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Nonlinear Dynamical Behavior in Neuron Model Based on Small World Network with Attack and Repair Strategy*

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Abstract *In this paper, we investigate the effect due to the change of topology structure of network on the nonlinear dynamical behavior, by virtue of the OFC neuron evolution model with attack and repair strategy based on the small world. In particular, roles of various parameters relating to the dynamical behavior are carefully studied and analyzed. In addition, the avalanche and EEG-like wave activities with attack and repair strategy are also explored in detail in this work.*

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Key words: attack, repair, self-organized criticality, small world network

1 Introduction

Recently, the small world network, which can be transferred from regular to random state by varying a single parameter, has been studied by Watts and Strogatz.[1] Since then, a great deal of articles concentrating on the small world network have emerged, varying from economy field to medicine area, due to the fact that a large number of complex systems around us, such as the social networks,[2] electric power grids, highways or subway systems[2] and neural networks,[3] can be regarded as small world networks. The basic idea of the small world network is to add some shortcuts into regular network, which impose some unique characters of the small world network as large clustering coefficient and small shortest path length. It should be also pointed out that there is another phenomenon called self-organized criticality (SOC) in the nature, such as earthquake, extinction events in biological evolution, and sandpile model, which can lead to scaling law in the dynamics of driven systems, evolving towards a critical state without the fine tuning of an outer control parameter.[4] An obvious feature of the system possessed SOC is that the scaling of the space-time is invariable for this system.

As is well known, the brain is consisted of $10^{10} \sim 10^{12}$ neurons.[5] However, the active neurons that are working for us constitute only a small portion of the whole neurons. Furthermore, it is reported that 100 000 neurons die each day. The mechanism accounting for the transmission of information in the brain can be explained as follows: Each neuron can hold two states, integrate and fire, which can be described by the membrane potential $V(t)$. In addition, the neurons have thousands of synapses (connections) with others typically. Once the input information is being processed, some particular currents flow into the neural system.[6] With the incentive increasing, the neuron that is stable at first becomes active gradually and eventually fires when the potential arrives at its threshold value, and hence the message will transfer to other neurons. This strategy is similar to that for the Olami–Feder–Christensen (OFC) earthquake model.

There are a large amount of papers on the small world network in terms of OFC model recently, including the effect induced by the variation of the topology structure of network on the dynamical behavior. In addition, the brain with the small world network in the framework of OFC neuron model has been investigated by our group,[7,8] the conclusion of which is that the distribution of avalanche size shows power-law behavior, namely, $P(S) \propto (S)^{-\tau}$. Furthermore, detailed analysis about the EEG-like wave, which denotes the average membrane potential of all neurons in the network each time, i.e. $\langle V(t)\rangle = (1/L^2)\sum_{i=1}^{L^2} V_i(t)$, is also presented in our previous study. One of the motivations of this work is initiated by the recent report that neuron can repair itself to a certain extent. In fact, the strategy of attack and repair has been wildly used in many other networks, such as the Internet. In view of this point, we introduce the attack and repair mechanism to the previous model in order to study its dynamical behavior and the connection with the working mechanism of the brain.

The structure of this paper is organized as follows. After this introduction, we firstly display the construction and the dynamical process of adopted model in Sec. 2. In Sec. 3, we show the simulation results: the influence of various parameters of the model on SOC behavior and that of on the EEG-like wave activities. The last section is devoted to our conclusion and outlook.

2 The Model

Firstly, we would like to present the construction of the network used in this work below.

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(i) Start with a two-dimensional regular square lattice with $L \times L$ sites. All bonds are present between the nearest neighbor sites. The boundary condition is open.

(ii) Randomly choose two sites of the lattice and place a bond between them excluding self-connections and duplicate links.

(iii) Repeat Step (ii) until the number of the bonds added is the fraction ϕ of all bonds of the original lattice, i.e. $2\phi L(L-1)$.

