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Influence of Auxiliary Equation on Wave Functions for Time-Dependent Pauli Equation in Presence of Aharonov–Bohm Effect

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Abstract *Invariant operator method for discrete or continuous spectrum eigenvalue and unitary transformation approach are employed to study the two-dimensional time-dependent Pauli equation in presence of the Aharonov–Bohm effect (AB) and external scalar potential. For the spin particles the problem with the magnetic field is that it introduces a singularity into wave equation at the origin. A physical motivation is to replace the zero radius flux tube by one of radius R, with the additional condition that the magnetic field be confined to the surface of the tube, and then taking the limit $R \to 0$ at the end of the computations. We point that the invariant operator must contain the step function $\theta(r - R)$. Consequently, the problem becomes more complicated. In order to avoid this difficulty, we replace the radius R by $\rho(t)R$, where $\rho(t)$ is a positive time-dependent function. Then at the end of calculations we take the limit $R \to 0$. The qualitative properties for the invariant operator spectrum are described separately for the different values of the parameter C appearing in the nonlinear auxiliary equation satisfied by $\rho(t)$, i.e., $C > 0$, $C = 0$, and $C < 0$. Following the C's values the spectrum of quantum states is discrete ($C > 0$) or continuous ($C \leq 0$).*

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1 Introduction

The scattering of charged particles by an infinitely long straight solenoid that encloses a magnetic flux known as the Aharonov–Bohm effect,[1] is of paramount interest in quantum physics. Even though the region containing the magnetic field is inaccessible to the particles, the magnetic flux inside the solenoid affects their propagation. The observed interference pattern cannot be explained within classical physics; it is a purely quantal effect, without classical correspondence. This fact differentiates the Aharonov–Bohm (AB) effect from other important processes, such as Rutherford scattering. The AB effect has been analyzed by many different approaches.[2−11] The scattering of an electron beam by a magnetic field existing in a region is restricted to the simplest situation of a straight filiform solenoid with constant magnetic flux ν.

The main part of the original AB paper[1] consisted of a calculation using a nonrelativistic spinless wave equation (i.e. Schrödinger equation). However, the elementary particles generally available for AB experiments all have spin one-half. In a series of articles,[3−5] Hagen shown that there are indeed observable effects associated with the spin degree of freedom by doing the calculation, in relativistic quantum mechanics, with the Dirac equation in presence of the AB effect. In nonrelativistic quantum mechanics when the spin is included, the situation becomes quite different. Here one is concerned with the Pauli equation.

A nonrelativistic charged spin-1/2 particle in external electromagnetic field is described by the Pauli Hamiltonian,

$$H_p = \frac{(\mathbf{p} - e\mathbf{A})^2}{2M} - \mu\sigma\mathbf{B} + U\,, \tag{1}$$

where $\mathbf{A}$ is electromagnetic vector potential; $\mathbf{B} = \operatorname{curl} \mathbf{A}$ is the external magnetic field, $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ is a vector consisting of three Pauli matrices, U is the external scalar potential, while M, e, and μ are respectively the mass, the charge and the magnetic moment of particle. The external scalar potential U can be chosen as a harmonic potential $U = M\omega^2 r^2/2$.

On the other hand, the time-dependent systems are still receiving considerable interest and used as models to describe several physical phenomena.[12−15] Effort was concentrated on spinless charged particle in a magnetic field,[16−27] However, the most frequently used charged particles have spin-1/2 and their physics is described in the presence of a magnetic field, by the Pauli equation in non-relativistic case. The energy levels of this system are called Landau levels. It is well known that such a system has played a fundamental role in physics and has a wide spectrum of applications.[28] For example, 2D electron systems have become an active research subject due to advances in nanofabrication technology like quantum wells, quantum wires, quantum dots, quantum Hall effect and high superconductivity.[30,31] Currently, the 2D harmonic potential $U = M\omega^2 r^2/2$ is often used to describe confined 2D systems in nonrelativistic case. For relativistic case, such systems are described using 2D Dirac oscillator.[30]

