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Criticality in Two-Variable Earthquake Model on a Random Graph

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Abstract *A two-variable earthquake model on a quenched random graph is established here. It can be seen as a generalization of the OFC models. We numerically study the critical behavior of the model when the system is nonconservative: the result indicates that the model exhibits self-organized criticality deep within the nonconservative regime. The probability distribution for avalanche size obeys finite size scaling. We compare our model with the model introduced by Stefano Lise and Maya Paczuski [Phys. Rev. Lett.* **88** *(2002) 228301], it is proved that they are not in the same universality class.*

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1 Introduction

The idea of self-organized criticality (SOC) was introduced by Bak and his co-workers in 1987,[1] as an attempt to explain the appearance of scale invariance in nature. Several typical types of models of self-organized criticality exist, such as sand-pile model,[2−4] forest-fire model,[5,6] earthquake model,[7,8] and so on. The majority of these simulations has been limited to conservative models in early times. But recently, there are more and more evidences indicating that some nonconservative earthquake models also display SOC.[9−13]

Earthquakes may be the most dramatic example of SOC that can be seen on earth. Most of the time, the crust of the earth is at rest, or quiescent. These periods of stasis are punctuated by sudden, thus far unpredictable, bursts or earthquake.[14]

In 1992, Olami, Feder, and Christensen proposed a simplified dissipative earthquake model (OFC model) on a two-dimensional regular square lattice,[7] where the OFC model displays criticality. Later, OFC model has been applied to some other networks. For example, Lise and Paczuski have proposed the OFC model on quenched random graph,[9] which displays criticality deep within the nonconservative regime; the earthquake model on small world networks also has been investigated,[10,11] and it displays criticality under some conditions; Hergarten and Neugebauer have studied the two-variable model on a two-dimensional regular square lattice,[12] C.J. Boulter and G. Miller have extended this model and proved that it displays criticality, even when the parameter measuring the level of dissipation equals 0 if sufficiently large systems are considered.[13] Although both the analytical and numerical evidence above in favor of criticality are quite convincing, the role played in these models by the nonconservative dynamics is still not clear. There are still many debates on it.[14,15]

2 Model

The purpose of this paper is to investigate the criticality of the two-variable model on a quenched random graph.

The quenched random graph is defined as a set of N sites connected by bonds randomly. Two connected sites are denoted as "nearest neighbor". The number of the nearest neighbors of every site is the same q (self-connections and duplicate edges are excluded).

The dynamical process of our model is as follows.

(i) With each site of the network are associated two variables u_i and w_i, the conservative variables are represented by u_i, while the nonconservative variables are represented by w_i. Initially, the two variables are chosen randomly from a uniform distribution between 0 and 1.

(ii) All the forces are increased uniformly and simultaneously at the same speed. This continues until at some site the energy reaches the threshold $u_i w_i = 1$, at which point an avalanche is initiated. The supercritical site relaxes according to

$$u_{\rm neigh} = u_{\rm neigh} + \frac{1}{q} u_i \,, \quad u_i \to 0 \,, \quad w_i \to e w_i \,, \qquad (1)$$

where neigh denotes the nearest neighbor site of i. In this paper, we choose $q = 4$ and $q = 6$. The variable w_i is not redistributed during the toppling, and the parameter ε, which is assumed to be in the range $0 \le \varepsilon < 1$, measures the level of dissipation. If $\varepsilon = 1$, both u_i and w_i are conservative during the toppling; conversely, if $\varepsilon < 1$, the toppling rule is dissipative in the variable w_i.

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(iii) Repeat Step (ii) until all sites of the network are stable. The sequence of the toppling of the unstable sites forms an avalanche. Define this process as one avalanche, the number of topplings during an earthquake defines its size s.

(iv) Begin Step (ii) again and another new avalanche begins.

In order to observe scaling in the avalanche distribution, the inhomogeneities must be introduced.[9] On the lattice, there is no critical behavior if the periodic boundary conditions are applied,[16−18] this is generally achieved by considering open boundary conditions, which imply that boundary sites have fewer nearest neighbors.

Inhomogeneities induce partial synchronization of the elements of the system building up long range spatial correlations and thereby creating a critical state. The mechanism of synchronization requires an underlying spatial structure and therefore cannot operate in an annealed RN model,[9] where each site is assigned new random neighbors at each update. It suffices to consider just two sites in the system with coordination $q-1$ on the random graph.