(iv) Find a node with the maximum degree and remove all its links. (If several nodes happen to have the same highest degree of connection, we randomly choose one of them.)[9]

(v) Reconnect the node with other nodes which are randomly chosen in the network, we reconnect them with repair probability θ, which decides the amount of the nodes to be connected to the node.

(vi) Repeat Steps (iv) and (v) for T times. This parameter decides the attack times.

The square lattice here denotes a sheet of neurons occurring in the cortex, while each node describes a neuron in fact. A connection between two nodes represents a synapse. As usual, once some nodes are attacked, the neurons are destroyed totally. However, the neuron possesses the ability of self-repair in some way, as mentioned above. Hence, it is necessary to study the model with attack and repair mechanism.

A kind of integrate-and-fire mechanism can be described in the following.[10] For any neuron sited at position i in the lattice, we give it a dynamical variable V_i ($V_i \geq 0$), which represents the membrane potential of the i-th neuron. When a neuron's dynamical variable V_i exceeds a threshold $V_{\rm th} = 1$, the neuron i is unstable and it will fire and return to a rest state (V_i returns to zero). Each of the nearest neighbors will receive a pulse (action potential) and its membrane potential V_j will be changed.

Here we give a simple description of our model as follows.

(i) Initialize the membrane potential of each neuron below $V_{\rm th}$.

(ii) Find out the maximal value of all V_i, $V_{\max}$, and add $V_{\rm th} - V_{\max}$ to all neurons.

(iii) If there exists any unstable neuron, $V_i \geq V_{\rm th}$, then redistribute the membrane potential V_i on the i-th neuron to its nearest neighbors:

$$V_j \to aV_j + \frac{b}{q_i}V_i\,, \quad V_i \to 0\,,$$

where a is a constant smaller than 1, denoting the remains of V_i due to its slow relaxation after the firing. The parameter b represents the pulse intensity, and q_i is the number of neighbors of neuron i.

(iv) Repeat Step (iii) until all the neurons of the lattice are stable. The progress is called an avalanche and the number of the unstable sites is called the size of the avalanche.

(v) Apply Step (ii) again and another new avalanche begins.

So far, we have finished the construction of a small world network in light of the OFC neuron model with the attack and repair mechanism. In the subsequent section, we will explore the effect resulting from the attack and repair strategy to the dynamical behavior of the network in detail.

3 Simulation Results

3.1 *Degree Distribution*

The degree for a definite site is defined as the total number of connections of a site. In addition, degree distribution is the simplest statistical character describing the properties of a network, and it is often introduced to figure the probability that the site in the network has k connections, which can be written as[11]

$$P(k_i) = \frac{k_i}{\sum\limits_j k_j}\,.$$

The degree distribution of the small world network with attack and repair mechanism has been presented in Fig. 1 with different values of connection probability ϕ. We take the attack times $T = 10$ and the repair probability $\theta = 0.002$. It can be easily observed that the degree distribution approaches to the classical Possion distribution. On the other hand, the curve of degree distribution becomes flat with the increment of ϕ, and the position of the peak moves from left to right.

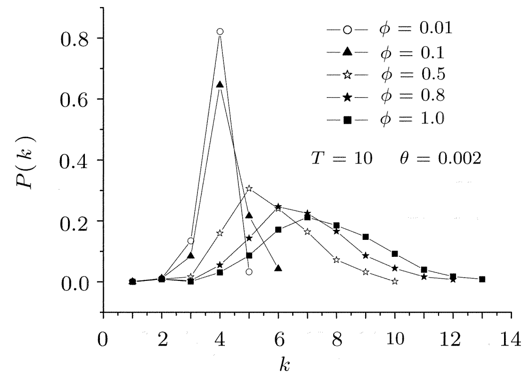

Fig. 1 The degree distribution of networks with attack and repair mechanism for different connection probability ϕ with $T = 10$, $\theta = 0.002$.

