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Various Methods for Constructing Auto-Bäcklund Transformations for a Generalized Variable-Coefficient Korteweg-de Vries Model from Plasmas and Fluid Dynamics*

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Abstract *In this paper, under the Painlevé-integrable condition, the auto-Bäcklund transformations in different forms for a variable-coefficient Korteweg-de Vries model with physical interests are obtained through various methods including the Hirota method, truncated Painlevé expansion method, extended variable-coefficient balancing-act method, and Lax pair. Additionally, the compatibility for the truncated Painlevé expansion method and extended variable-coefficient balancing-act method is testified.*

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1 Introduction

Since the discovery of the soliton,[1] it has been a major concern to study the nonlinear evolution equations (NLEEs) and solitons.[2−17] Originating from the investigation of the surfaces of constant negative curvature, the auto-Bäcklund transformation provides an effective means of constructing multi-soliton solutions for a wide class of integrable NLEEs.[2,3] Recent investigations have shown that much attention has been paid to the study of NLEEs with variable coefficients and/or with additional terms.[4−17]

In this paper, we would like to investigate the damped variable-coefficient Korteweg-de Vries (vcKdV) model,[9]

$$u_t + f(t)\, u\, u_x + g(t) u_{xxx} + l(t) u = 0\,, \tag{1}$$

aiming at constructing its auto-Bäcklund transformations through various methods under the Painlevé-integrable condition, where the wave amplitude $u(x,t)$ is a function of the scaled "space" x and scaled "time" t, the real functions $f(t) \neq 0$, $g(t) \neq 0$ and $l(t)$ represent the coefficients of the nonlinear, dispersive and damped terms, respectively. Equation (1) can be widely used to describe the nonlinear physical phenomena such as nonlinear excitations of a Bose gas of impenetrable bosons, propagation of weakly nonlinear solitary waves in a varied-depth shallow-water tunnel, evolution of internal gravity waves, etc.

We notice that some other physically interesting vcKdV models are actually transformable into the damped vcKdV model without any constraint. For instance, describing the nonlinear waves in a fluid-filled tube[8] and trapped quasi-one-dimensional Bose–Einstein condensates,[13] the following vcKdV model with dissipative and damped terms

$$v_\tau + f(\tau)\, v\, v_\zeta + g(\tau) v_{\zeta\zeta\zeta} + l(\tau) v + q(\tau) v_\zeta = 0\,, \tag{2}$$

can be transformed into Eq. (1) through the transformation

$$x = \zeta - \int q(t)\,\mathrm{d}t\,, \quad t = \tau\,, \quad u(x,t) = v(\zeta,\tau)\,. \tag{3}$$

Another vcKdV model with dissipative, damped and external-force terms[23]

$$v_\tau + f(\tau) v v_\zeta + g(\tau) v_{\zeta\zeta\zeta} + l(\tau) v + q(\tau) v_\zeta = h(\tau)\,, \tag{4}$$

which governs the pulse wave propagation in blood vessels and dynamics in the circulatory system,[6] is also equivalent to Eq. (1) with the following transformation

$$v(\zeta,\tau) = \mathrm{e}^{-\int l(t)\,\mathrm{d}t} \int \mathrm{e}^{\int l(t)\,\mathrm{d}t} h(t)\,\mathrm{d}t + u(x,t)\,, \tag{5}$$

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with

$$x = \zeta - \int\Big[\mathrm{e}^{-\int l(t)\,\mathrm{d}t} f(t) \int \mathrm{e}^{\int l(t)\,\mathrm{d}t} h(t)\,\mathrm{d}t\Big]\mathrm{d}t - \int q(t)\,\mathrm{d}t\,, \quad t = \tau\,,$$

where $v(\zeta,\tau)$ satisfies Eq. (4), while $u(x,t)$ satisfies Eq. (1).