The extension of time-dependent systems to spinning charged particle remains unexplored and the few problems solved for nonstationary Pauli equation in the presence of

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time-varying electromagnetic fields has been investigated using supersymmetry.[32,33]

Recently, Bouguerra *et al.*[34] extended a previous work[29] to the case of a nonrelativistic charged spin-1/2 particle with time-dependent mass and frequency confined to (x, y) plane in the presence of the AB effect and a two-dimensional (2D) time-dependent harmonic potential $U = M(t)\omega^2(t)r^2/2$. The corresponding time-dependent Pauli Hamiltonian is

$$H_p(t) = \frac{(\mathbf{p} - e\mathbf{A})^2}{2M(t)} - \frac{e\hbar B}{2M(t)}\sigma^3 + \frac{1}{2}M(t)\omega^2(t)r^2 . \quad (2)$$

The magnetic field $\mathbf{B}$, associated to the AB effect, assumed to be perpendicular to the plane and confined to a thin magnetized filament, is given by

$$e\mathbf{B} = -\frac{\nu}{r}\delta(r)\mathbf{u}_z , \quad (3)$$

where ν is a finite and nonzero flux parameter and $\delta(r)$ is the Dirac's function,

$$\delta(r) = \begin{cases} 0, & r \neq 0 , \\ \infty, & r = 0 . \end{cases} \quad (4)$$

The vector potential $\mathbf{A}$ that gives this magnetic field is found to be

$$e\mathbf{A} = -\frac{\nu}{r}\theta(r)\mathbf{u}_\varphi , \quad (5)$$

where (r, φ, z) are cylindrical coordinates, $\mathbf{u}_\varphi$ denotes the unit vector in the φ-direction and $\theta(r)$ is the step function defined by

$$\theta(r) = \begin{cases} 1, & r > 0 , \\ 0, & r < 0 . \end{cases} \quad (6)$$

Since $\theta'(r) = \delta(r)$ it is easy to verify that the above $\mathbf{A}$ gives $\mathbf{B}$ correctly. Again, $\mathbf{A}$ satisfies the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. The problem with the magnetic field (3) is that it introduces a singularity into wave equation at the origin for spin particles.

There were attempts by Hagen[3–5] to provide a physical motivation for the choice of the boundary conditions among the admissible ones. He computed the eigenfunctions using the procedure of replacing the thread of flux by a fictitious flux tube of radius R (thus removing the ambiguity) and then taking the limit $R \to 0$ at the end of the computations. Such a model has in fact been presented in the context of obtaining the solution of the spin-1/2 AB scattering amplitude. His starting point was the replacement of the point like thread of flux by a magnetic field concentrated on the surface of a tube of radius R and in this case Eqs. (3) and (5) are replaced by

$$e\mathbf{B} = -\frac{\nu}{r}\delta(r - R)\mathbf{u}_z , \quad (7)$$

$$e\mathbf{A} = -\frac{\nu}{r}\theta(r - R)\mathbf{u}_\varphi . \quad (8)$$

Using the invariant operator theory and the three elements of standard representation of the group SL(2,R), namely,

$$T_1^\nu = \frac{1}{2}\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}(p_\varphi + \nu)^2 - \frac{\nu\hbar}{r}\delta(r)\sigma^3\Big] ,$$

$$T_2 = \frac{1}{2}[rp_r + p_r r], \quad T_3 = \frac{1}{2}r^2 ,$$

with the quantum-mechanical commutators relations

$$[T_1^\nu, T_2] = -2i\hbar T_1^\nu, \quad [T_2, T_3] = -2i\hbar T_3 ,$$
$$[T_1^\nu, T_3] = -i\hbar T_2 ,$$

where

$$p_r = -i\hbar\Big(\frac{\partial}{\partial r} + \frac{1}{2r}\Big), \quad p_\varphi = -i\hbar\frac{\partial}{\partial\varphi} ,$$
$$[r, p_r] = [\varphi, p_\varphi] = i\hbar ,$$