3.2 *Power-Law Behavior and Influence of Different Parameters*

As can be observed from our previous work, the distribution of the avalanche size obeys a power-law, i.e. $P(S) \propto S^{-\tau}$, where the value of the parameter τ becomes large with the increment of ϕ, and the value of τ is between 1.15 and 1.40, with the ϕ falling into the following range $[0, 1]$. It needs to be noticed that $\phi = 1$ represents a fully connected network, on the contrary, the network is regular with $\phi = 0$. As for the case that the value of ϕ is between 0 and 1, the network denotes the small world effect. The problem is how this conclusion will change with the attack and repair mechanism.

3.2.1 *The Influence of Connection Probability ϕ*

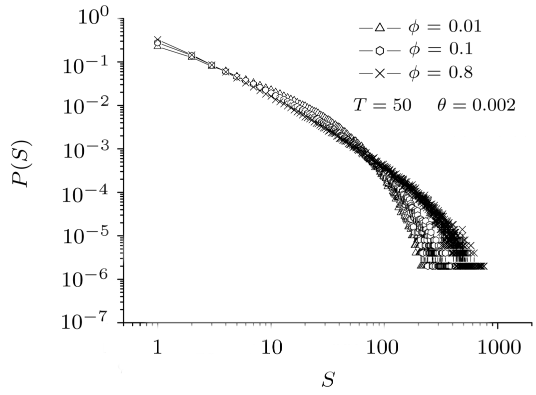

Fig. 2 Power-law behavior for networks with attack and repair at different connection probability ϕ with $T = 50$, $\theta = 0.002$.

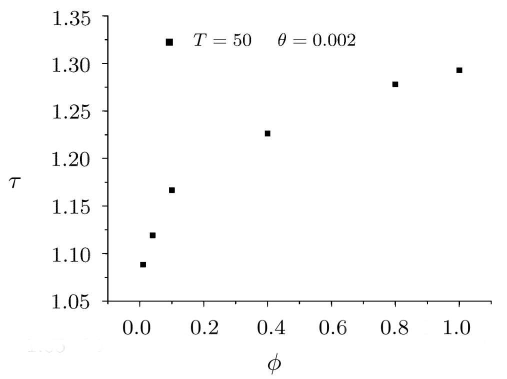

Fig. 3 The power-law exponent τ of the avalanche size distribution as a function of ϕ for $L = 35$, $T = 50$, $\theta = 0.002$.

In particular, we fix two parameters T and θ characterizing attack and repair mechanism, only with variation of the parameter ϕ. We take $T = 50$, $\theta = 0.002$, and $L = 35$ which is the size of lattice. Figure 2 shows that the distribution of the avalanche size also follows the power-law behavior. It can be also found that the cutoff of the avalanche distribution grows large with the increment of parameter ϕ. It needs to be pointed out that some changes on the parameter τ have emerged, even if the work still satisfies the formula $P(S) \propto S^{-\tau}$. The impact of the variation of ϕ on the parameter τ is displayed in Fig. 3, from which we can find that the larger the parameter ϕ, the larger the τ will be. However, the value of τ is a little smaller than that of the previous model[8] without attack and repair mechanism with the same ϕ, which can be indicated by the simple truth that the value of τ is equal to 1.29 at $\phi = 1$ here, but it is 1.4 in the previous model. It is reasonable that with ϕ increasing, the shortcuts also increase, so it is easy to transfer message to more sites and to further sites. The size of large avalanche increases, and the power-law exponent τ increases. But the value of τ is a little smaller at the same ϕ because of attack.

3.2.2 *The Influence of the Size of Lattice*

We fix the parameters relating to the attack and repair strategy to simplify and explore the impact due to change of the size of lattice L on the dynamical behavior of network. As shown in Fig. 4, we can find that the larger the size L, the large size avalanche will be more possible, which also indicates that it is the behavior of SOC rather than local behavior. On the other hand, it is also confirmed by the biology experiment that the transfer of information between neurons will be much more convenient, if the more neurons are used in practice. Therefore, our conclusion is reasonable.