Reference [16] has addressed that equation (1) has the Painlevé property only when

$$g(t) = f(t)\,\mathrm{e}^{-\int l(t)\,\mathrm{d}t}\Big[c_1 c_2 + c_2 \int f(t)\,\mathrm{e}^{-\int l(t)\,\mathrm{d}t}\,\mathrm{d}t\Big]\,, \quad (6)$$

where c_1 and c_2 are both arbitrary real constants with $c_1^2 + c_2^2 \neq 0$. Under Condition (6), we plan to construct the auto-Bäcklund transformations for Eq. (1) through various methods.

The rest of this paper is organized as follows. In Sec. 2, the auto-Bäcklund transformations in two different forms for Eq. (1) will be derived. In Sec. 3, the auto-Bäcklund transformation for Eq. (1) will be presented through the truncated Painlevé expansion method and the compatibility of such method will be testified. In Sec. 4, the auto-Bäcklund transformation for Eq. (1) will be obtained by the extended variable-coefficient balancing-act method. In Sec. 5, the auto-Bäcklund transformation for Eq. (1) in the accepted form will be given. Section 6 will be the discussions and conclusions for this paper.

2 Auto-Bäcklund Transformations in Bilinear and Lax Pair Forms

In this section, with the help of symbolic computation,[5,17] we will derive out the auto-Bäcklund transformations in bilinear and Lax pair forms for Eq. (1).

Introducing

$$u = \frac{12\,g(t)}{f(t)}\,\frac{\partial^2}{\partial x^2}\ln[\tau(x,t)]$$

directly into Eq. (1) with Condition (6), we can get the following general variable-coefficient bilinear form

$$[D_x\,D_t + g(t)\,D_x^4](\tau\cdot\tau) + \frac{2\,A'(t)}{A(t)}\,\tau\,\tau_x = 0\,, \quad (7)$$

where in the following analysis,

$$A(t) = \Big[c_1 + \int \mathrm{e}^{-\int l(t)\mathrm{d}t} f(t)\mathrm{d}t\Big]$$

and the prime sign denotes the differential with respect to t, while $D_x D_t$ and D_x^4 are both the bilinear operators defined in Ref. [18] as

$$D_x^m D_t^n(a\cdot b) \equiv \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^m \Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^n a(x,t)\,b(x',t')\Big|_{x'=x,\,t'=t}\,. \quad (8)$$

Let $\tau_1(x,t)$ and $\tau_2(x,t)$ be two distinct solutions for Eq. (7), the following equation

$$P \equiv {\tau_1}^2\,[D_x\,D_t + g(t)\,D_x^4]\,(\tau_2\cdot\tau_2) - {\tau_2}^2\,[D_x\,D_t + g(t)\,D_x^4]\,(\tau_1\cdot\tau_1) + \frac{A'(t)}{A(t)}{\tau_1}^2\tau_2\tau_{2,\,x} - \frac{A'(t)}{A(t)}\,{\tau_2}^2\tau_1\tau_{1,\,x} = 0\,, \quad (9)$$

can be regarded as the auto-Bäcklund transformation for Eq. (7) in bilinear form. By virtue of the properties of bilinear operators,[18−20] P can be transformed into

$$2D_x\Big\{\Big[D_t + 3\lambda(t)D_x + g(t)D_x^3 + \frac{A'(t)\,x}{2A(t)}D_x\Big](\tau_2\cdot\tau_1)\Big\}\cdot(\tau_1\tau_2) + 6D_x\Big\{\Big[g(t)D_x^2 - \lambda(t) - \frac{A'(t)\,x}{6\,A(t)}\Big](\tau_2\cdot\tau_1)\Big\}\cdot(D_x\tau_1\cdot\tau_2) = 0\,. \quad (10)$$

Accordingly, equation (10) could be split into two parts:

$$\Big[D_t + 3\lambda(t)D_x + g(t)D_x^3 + \frac{A'(t)\,x}{2\,A(t)}D_x\Big](\tau_2\cdot\tau_1) = 0\,, \quad (11)$$

$$\Big[g(t)D_x^2 - \lambda(t) - \frac{A'(t)\,x}{6\,A(t)}\Big](\tau_2\cdot\tau_1) = 0\,, \quad (12)$$

which constitute the auto-Bäcklund transformation for Eq. (1) in bilinear form, where $\lambda(t)$ is an arbitrary function.