Bouguerra *et al.*[34] have recently studied the nonrelativistic charged spin-1/2 particle with time-dependent mass and frequency confined to (x, y) plane in the presence of the AB effect and a two-dimensional (2D) time-dependent harmonic potential. Such a model has in fact been presented in the context of obtaining the solution corresponding to zero radius of filament containing the Aharonov-Bohm flux. The invariant operator $I(t) = \sum_{i=1}^{3}\mu_i T_i$ written as a linear combination of the three elements of standard representation of the group SL(2,R) satisfies the Liouville-Von Neumann equation

$$\frac{\partial I(t)}{\partial t} = \frac{i}{\hbar}[I(t), H] . \quad (9)$$

On the other hand, if we replace the zero radius flux tube by one of radius R, with the additional condition that the magnetic field be confined to the surface of the tube, we point that the part of the operator T_1^ν, which depends on the space-coordinates must contain the step function $\theta(r - R)$. Consequently, the problem becomes more complicated because the algebra based on

$$T_1^\nu = \frac{1}{2}\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}(p_\varphi + \nu\theta(r - R))^2 - \frac{\nu\hbar}{r}\delta(r - R)\sigma^3\Big] ,$$

T_2 and T_3 is not closed and it is difficult to construct an exact invariant operator. In order to avoid this difficulty, it is convenient to replace the radius R by $\rho(t)R$, where $\rho(t)$ is a positive time-dependent function. The limit $R \to 0$ will be taken only after all calculations are carried out, to avoid a singular magnetic field at the singular point $r = 0$ of the coordinate system.

We consider the Pauli equation of the nonrelativistic charged spin-1/2 particle with time-dependent mass and frequency confined to (x, y) plane in the presence of the AB effect and a two-dimensional (2D) time-dependent harmonic potential,

$$i\hbar\frac{\partial}{\partial t}\psi = H_p(t)\psi , \quad (10)$$

with the Pauli Hamiltonian in the polar coordinates

$$H_p(t) = \frac{1}{2M(t)}\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}\Big(p_\varphi + \nu\theta\Big(\frac{r}{\rho(t)} - R\Big)\Big)^2 - \frac{\nu\hbar}{\rho(t)r}\delta\Big(\frac{r}{\rho(t)} - R\Big)\sigma^3\Big] + \frac{1}{2}M(t)\omega^2(t)r^2 , \quad (11)$$

and where $\psi = \binom{\psi_1}{\psi_2}$ is the two-component spinors.

The corresponding magnetic field $\mathbf{B}$ and vector potential $\mathbf{A}$ are respectively given by:

$$e\mathbf{B} = -\frac{\nu}{\rho(t)r}\delta\Big(\frac{r}{\rho(t)} - R\Big)\mathbf{u}_z , \quad (12)$$

$$e\mathbf{A} = -\frac{\nu}{r}\theta\Big(\frac{r}{\rho(t)} - R\Big)\mathbf{u}_\varphi\,, \tag{13}$$

where we have used the following propriety of the step function $\theta(r-\rho(t)R)$

$$\theta(r-\rho(t)R) = \theta\Big(\frac{r}{\rho(t)} - R\Big) = \begin{cases} 1 & \text{for } \dfrac{r}{\rho(t)} > R\,, \\ 0 & \text{for } \dfrac{r}{\rho(t)} < R\,. \end{cases} \tag{14}$$

This paper is organized as follows. In Sec. 2, we introduce and derive an appropriate invariant operator associated to the time-dependent Hamiltonian (11). The complete quantum solutions of the system will be investigated in Sec. 3. The concluding remarks are given in Sec. 4.

