics and National Key Laboratory for Laser Technique, Huazhong University of Science and Technology, Wuhan 430074, China

[2]Wuhan Institute of Physics and Mathematics, the Chinese Academy of Sciences, Wuhan 430071, China

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***Abstract*** *We show that the nonlinear equation governing wave propagation in a loss fibre system considered by Nakkeeran in J. Phys.* **A34** *(2001) 5111 can be brought into the standard nonlinear Schrödinger equation by a simple transformation.*

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Nakkeeran[1] has recently considered the following nonlinear equation describing wave propagation in a loss fibre system

$$\mathrm{i}q_z + q_{tt} + 2|q|^2 q + \beta^2 t^2 q + \mathrm{i}\beta q = 0 \tag{1}$$

with a constant $\beta$, and has in the same paper shown its complete integrability and obtained the soliton solutions by constructing its Lax pair and Bäcklund transformation. However, we shall provide an alternative and simpler method to do so by showing that this equation can in fact be transformed into the standard nonlinear Schrödinger equation (NLS)

$$\mathrm{i}Q_\xi + Q_{\tau\tau} + 2|Q|^2 Q = 0 \tag{2}$$

by introducing the following transformations

$$q = Q(\xi,\tau)\exp[-a(z)]\exp\{\mathrm{i}[b(z)t^2 + c(z)t + d(z)]\} \tag{3}$$

with

$$\xi = \frac{1 - \mathrm{e}^{-4\beta z}}{4\beta}, \qquad \tau = \mathrm{e}^{-2\beta z}(t - \mathrm{e}^{-2\beta z}), \tag{4a}$$

$$a(z) = 2\beta z, \qquad b(z) = \beta/2, \qquad c = \mathrm{e}^{-2\beta z},$$
$$d(z) = \frac{\mathrm{e}^{-4\beta z} - 1}{4\beta}, \tag{4b}$$

when $\beta$ is a constant.

Let us now give the explicit derivation of the transformation results in Eqs. (2) $\sim$ (4). Substituting $q(x,t)$ in Eq. (3) but with

$$\xi = \eta(z), \qquad \tau = \omega(z)[t - f(z)], \tag{5}$$

it is readily shown that the function $Q(\xi,\tau)$ satisfies the following equation

$$\mathrm{i}\eta_z Q_\xi + \omega^2 Q_{\tau\tau} + 2\,\mathrm{e}^{-2a}|Q|^2 Q + AQ + \mathrm{i}BQ_\tau = 0 \tag{6}$$

with

$$B = (\omega_z + 4\omega b)t + (2c\omega - \omega_z f - \omega f_z)$$

and

$$A = -(4b^2 + b_z - \beta^2)t^2 - (4bc + c_z)t + [\mathrm{i}(2b - a_z + \beta) - (c^2 + d_z)].$$

Our task is to find the explicit forms of the functions $a(z)$, $b(z)$, $c(z)$, $d(z)$, $f(z)$, $\eta(z)$, and $\omega(z)$ such that $Q(\xi,\tau)$ satisfies the standard nonlinear Schrödinger equation (2). Obviously, such a requirement will be satisfied if $A = B = 0$ and

$$\eta_z = \mathrm{e}^{-2a}, \qquad \omega = \mathrm{e}^{-a}. \tag{7}$$

$B = 0$ and $A = 0$ respectively lead to

$$(\ln\omega)_z = -4b(z), \tag{8a}$$

$$2c(z) - 2(\ln\omega)_z f(z) - f_z = 0, \tag{8b}$$

$$a_z = 2b(z) + \beta, \qquad d_z = -c^2(z), \tag{9a}$$

$$(\ln c)_z = -4b(z), \qquad b_z + 4b^2(z) = \beta^2. \tag{9b}$$

Equations (7), (8a), and (9a) immediately give the results $b = \beta/2$ and $a = 2\beta z$. Equations (9b) and (9a) give $c = f = \exp(-2\beta z)$ and $d = (\mathrm{e}^{-4\beta z} - 1)/(4\beta)$. These results together with Eqs. (5) and (7) are nothing but those given by Eq. (4).

It is well known[2] that the standard NLS is completely integrable, and the explicit forms of its solition solutions have been known for a long time. In this paper, we have shown that the nonlinear equation (1) can be transformed into the standard NLS (2), and have thus shown the complete integrability of the nonlinear equation (1). In addition, the explicit forms of its soliton solutions can be easily obtained by borrowing the soliton solutions of the standard NLS,[2] and hence we omitted here.

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Our approach provides not only an alternative method to show the complete integrability and obtain the exact soliton solutions of Eq. (1) but also an explicit form to see how the $N$-soliton solutions of the nonlinear equation (1) depend on the variables $z$ and $t$ through the combination variables $\xi$ and $\tau$. What is more important, the results in this paper offer another significant merit over the previous study[1] in which the relation between the nonlinear equation (1) and the standard NLS (2) also allows one to borrow more general conclusions of the much thoroughly studied standard NLS beyond soliton excitations, for instance, nonlinear dynamics for general initial conditions which may or may not involve soliton(s). Before ending this paper, we mention that the nonlinear equations

$$\mathrm{i}q_{jz} + q_{jtt} + 2\left(\sum_{j=1}^{N}|q_j|^2\right)q_j + \beta^2 t^2 q_j + \mathrm{i}\beta q_j = 0 \quad (10)$$

with $j = 1, \ldots, N$ considered also in Ref. [1] can be transformed into the forms

$$\mathrm{i}Q_{j\xi} + Q_{j\tau\tau} + 2\left(\sum_{j=1}^{N}|Q_j|^2\right)Q_j = 0\,, \quad j = 1, \ldots, N \quad (11)$$

by the transformations (3) and (4) except the replacements $q \to q_j$ and $Q \to Q_j$.

## References

[1] K. Nakkeeran, J. Phys. **A34** (2001) 5111.

[2] See, for example, G.L. Jr. Lamb, *Elements of Soliton Theory*, Wiley, New York (1980); M.J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981); V.E. Zakharov and S. Wabnitz, *Optical Solitons: Theoreticl Challenges and Industrial Perspectives*, Berlin, Springer (1998).