Furthermore, if we let

$$\tau_2(x,t) = \tau_1(x,t)\Psi(x,t)\,, \quad (13)$$

and substitute it into Eqs. (11) and (12), the Lax pair for Eq. (1) is derived as follows:

$$L = 6g(t)\partial_x^2 + f(t)u_1(x,t) - \frac{A'(t)}{A(t)}x\,, \quad (14)$$

$$M = -\frac{1}{3}\Big[f(t)u_1(x,t) - 2\lambda(t) + \frac{2\,A'(t)\,x}{A(t)}\Big]\partial_x + \frac{1}{6}\Big[f(t)\frac{\partial u_1(x,t)}{\partial x} - \frac{A'(t)}{A(t)}\Big]\,, \quad (15)$$

where the operators L and M satisfy $L\Psi(x,t) = -\lambda(t)\Psi(x,t)$ and $M\Psi(x,t) = \partial_t\Psi(x,t)$, respectively. With symbolic computation, under Condition (6), it is easy to verify that $L_t - [M,L] + \lambda'(t) = 0$ with $\lambda(t) = c_0\,g(t)/A(t)^2$ (c_0 is an arbitrary parameter). Equivalently, the Lax pair can also be written in the form

$$\Psi_{xx} = -\frac{1}{6\,g(t)}\Big[f(t)u_1(x,t) - \frac{A'(t)}{A(t)}\,x + \lambda(t)\Big]\Psi\,, \quad (16)$$

$$\Psi_t = -\frac{1}{3}\Big[f(t)u_1(x,t) - 2\lambda(t) + \frac{2\,A'(t)\,x}{A(t)}\Big]\Psi_x + \frac{1}{6}\Big[f(t)\frac{\partial u_1(x,t)}{\partial x} - \frac{A'(t)}{A(t)}\Big]\Psi\,. \quad (17)$$

Since $\tau_1(x,t)$ and $\tau_2(x,t)$ are both solutions for Eq. (1) in bilinear form, hereby, the following relation

$$u_2(x,t) = u_1(x,t) + \frac{12\,g(t)}{f(t)}\,\frac{\partial^2}{\partial x^2}\ln[\Psi(x,t)]\,, \quad (18)$$

can also be regarded as the auto-Bäcklund transformation in the Lax pair form for Eq. (1), where

$$u_i(x,t)=\frac{12\,g(t)}{f(t)}\frac{\partial^2}{\partial x^2}\ln[\tau_i(x,t)]\quad(i=1,2)\,,$$

while $\Psi(x,t)$ satisfies Eqs. (16) and (17).

3 Auto-Bäcklund Transformation Through Truncated Painlevé Expansion Method

In this section, we will determine the auto-Bäcklund transformation for Eq. (1) through the truncated Painlevé expansion method.

It is known that the necessary condition for Eq. (1) to be completely integrable is that it possesses the Painlevé property[21] when the solutions written as

$$u(x,t)=\phi^{-J}(x,t)\sum_{i=0}^{\infty}u_i(x,t)\phi^i(x,t)\,,\tag{19}$$

are single-valued in the neighborhood of a noncharacteristic, where J is a natural number to be determined, $u_i(x,t)$ and $\phi(x,t)$ are both analytic functions with $u_0(x,t)\neq 0$.