2 Invariant Operator and Quantum Phase

Let us first recall the general method to introduce the invariant operator theory for systems whose invariant operator has a discrete[35] or continuous[36] eigenvalues. For a system whose invariant has a completely discrete spectrum, specified by a time-independent eigenvalue $\lambda_{n,m}$ and a corresponding eigenvalue equation

$$I(t)\phi_{n,m}(r,\varphi,t) = \lambda_{n,m}\ \phi_{n,m}(r,\varphi,t)\,, \tag{15}$$

where n and m denote a quantum numbers. An invariant possesses a remarkable property that any eigenstate of $I(0)$ evolves into an eigenstate of $I(t)$. Then, if the set of reference eigenstates bispinors

$$\Big\{\phi_{n,m}(r,\varphi,t) = \begin{pmatrix}\phi^1_{n,m}\\ \phi^2_{n,m}\end{pmatrix}(r,\varphi,t)\Big\}$$

for the operator $I(t)$ are continuous with respect to t (all eigenstates are associated with the same time-independent eigenvalue $\lambda_{n,m}$), the corresponding global phases $\alpha_{n,m}(t)$ are defined by the relation associated to the wave functions $\psi_{n,m}(r,\varphi,t)$:

$$\psi_{n,m}(r,\varphi,t) = \exp[i\alpha_{n,m}(t)]\phi_{n,m}(r,\varphi,t)\,. \tag{16}$$

It follows from the Pauli equation (10) for $\psi_{n,m}(r,\varphi,t)$ that $\alpha_{n,m}(t)$ satisfies the relation

$$\hbar\frac{d}{dt}\alpha_{n,m}(t) = \Big\langle\phi_{n,m}(t)\Big|i\hbar\frac{\partial}{\partial t} - H\Big|\phi_{n,m}(t)\Big\rangle\,. \tag{17}$$

Very recently, Maamache and Saadi[36] have presented a straightforward, yet rigorous, proof of the exact quantum evolution for systems whose invariant operator $I(t)$ verifying (9) has a completely continuous spectrum, i.e.; its eigenvalues $\lambda_{k,m}$ are purely continuous and constants

$$I(t)\phi_{k,m}(r,\varphi,t) = \lambda_{k,m}\phi_{k,m}(r,\varphi,t)\,, \tag{18}$$

and have found that the eigenfunctions in a continuous spectrum $\phi_{k,m}(r,\varphi,t)$ of the invariant operator $I(t)$ and the solution $\psi_{k,m}(r,\varphi,t)$ of the Pauli equation are in the form

$$\psi_{k,m}(r,\varphi,t) = \exp[i\alpha_{k,m}(t)]\phi_{k,m}(r,\varphi,t)\,, \tag{19}$$

where the global phase $\alpha_{k,m}(t)$ is given by

$$\hbar\frac{d}{dt}\alpha_{k,m}(t) = \int_{-\infty}^{+\infty}\Big\langle\phi_{k',m}(t')\Big|i\hbar\frac{\partial}{\partial t'} - H(t')\Big|\phi_{k,m}(t')\Big\rangle dk'\,. \tag{20}$$

Now, we look for the invariant in the form

$$\begin{aligned} I(t) = {}& \frac{\mu_1(t)}{2}\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}\Big(p_\varphi + \nu\theta\Big(\frac{r}{\rho(t)} - R\Big)\Big)^2 \\ & - \frac{\nu\hbar}{\rho(t)r}\delta\Big(\frac{r}{\rho(t)} - R\Big)\sigma^3\Big] \\ & + \frac{\mu_2(t)}{2}[rp_r + p_r r] + \frac{\mu_3(t)}{2}r^2\,, \end{aligned} \tag{21}$$

where $\mu_i(t)$, $i=1,2,3$ are time-dependent functions which should be determined. Inserting Eqs. (11) and (21) into Eq. (9), we find that the time-dependent parameters are given by

$$\mu_1(t) = \rho(t)^2\,, \tag{22}$$

$$\mu_2(t) = -M(t)\rho(t)\dot\rho(t)\,, \tag{23}$$

$$\mu_3(t) = \frac{1}{\rho(t)^2}[C + (M(t)\rho(t)\dot\rho(t))^2]\,, \tag{24}$$

where

$$C = \mu_1(t)\mu_3(t) - \mu_2(t)^2 \tag{25}$$

is a real constant (this can be checked by direct differentiation of C with respect to time) and the time-dependent function $\rho(t)$ obeys the following auxiliary equations:

$$\ddot\rho(t) + \frac{\dot M(t)}{M(t)}\dot\rho(t) + \omega(t)^2\rho(t) = \frac{C}{M(t)^2\rho(t)^3}\,. \tag{26}$$