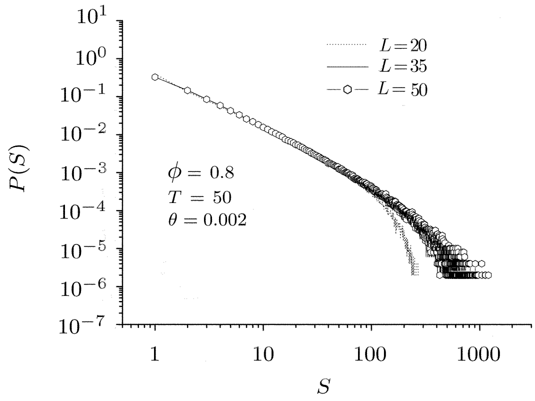

Fig. 4 Power-law behavior for networks with attack and repair at different sizes of lattice $L = 20$, 35 and 50 with $\phi = 0.8$, $T = 50$, $\theta = 0.002$.

3.2.3 *The Influence of Attack Times*

To focus on effect of attack, we fix the other parameters. The parameter ϕ is chosen as $\phi = 0.01$ (close to regular state) and 0.8 (close to full connection state) respectively, while the repair parameter θ is set as $\theta = 0.002$. In general, the value of θ should not be too large, since the repairing ability of the cerebra is very limited. The simulation results in this case have been shown in Fig. 5, from which we can conclude that the cutoff in the avalanche size distribution decreases with the increment of T. Another important point that can be read from this figure is that the change of the curve between $T = 10$ and $T = 50$ is larger for the case of network with less shortcuts ($\phi = 0.01$), while the change is smaller for the network with more shortcuts ($\phi = 0.8$). It means that the cutoff in the avalanche size distribution changes more quickly at $\phi = 0.01$. Especially, the effect of the attack is more obvious in the network with less shortcuts ($\phi = 0.01$) compared network with more shortcuts ($\phi = 0.8$). In other words, the network with less shortcuts is much more difficult to transfer information than that with more shortcuts under the condition that both the two networks are attacked to the same extent.

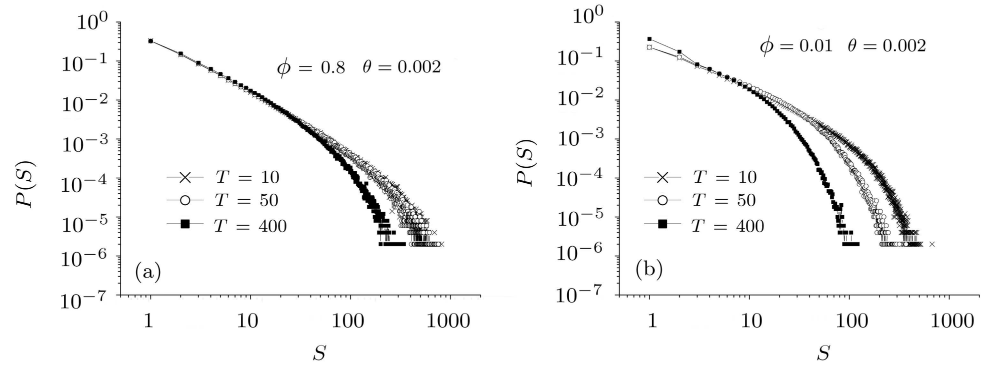

Fig. 5 (a) Power-law behavior for networks with attack and repair at different T, $T = 10$, 50, and 400 with $\phi = 0.8$, $\theta = 0.002$; (b) Power-law behavior for networks with attack and repair at different T, $T = 10$, 50, and 400 with $\phi = 0.01$, $\theta = 0.002$.

3.2.4 *The Influence of the Repair*

Now we are going to concentrate on the repair mechanism. We take $L = 35$, and the parameter ϕ is selected as $\phi = 0.01$ and 0.8 separately. Generally speaking, the trend of the change corresponding to the cutoff in the avalanche size distribution is that the larger the repair parameter θ, the larger cutoff. As presented in Fig. 6, it seems that the avalanche does not change too much at $\phi = 0.8$, however, the avalanche varies remarkably at $\phi = 0.01$ between $\theta = 0$ and $\theta = 0.004$. Thanks to the self-repair ability of neuron, the brain can work normally, although about 100 000 neurons die each day. In a word, the brain is robust because of repair mechanism and shortcuts. On the other hand, we may suspect that the topology structure of our brain is a small world network with more shortcuts.