According to the leading-order analysis of Eq. (1), we obtain the truncated Painlevé expansion as

$$u(x,t)=u_0(x,t)\phi^{-2}(x,t)+u_1(x,t)\phi^{-1}(x,t)+u_2(x,t)\,,\tag{20}$$

which is substituted into Eq. (1), yielding

$$\begin{aligned}&\phi^{-5}[-2f(t){u_0}^2\phi_x-24g(t)u_0{\phi_x}^3]\\&+\phi^{-4}[f(t)u_0u_{0,\,x}-3f(t)u_0u_1\phi_x+18g(t)u_{0,\,x}{\phi_x}^2-6g(t)u_1{\phi_x}^3+18g(t)u_0\phi_x\phi_{xx}]\\&+\phi^{-3}[-2u_0\phi_x+f(t)u_1u_{0,\,x}+f(t)u_0u_{1,\,x}-f(t){u_1}^2\phi_x-2\,f(t)u_0u_2\phi_x\\&+6g(t)u_{1,\,x}{\phi_x}^2-6g(t)\phi_xu_{0,\,xx}-6g(t)u_{0,x}\phi_{xx}+6g(t)u_1\phi_x\phi_{xx}-2g(t)u_0\phi_{xxx}]\\&+\phi^{-2}[l(t)u_0+u_{0,\,t}-u_1\phi_t+f(t)u_2u_{0,\,x}+f(t)u_1u_{1,\,xx}+f(t)u_0u_{2,\,x}-f(t)u_1u_2\phi_x\\&-3g(t)\phi_xu_{1,\,xx}-3g(t)u_{1,x}\phi_{xx}+g(t)u_{0,\,xxx}-g(t)u_1\phi_{xxx}]\\&+\phi^{-1}[l(t)u_1+u_{1,\,t}+f(t)u_2u_{1,\,x}+f(t)u_1u_{2,\,x}+g(t)u_{1,\,xxx}]\\&+u_{2,\,t}+f(t)u_2u_{2,\,x}+g(t)u_{2,\,xxx}+l(t)u_2=0\,.\end{aligned}\tag{21}$$

Let the coefficients of ϕ^{-5} and ϕ^{-4} in Eq. (21) be zero, we know that

$$u_0(x,t)=\frac{-12g(t)}{f(t)}{\phi_x}^2\,,\quad u_1(x,t)=\frac{12g(t)}{f(t)}\phi_{xx}\,.\tag{22}$$

Thus, the truncated Painlevé expansion (20) becomes

$$u(x,t)=u_2(x,t)+\frac{12g(t)}{f(t)}\,(\ln\phi)_{xx}\,,\tag{23}$$

which constitutes the auto-Bäcklund transformation for Eq. (1), while $u_2(x,t)$ is a solution for Eq. (1) and $\phi(x,t)$ satisfies

$$\phi_x\phi_t+f(t)u_2{\phi_x}^2-3g(t){\phi_{xx}}^2+4g(t)\phi_x\phi_{xxx}=0\,,\tag{24}$$

$$\phi_{xt}+f(t)u_2\phi_{xx}+g(t)\phi_{xxxx}+\frac{A'(t)}{A(t)}\phi_x=0\,.\tag{25}$$

In the following analysis, with the help of Schwarzian derivative-scattering method,[22] we will testify the compatibility of Eqs. (24) and (25).

Firstly, eliminating $u_2(x,t)$ in Eqs. (24) and (25) yields

$$\frac{\partial}{\partial x}\Big[\frac{\phi_t}{\phi_x}+g(t)H(x,t)+\frac{A'(t)}{A(t)}x\Big]=0\,,\tag{26}$$

with

$$H(x,t)=\Big\{\phi:x\Big\}\equiv\frac{\partial}{\partial\,x}\Big(\frac{\phi_{xx}}{\phi_x}\Big)-\frac{1}{2}\Big(\frac{\phi_{xx}}{\phi_x}\Big)^2\,,\tag{27}$$

where $\{\phi:x\}$ is called the Schwarzian derivative.[22] Integrating Eq. (26) with respect to x, we have

$$\frac{\phi_t}{\phi_x}+g(t)\,H(x,t)+\frac{A'(t)}{A(t)}x=\eta_0(t)\,,\tag{28}$$

where $\eta_0(t)$ is an integration function of t.