Consequently, the invariant operator can be constructed in the form

$$\begin{aligned} I(t) = {}& \frac{1}{2}\rho(t)^2\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}\Big(p_\varphi + \nu\theta\Big(\frac{r}{\rho(t)} - R\Big)\Big)^2 \\ & - \frac{\nu\hbar}{\rho(t)r}\delta\Big(\frac{r}{\rho(t)} - R\Big)\sigma^3\Big] \\ & - M(t)\rho(t)\dot\rho(t)[rp_r + p_r r] \\ & + \frac{1}{\rho(t)^2}[C + (M(t)\rho(t)\dot\rho(t))^2]r^2\,. \end{aligned} \tag{27}$$

3 Quantum Solutions

To find the eigenstates of $I(t)$, we look for a time-dependent unitary transformations $U(t)$ and $V_{\rho(t)}$ such that

$$\phi_{\gamma,m}(r,\varphi,t) = e^{im\varphi}U^\dagger(t)V^+_{\rho(t)}\chi_{\gamma,m}(r) \tag{28}$$

i.e., $U(t)$ and $V_{\rho(t)}$ bring any solution of the operator eigenvalue equation

$$I(t)\phi_{\gamma,m}(r,\varphi,t) = \lambda_{\gamma,m}\ \phi_{\gamma,m}(r,\varphi,t)\,, \tag{29}$$

(the subscript γ depends on the nature of the spectrum discrete ($\gamma = n$) or continuous ($\gamma = k$)), into a solution of the operator eigenvalue equation

$$I\,'(t)\chi_{\gamma,m}(r) = \lambda_{\gamma,m}\ \chi_{\gamma,m}(r)\,, \tag{30}$$

where the functional forms of $V_{\rho(t)}$ and $U(t)$ are

$$V_{\rho(t)} = \exp\Big(\frac{i\ln\rho(t)}{2\hbar}[rp_r + p_r r]\Big)\,, \tag{31}$$

$$U(t) = \exp\Big[-i\frac{M(t)\dot\rho(t)}{2\hbar\rho(t)}r^2\Big]\,. \tag{32}$$

It can easily be shown that under these transformations the coordinate and momentum operators change according to

$$V_{\rho(t)} r V^{\dagger}_{\rho(t)} = \rho(t) r\,, \tag{33}$$

$$U(t) p_r U^{\dagger}(t) = p_r + \frac{M(t)\dot{\rho}(t)}{\rho(t)} r\,, \tag{34}$$

$$V_{\rho(t)} p_r V^{\dagger}_{\rho(t)} = \frac{1}{\rho(t)} p_r\,. \tag{35}$$

An important property of the transformation $V_{\rho(t)}$, the action of which on a wave function in the r representation reads

$$V_{\rho(t)}\ \chi_{\gamma,m}(r) = \exp\Big(\frac{i\ln\rho(t)}{2\hbar}\Big)\chi_{\gamma,m}(r\rho(t))\,, \tag{36}$$

and therefore it defines a dilation.

One sees that the transformed invariant $I'(t)$ must satisfy the relation

$$I'(t) = V_{\rho(t)} U(t) I(t) U^{\dagger}(t) V^{\dagger}_{\rho(t)} = \frac{1}{2}\Big\{\Big[p_r^2 - \frac{\hbar^2}{4r^2} + \frac{1}{r^2}(p_\varphi + \nu\theta(r-R))^2 - \frac{\nu\hbar}{r}\delta(r-R)\sigma^3\Big] + Cr^2\Big\}\,. \tag{37}$$

By using $\delta(\xi - R)/\xi = \delta(\xi - R)/R$, Eq. (30) takes the following form