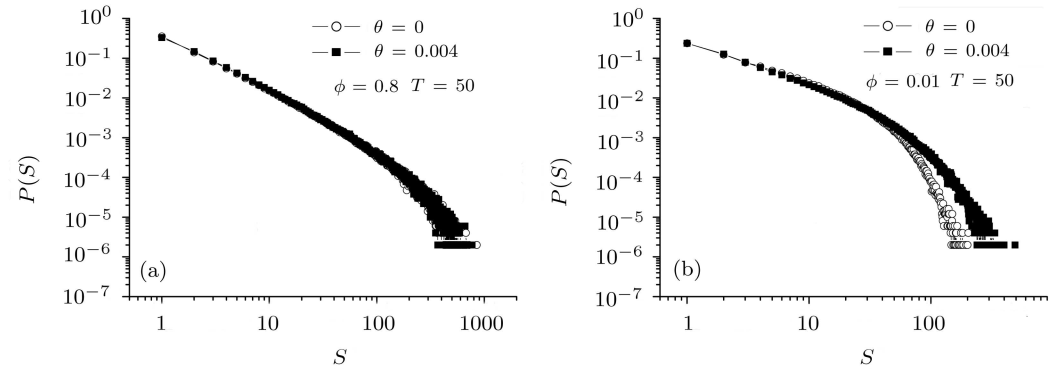

Fig. 6 (a) Power-law behavior for networks with attack and repair at different θ, $\theta = 0$ and 0.004 with $\phi = 0.8$, $T = 50$; (b) Power-law behavior for networks with attack and repair at different θ, $\theta = 0$, and 0.004 with $\phi = 0.01$, $T = 50$.

3.3 *EEG-LIKE Wave*

This section is contributed to the investigation of the temporal sequence of avalanches for the sake of exploring the temporal character of the neuron network. In the first step, we have to calculate the function of the average membrane potential of all lattice sites, which can be defined by averaging all the sites of the V after each avalanche, with the time being the numbers of avalanches. It can also be written in a mathematical form as

$$\langle V(t)\rangle = \frac{1}{L^2}\sum_{i=1}^{L^2} V_i(t)\,.$$

In this work, 2000 times of the $\langle V\rangle$ are simulated in order to obtain Fig. 7, from which we can find that the amplitude of the signal is smaller in model with attack and repair mechanism than that in the previous one. In fact, the biology experiment shows that the brain can work more actively with the larger amplitude of the electroencephalography. We can conclude that the more compact the network is, the more effective the brain works. If the brain is attacked, the amplitude of the electroencephalography becomes small and it works less effectively.

In order to explore the effect induced by the attack and repair mechanism on EEG-like wave more carefully, the two parameters T and θ are considered separately. The value of ϕ is adopted as 0.01 in both cases. For one thing, we choose $T = 10$ and 400 with invariant θ respectively to simulate the EEG-like wave of neuron network, the results of which have been shown in Fig. 8. It can be easily found that the amplitude of electroencephalography becomes small with T increasing, which indicates that the brain works less effectively. In the next place, we fix the number of T as 50, with the value of θ being 0.002 and 0.01 separately, the simulation results are presented in Fig. 9, from which we can observe that the larger θ corresponds to the larger amplitude of electroencephalography and the more active of our brain. In a short summary, if the network is much tighter, then the larger amplitude will be and hence the brain can work more actively.