Secondly, let $\phi(x,t)=\psi_1(x,t)/\psi_2(x,t)$ and require $\psi_i(x,t)$ $(i=1,2)$ to satisfy the following scattering problem:

$$\psi_{i,\,xx}=U(x,t)\psi_i\,,\tag{29}$$

$$\psi_{i,\,t}=V(x,t)\psi_{i,\,x}+W(x,t)\psi_i\,,\tag{30}$$

where $U(x,t)$, $V(x,t)$, and $W(x,t)$ are all real functions of x and t to be determined.

Introducing $\phi(x,t)=\psi_1(x,t)/\psi_2(x,t)$ into Eq. (28), we have

$$V-2g(t)\,U+\frac{A'(t)}{A(t)}x=\eta_0(t)\,.\tag{31}$$

The compatibility of Eqs. (29) and (30), i.e., $\psi_{i,\,xxt}=\psi_{i,\,txx}$ $(i=1,2)$ gives rise to

$$U_t=V\,U_x+2\,U\,V_x+W_{xx}\,,\tag{32}$$

$$W_x=-\frac{1}{2}\,V_{xx}\,.\tag{33}$$

Substitution of Eqs. (31) and (33) into Eq. (32) yields

$$\begin{aligned}U_t&=6g(t)\,U\,U_x-g(t)\,U_{xxx}-\frac{A'(t)}{A(t)}\,x\,U_x\\&+\eta_0(t)U_x-2\frac{A'(t)}{A(t)}U\,.\end{aligned}\tag{34}$$

Finally, we assume

$$U(x,t)=\frac{f(t)}{6g(t)}u(x,t)+Y(x,t)\,,$$

and substitute it into Eq. (34), yielding

$$Y(x,t)=\frac{1}{6g(t)}\Big[\frac{A'(t)}{A(t)}\,x+6\eta_0(t)\Big]\,,\tag{35}$$

$$U(x,t) = -\frac{1}{6g(t)}\Big[f(t)u(x,t) - \frac{A'(t)}{A(t)}\,x - 6\eta_0(t)\Big]\,, \tag{36}$$

where $u(x,t)$ satisfies Eq. (1). From Eqs. (31) and (33), we obtain

$$V(x,t) = -\frac{1}{3}\Big[f(t)u(x,t) + 2\frac{A'(t)}{A(t)}\,x + 12\eta_0(t)\Big]\,, \tag{37}$$

$$W(x,t) = \frac{1}{6}\Big[f(t)\frac{\partial u(x,t)}{\partial x} + 2\frac{A'(t)}{A(t)}\Big] + \eta_1(t)\,, \tag{38}$$

where $\eta_1(t)$ is an integration function of t.

We note that if the integration functions $\eta_0(t) = -6\lambda(t)$ with arbitrary function $\lambda(t)$ and $\eta_1(t) = -A'(t)/2A(t)$, then equations (29) and (30) become

$$\psi_{i,\,xx} = -\frac{1}{6g(t)}\Big[f(t)u(x,t) - \frac{A'(t)}{A(t)}\,x + \lambda(t)\Big]\psi_i\,, \tag{39}$$

$$\psi_{i,t} = -\frac{1}{3}\Big[f(t)u(x,t) - 2\lambda(t) + \frac{2\,A'(t)\,x}{A(t)}\Big]\psi_{i,\,x} + \frac{1}{6}\Big[f(t)\frac{\partial u(x,t)}{\partial x} - \frac{A'(t)}{A(t)}\Big]\psi_i\,, \tag{40}$$

which are just the Lax pair for Eq. (1) and identical to Eqs. (16) and (17).

Furthermore, by choosing $\lambda(t) = c_0 g(t)/A(t)^2$ with arbitrary parameter c_0 and substituting it into the Lax pair (39) and (40) under Condition (6), we have

$$\psi_{i,\,xxt} - \psi_{i,\,txx} = \frac{-\,\mathrm{e}^{\int l(t)\,\mathrm{d}t}\psi_i(x,t)}{6\,c_2\,A(t)} \times [u_t + f(t)uu_x + g(t)u_{xxx} + l(t)u] = 0\,. \tag{41}$$

Hereby, the compatibility of Eqs. (24) and (25) is testified.