3 Simulations and Results

After a sufficiently long transient time, the system settles into a statistically stationary state. The size of an avalanche can be defined in several ways: the number of topplings s, the avalanche time duration t, and the avalanche area a. Here we mainly concentrate on the nonconservative case. We focus on the probability distribution of earthquake sizes, s in a system of size N, $P(s,N)$. At first, we report the probability distribution of avalanche sizes with parameter $\varepsilon = 0.2$ for different system sizes. The statistics are collected in the critical state for 10^6 non-zero avalanches for each system size. From Fig. 1, we can see that the distribution scales with system size, which is indicative of a critical state. In fact, the largest avalanche roughly coincides with system size.

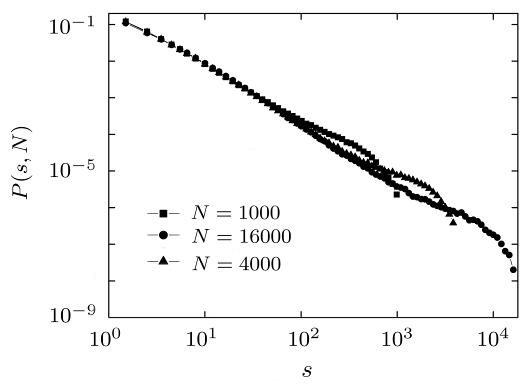

Fig. 1 Log-log plot of the probability distribution $P(s,N)$ for the number of nearest neighbor sites $q = 4$, the parameter $\varepsilon = 0.2$ with different system sizes (from left to right, the system sizes $N = 1000$, 4000, 16000). The data have been binned over exponentially increasing sizes with base 1.1.

In order to characterize the critical behavior of the model, a finite size scaling (FSS) ansatz is used.

$$P(s,N) \approx N^{-\beta} f \frac{s}{N^D}\,, \tag{2}$$

where f is a so-called universal scaling function, β and D are critical exponents describing the scaling of the distribution function. The critical exponent D expresses how the finite-size cutoff scales with the system size, while the critical exponent β is related to the normalization (or rather renormalization) of the distribution function.

As shown in Fig. 2, an FSS collapse of $P(s,N)$ for different q is shown. We can see that the probability distribution $P(s,N)$ satisfies the FSS hypothesis reasonably well. The critical exponents derived from the fit of Fig. 2 are $\beta = 1.71$, $D = 1$. The FSS hypothesis implies that, for asymptotically large N, the value of the exponent is $\tau = \beta/D \approx 1.71$. It is different from the one for the quenched random graph model ($\tau = 1.65$).

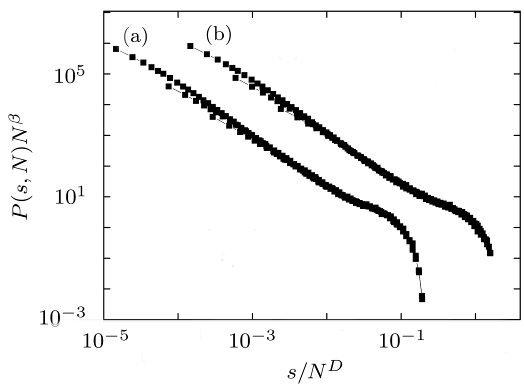

Fig. 2 Data collapse analysis of the case with the parameter$\varepsilon = 0.2$, the number of nearest neighbors $q = 4$ (a) and $q = 6$ (b) with different system sizes. The critical exponents are $\beta = 1.71$ and $D = 1$. For visual clarity, curves (b) have been shifted along the x axis.

It is verified that the statistical properties of the system are independent of the actual realization of the random graph, as long as the coordination number q is the same.[9] And it is also a strong evidence of universality for all dissipative choices of ε.

We now discuss the time properties of the avalanches. In Fig. 3 we report the average size of an avalanche stopping at time t, $\langle s\rangle_t$, as a function of the rescaled time $\bar{t} = t + 10$ (as we are mainly interested at large values of t, the constant should be irrelevant). We have studied the relationship between the time t and the average size of the avalanche stopping at time t for different parameter ε, the result indicates that the distribution between the two variables obeys a power-law behavior, and the straight line of the two curves (a) and (b) in Fig. 3 is in parallel. The curves for different system sizes overlap (deviations can be attributed to finite-size effects) and we observe that $\langle s \rangle_t \approx t^\gamma$, where $\gamma \approx 1.8$, providing further evidence of criticality in the nonconservative system. But it is different from the one for the quenched random graph model ($\gamma = 2.1$), this also indicates that the two models are not in the same universality class.