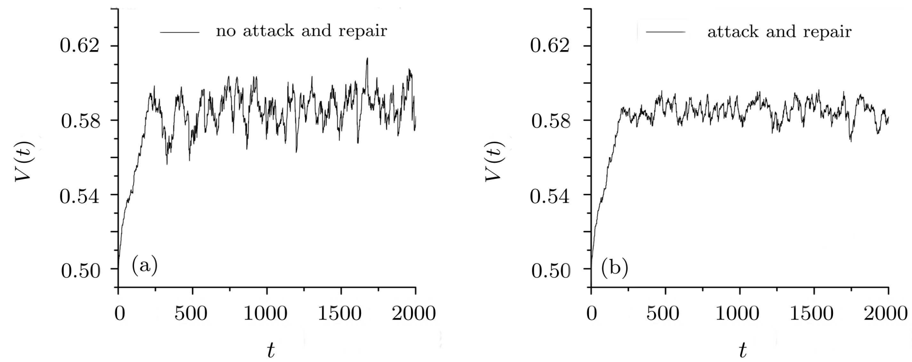

Fig. 7 EEG-like wave of the neuron networks. (a) EEG-like wave of the neuron network without attack and repair mechanism with $\phi = 0.01$; (b) EEG-like wave of the neuron network with attack and repair mechanism with $\phi = 0.01$, $T = 50$, $\theta = 0.002$.

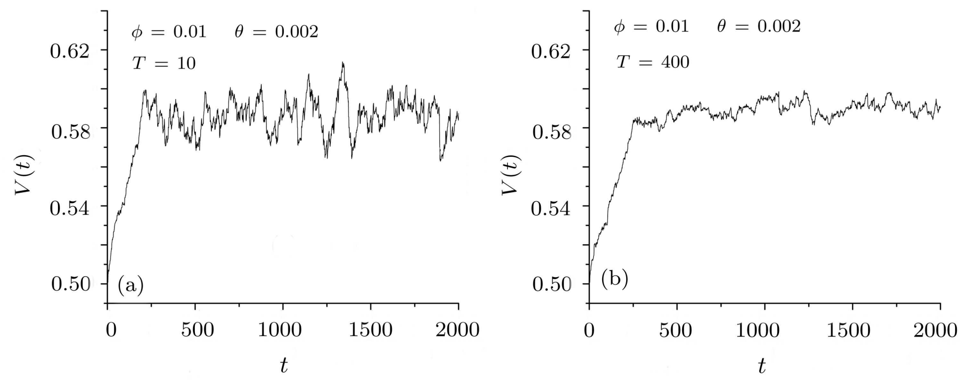

Fig. 8 EEG-like wave of the neuron networks at different T, $T = 10$ and 400 with $\phi = 0.01$, $\theta = 0.002$.

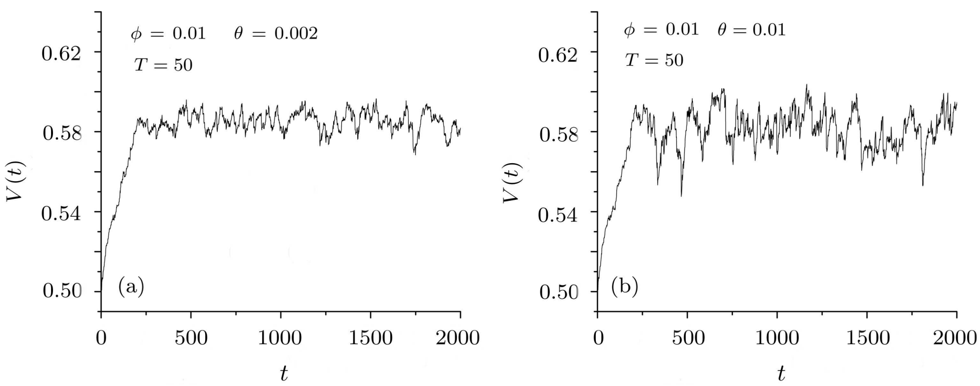

Fig. 9 EEG-like wave of the neuron networks at different θ, $\theta = 0.002$ and 0.01 with $\phi = 0.01$, $T = 50$.