4 Auto-Bäcklund Transformation Through Extended Variable-Coefficient Balancing-Act Method

We know that the extended variable-coefficient balancing-act method is an effective means to construct the auto-Bäcklund transformation for a given system of NLEEs.[3] So, utilizing such method, we can seek for the general auto-Bäcklund transformation for Eq. (1) in the form

$$u(x,t) = k(t)\frac{\partial^2}{\partial x^2}F[\varphi(x,t)] + u_0(x,t)\,, \tag{42}$$

where $k(t)$, $F(\varphi)$, $\varphi(x,t)$, and $u_0(x,t)$ are all differentiable functions to be determined.

Substituting Eq. (42) into Eq. (1) yields

$$\begin{aligned} &l(t)u_0 + u_{0,\,t} + f(t)u_0u_{0,\,x} + k(t)\,l(t)\,F''\varphi_x{}^2 + k'(t)\,F''\varphi_x{}^2 + k(t)\,F^{(3)}\varphi_t\,\varphi_x{}^2 \\ &+ f(t)k(t)F''u_{0,x}\varphi_x{}^2 + f(t)k(t)u_0F^{(3)}\varphi_x{}^3 + f(t)k(t)^2F''F^{(3)}\varphi_x{}^5 + g(t)k(t)F^{(5)}\varphi_x{}^5 \\ &+ 2k(t)F''\varphi_x\varphi_{xt} + k(t)l(t)F'\varphi_{xx} + F'k'(t)\varphi_{xx} + k(t)F''\varphi_t\varphi_{xx} + f(t)k(t)F'u_{0,x}\varphi_{xx} \\ &+ 3f(t)k(t)u_0F''\varphi_x\varphi_{xx} + 3f(t)k(t)^2F''^2\varphi_x{}^3\varphi_{xx} + f(t)k(t)^2F'F^{(3)}\varphi_x{}^3\varphi_{xx} \\ &+ 10g(t)k(t)F^{(4)}\varphi_x{}^3\varphi_{xx} + 3f(t)k(t)^2F'F''\varphi_x\varphi_{xx}{}^2 + 15g(t)k(t)F^{(3)}\varphi_x\varphi_{xx}{}^2 \\ &+ k(t)F'\varphi_{xxt} + g(t)u_{0,xxx} + f(t)k(t)u_0F'\varphi_{xxx} + f(t)k(t)^2F'F''\varphi_x{}^2\varphi_{xxx} \\ &+ 10g(t)k(t)F^{(3)}\varphi_x{}^2\varphi_{xxx} + f(t)k(t)^2F'^2\varphi_{xx}\varphi_{xxx} + 10g(t)k(t)F''\varphi_{xx}\varphi_{xxx} \\ &+ 5g(t)k(t)F''\varphi_x\varphi_{xxxx} + g(t)k(t)F'\varphi_{xxxxx} = 0\,, \end{aligned} \tag{43}$$

where $k'(t) = (\mathrm{d}/\mathrm{d}t)k(t)$ and $F^{(j)} = \mathrm{d}^jF[\varphi(x,t)]/\mathrm{d}\varphi^j$.

To simplify Eq. (43), we set the coefficient of φ_x^5 to be zero and obtain

$$f(t)k(t)^2\,F''\,F^{(3)} + g(t)\,k(t)\,F^{(5)} = 0\,, \tag{44}$$

which has a solution

$$F[\varphi(x,t)] = 12\ln\varphi(x,t)\,,$$

with

$$k(t) = \frac{g(t)}{f(t)}\,. \tag{45}$$

Thus, the general auto-Bäcklund transformation (42) becomes

$$u(x,t) = u_0(x,t) + \frac{12g(t)}{f(t)}\,(\ln\varphi)_{xx}\,, \tag{46}$$

where $u_0(x,t)$ is an arbitrary solution for Eq. (1), which is the same as truncated Painlevé expansion (23) and $\varphi(x,t)$ satisfies the following equations

$$\varphi_x\varphi_t + f(t)u_0\varphi_x{}^2 - 3g(t)\varphi_{xx}{}^2 + 4g(t)\varphi_x\varphi_{xxx} = 0\,, \tag{47}$$

$$\varphi_{xt} + f(t)u_0\varphi_{xx} + g(t)\varphi_{xxxx} + \frac{A'(t)}{A(t)}\varphi_x = 0\,. \tag{48}$$

The compatibility of Eqs. (47) and (48) under condition (6) has been testified in Sec. 3.