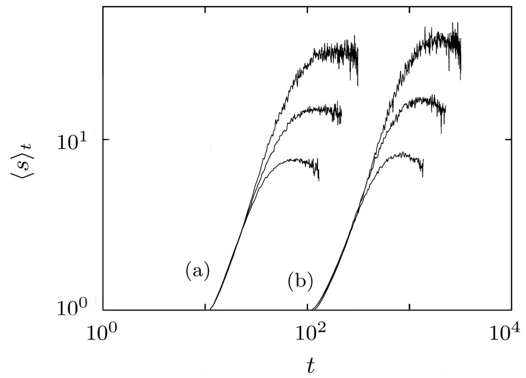

Fig. 3 Average size of an avalanche lasting t time steps as a function of t for the number of nearest neighbors $q = 4$, the parameter $\varepsilon = 0.2$ (a) $\varepsilon = 0.8$ (b) respectively. Different curves correspond, from bottom to top, to system sizes $N = 1000$, 4000, 16000. For visual clarity, curves (b) have been shifted along the x axis. The slope of the straight line is $\gamma = 1.8$.

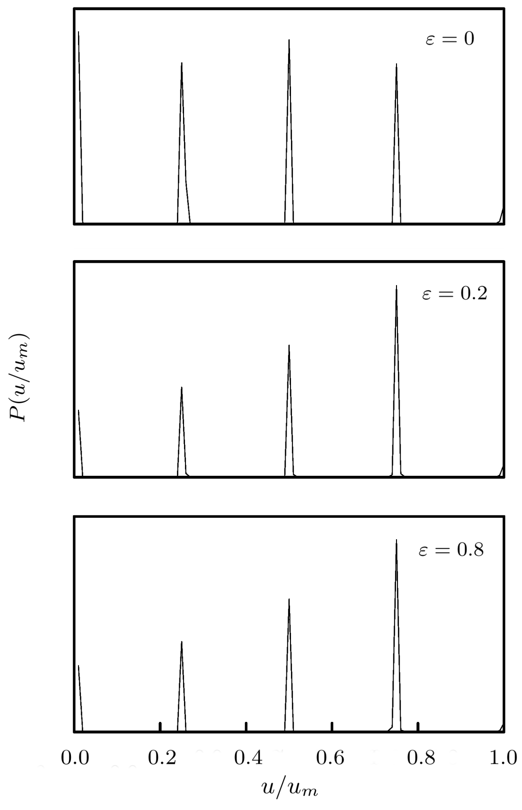

Fig. 4 Probability distributions of values u/u_m from simulations with $N = 1000$.

In order to understand the model better we investigate the distributions of the variables . We have simulated the distributions of the variable u_i with different parameters respectively. It seems that the distributions of the variable u_i with different parameter ε obey some laws in Fig. 4, it is clear that the distributions of the variable with different parameter ε are almost the same, with several significant peaks, and we further measure the peaks at $u/u_m \approx 0$, 1/4, 1/2, 3/4, where u_m is the maximum u value measured across the system, i.e., the parameter ε does not affect the distribution of variable u in the system.

When the system settles into a stationary state, almost every site has already toppled, during an avalanche, the toppling sites always take a u value that approximately equals u_m. Furthermore, the driving phase between two avalanches has little effect on the distribution of u. So, we can see that the system organizes in such a way that all sites take a u value that is approximately an integer multiple of $u_m/4$. We think it is still the evidence of universality for all dissipative choices of ε. This is different with the two-variable model on square lattice.

We have also simulated the distribution of variable w_i, but it seems disorderly and unsystematic, and it is complicated to explain the distribution.

4 Conclusions

In this paper, we have presented a generalized two-variable earthquake model based on a quenched random graph, on which every site has the same number nearest neighbors q with two sites whose number of nearest neighbors is $q - 1$. The introduction of the inhomogeneities in our model is the necessary condition that the system can reach critical state.

The probability distribution $P(s, N)$ displays power-law behavior, and $P(s, N)$ satisfies the FSS hypothesis when the system is nonconservative. This is the same as the quenched random graph model, but they have different critical behavior, that is to say, they are not in the same universality class. It seems that the toppling mechanism of the system has affected the critical behavior of the system. Contrarily, the different spatial topology does not alter the critical behavior of the system.

We have also studied the time properties of the avalanches, and the distribution of variable u_i. A power law relation between the size and the duration of an avalanche exists. The distributions of variable u_i have several significant peaks, and the locations of the peaks are almost the same, having no relations with the parameter. Both of them provide a strong evidence that the model is critical and universality for all dissipative choices of ε.

We also compared the critical behavior of our model with different number of nearest neighbors. It is shown that different spatial topology does not alter the critical behavior of the system.

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