In addition, it is found that the spectra of EEG signals often present the $1/f$ power-law distribution with frequencies.[12] Lately, our group[13] also found the same rule in small world network with OFC model. In this work, we also study the spectra of EEG signals under the attack and repair strategy, and it is indeed obtained the $1/f$ power-law distribution. Specially, the distribution showed the behavior as $S(f) \propto f^{-\tau}$, with the following parameters $\phi = 0.01$, $T = 50$, $\theta = 0.002$, and τ is near 1.785. We also find that when the number of T is set as 50 and 300 respectively, the power-law will be more exact with increment of T as shown in Fig. 10. In addition, it can be also learnt from Fig. 10 that the power-law can become better corresponding to the smaller number of θ.

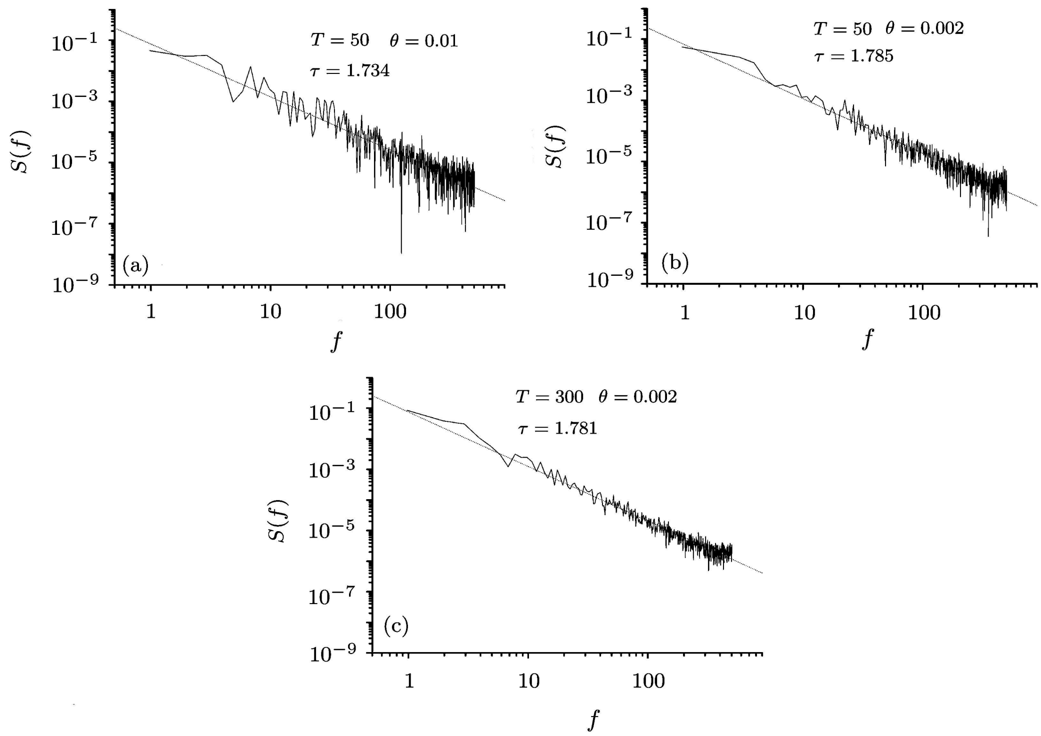

Fig. 10 The power spectrum of the average membrane potential at different θ, $\theta = 0.01$ (a) and 0.002 (b) with $\phi = 0.01$, $T = 50$. The power spectrum of the average membrane potential at different T, $T = 50$ (b) and 300 (c) with $\phi = 0.01$, $\theta = 0.002$.

4 Conclusion

Due to the new finding in the medical field that the neurons can repair themselves in a way, we introduce this mechanism into neuron model based on small world network to explore the effect resulting from this ingredient in this work. The dynamical behavior of neuron network and complex presentation of EEG-like activities are investigated in detail with this mechanism. With the increment of attack times T, the cutoff in the avalanche size distribution and the amplitude of EEG decrease, while they all increase with the increment of repair probability θ. The conclusion we obtained are reasonable. Based on this work, effect induced by variation of other dynamical parameters on kinds of models can be further studied.

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