5 Auto-Bäcklund Transformation Expressed as Wahlquist–Estabrook (WE) Form for Eq. (1)

In the above sections, the auto-Bäcklund transformations for Eq. (1) in various forms have been presented. In this section, we will write the auto-Bäcklund transformation in bilinear form as the WE[24] form, i.e.,

$$(w_2 + w_1)_x = \Gamma(w_1, w_2, x, t)\,, \tag{49}$$

$$(w_2 - w_1)_t = \Pi(w_1, w_2, x, t)\,, \tag{50}$$

where $w_i(x,t) = \partial\,u_i(x,t)/\partial x$ $(i = 1, 2)$ satisfy the potential vcKdV equation for Eq. (1), while $\Gamma(w_1, w_2, x, t)$ and $\Pi(w_1, w_2, x, t)$ are both analytic functions to be determined.

In order to write the auto-Bäcklund transformation in the WE form, we introduce the relations

$$\tau_1(x,t) = \mathrm{Exp}\Big[\frac{\rho(x,t)-\xi(x,t)}{2}\Big]\,, \tag{51}$$

$$\tau_2(x,t) = \mathrm{Exp}\Big[\frac{\rho(x,t)+\xi(x,t)}{2}\Big]\,, \tag{52}$$

where $\tau_1(x,t)$ and $\tau_2(x,t)$ are two distinct solutions of the auto-Bäcklund transformation in bilinear form, while $\rho(x,t)$ and $\xi(x,t)$ are both differentiable functions.

Substituting expressions (51) and (52) into Eqs. (11) and (12), i.e., the auto-Bäcklund transformation in bilinear form, we obtain

$$-\lambda(t) - \frac{x\,A'(t)}{6\,A(t)} + g(t)\,({\rho_x}^2 + \xi_{xx}) = 0\,, \tag{53}$$

$$\begin{aligned} &-\rho_t - 3\lambda(t)\rho_x - \frac{x\,A'(t)\rho_x}{2\,A(t)} \\ &\qquad - g(t)\Big({\rho_x}^3 + 3\rho_x\xi_{xx} + \rho_{xxx}\Big) = 0\,. \end{aligned} \tag{54}$$

Taking derivative of both sides in Eq. (54) with respect to x yields

$$-\rho_{xt} - 3\lambda(t)\rho_{xx} - \frac{A'(t)\rho_x}{2\,A(t)} - \frac{x\,A'(t)\rho_{xx}}{2\,A(t)} - g(t)\Big(3{\rho_x}^2\rho_{xx} + 3\rho_{xx}\xi_{xx} + 3\rho_x\xi_{xxx} + \rho_{xxxx}\Big) = 0\,. \tag{55}$$

We know that $\tau_1(x,t)$ and $\tau_2(x,t)$ are two distinct solutions for Eq. (7), hereby the following equations are satisfied,

$$\rho_x = \frac{\tau_{2,\,x}}{\tau_2} - \frac{\tau_{1,\,x}}{\tau_1} = \frac{\partial \ln\tau_2}{\partial\,x} - \frac{\partial \ln\tau_1}{\partial\,x} = \frac{f(t)}{12g(t)}\Big[w_2(x,t) - w_1(x,t)\Big]\,, \tag{56}$$

$$\xi_x = \frac{\tau_{1,\,x}}{\tau_1} + \frac{\tau_{2,\,x}}{\tau_2} = \frac{\partial \ln\tau_2}{\partial\,x} + \frac{\partial \ln\tau_1}{\partial\,x} = \frac{f(t)}{12g(t)}\Big[w_2(x,t) + w_1(x,t)\Big]\,. \tag{57}$$

Substituting Eqs. (56), (57), and their derivatives into Eqs. (53) and (55), we can get the auto-Bäcklund transformation for Eq. (1) in the WE form below

$$(w_2 + w_1)_x = -\,\frac{f(t)}{12g(t)}\Big(w_2 - w_1\Big)^2 + \frac{2\,A'(t)\,x}{A(t)\,f(t)} + \frac{12\lambda(t)}{f(t)}\,, \tag{58}$$

$$\begin{aligned} (w_2 - w_1)_t = &-\Big[\frac{A'(t)}{2\,A(t)} + \frac{6g(t)\,A'(t)}{A(t)\,f(t)}\Big](w_2 - w_1) + \Big[\frac{f(t)}{2} - \frac{f(t)^2}{48g(t)}\Big](w_2 - w_1)^2\,(w_2 - w_1)_x \\ &- \Big[3\lambda(t) + \frac{A'(t)\,x}{2\,A(t)}\Big](w_2 - w_1)_x - \frac{f(t)}{4}\,(w_{2,\,x}^2 - w_{1,\,x}^2) - g(t)\,(w_{2,\,xxx} - w_{1,\,xxx})\,, \end{aligned} \tag{59}$$

where equations (58) and (59) equate with the "space" part and "time" part, respectively.

6 Discussions and Conclusions

The vcKdV models with additional terms, i.e., Eqs. (1), (2), and (4), have been widely used in physical and engineering sciences. For instance, those models can describe the trapped quasi-one-dimensional Bose–Einstein condensates, water waves in a channel with an uneven bottom and/or deformed walls, nonlinear excitations of a Bose gas of impenetrable bosons with longitudinal confinement, dynamics of a circular rod composed of a general compressible hyperelastic material with variable cross-sections and material density, propagation of weakly nonlinear solitary waves in a varied-depth shallow-water tunnel. In the above sections, by using *Mathematica*, the auto-Bäcklund transformations in different forms for the damped vcKdV model have been obtained through various methods.

The discussions and conclusions of this paper are as follows.

(i) Without any constraint condition, some other vcKdV models are shown to be transformable into the damped vcKdV equation, e.g., Eqs. (2) and (4). Hereby, the above different expressions of auto-Bäcklund transformations for Eq. (1) can be mapped to Eqs. (2) and (4) respectively through Transformations (3) and (5). Meanwhile, the Painlevé-integrable conditions of Eqs. (2) and (4) are the same as Condition (6).

(ii) Different from the classical auto-Bäcklund transformation for constant-coefficient KdV model[18] and the one in bilinear form presented in Ref. [25] for vcKdV models under Condition (6) with $c_2 = 0$, the obtained auto-Bäcklund transformations in various forms in this paper include arbitrary functions of t, as seen in Eqs. (11), (12), (58), and (59).

(iii) In Secs. 3 and 4, through the truncated Painlevé expansion method and extended variable-coefficient balancing-act method, we derive the auto-Bäcklund transformation for Eq. (1). It is found that the results via the two methods are in accord with each other. The primary reason is that balancing $f(t)uu_x$ with $g(t)u_{xxx}$ in Eq. (1) leads to $F[\varphi(x,t)]$ being a logarithmic-type function in the extended variable-coefficient balancing-act method, which makes Eq. (42) have the same form as Eq. (23). Meanwhile, the compatibility of Eqs. (24) and (25) (or equivalent Eqs. (47) and (48)) has been testified through the Schwarzian derivative-scattering method and Lax equation firstly.

(iv) As we know, the above auto-Bäcklund transformations in different forms for Eq. (1) are based on the Painlevé-integrable condition. In addition, many other remarkable properties such as the N-soliton (or N-soliton-like) solution, Wronskian expression, nonlinear superposition formula, Lax pair and Darboux transformation can also be constructed under this integrable condition. Physically speaking, the Painlevé-integrable condition for a certain vcKdV model should be detailed as the suitable dynamical conditions as depicted in Ref. [23].

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