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\Big(-i\hbar\frac{\partial}{\partial\varphi} + \nu\theta(r-R)\Big)^2 - \frac{\nu\hbar}{R}\delta(r-R)\sigma^3 + C\ r^2\Big]\chi_{\gamma,m}(r,\varphi) = \lambda_{\gamma,m}\chi_{\gamma,m}(r,\varphi)\,. \tag{38}$$

in terms of two-component spinors $\chi_{\gamma,m} = \begin{pmatrix}\chi^1_{\gamma,m}\\ \chi^2_{\gamma,m}\end{pmatrix}$. The important point is that the transformed eigenvalue problem (38) is an ordinary time-independent Pauli equation of the nonrelativistic charged spin-1/2 particle confined to (x,y) plane in the presence of the AB effect and a harmonic potential. The fact that C is constant enables us to investigate the system separately for three cases, where $C > 0$, $C = 0$ and $C < 0$. The eigenstate of the system is discrete for $C > 0$ since the transformed invariant operator Eq. (38) corresponds to that of the oscillating system while that of the other two cases are continuous.

In order to solve the quantum problem and determine the solution $\psi_{\gamma,m}(r,\varphi,t)$ of the time-dependent Pauli equation, firstly let us calculate the quantum phases $\alpha_{\gamma,m}(t)$ (17) and (20). Carrying out the unitary transformations (31) and (32), the right-hand side of Eqs. (17) and (20) becomes

$$\Big\langle\phi_{\gamma',m}(t)\Big|i\hbar\frac{\partial}{\partial t} - H(t)\Big|\phi_{\gamma,m}(t)\Big\rangle = -\frac{1}{M(t)\rho^2(t)}\langle\chi_{\gamma',m}|I\,'|\chi_{\gamma,m}\rangle\,, \tag{39}$$

which gives

$$\Big\langle\phi_{\gamma',m}(t)\Big|i\hbar\frac{\partial}{\partial t} - H(t)\Big|\phi_{\gamma,m}(t)\Big\rangle = -\lambda_{\gamma,m}\frac{\langle\chi_{\gamma',m}|\chi_{\gamma,m}\rangle}{M(t)\rho^2(t)}\,. \tag{40}$$

The scalar product $\langle\chi_{\gamma',m}|\chi_{\gamma,m}\rangle$ depends on the nature of the spectrum of the invariant i.e.,

$$\langle\chi_{\gamma',m}|\chi_{\gamma,m}\rangle = \delta_{\gamma'\gamma} I \tag{41}$$

for discrete one, or

$$\langle\chi_{\gamma',m}|\chi_{\gamma,m}\rangle = \delta(\gamma' - \gamma) I \tag{42}$$

for continuous one.

Thus, the global phase can be represented by

$$\hbar\alpha_{\gamma,m}(t) = -\lambda_{\gamma,m}\int_{t_0}^{t}\frac{dt'}{M(t')\rho^2(t')}\,. \tag{43}$$

Let us write

$$\chi_{\gamma,m} = f_{\gamma,m}(r)\exp(im\varphi)\,,$$

equation (38) of the two components $\chi^{1,2}_{\gamma,m}$ is reduced to

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Big) + \frac{\hbar^2}{r^2}\Big(m + \frac{\nu}{\hbar}\theta(r-R)\Big)^2 + C\ r^2 - \frac{\nu\hbar s}{R}\delta(r-R)\Big] f^{1,2}_{\gamma,m} = \lambda_{\gamma,m}\ f^{1,2}_{\gamma,m}, \quad s = \pm 1\,, \tag{44}$$

where $s = \pm 1$ is twice the spin value ($+1$ for spin "up" and -1 for spin-down). For all what follows we adopt the following condition:

$$\text{If}\quad f^1_{\gamma,m}\quad \text{corespond to spin-up}\quad (s = +1)\quad \text{then}\quad f^2_{\gamma,m}\quad \text{is associated to spin-down}\quad (s = -1)\,.$$

We note that the last equation, except the inclusion of the oscillator term, is identical to that derived by Hagen[3−5] for a Dirac particle of mass M. In Ref. [34] an approach was taken, such an ansatz, which was based on the physically reasonable modifiaction of the vector potential to obtain this equation.

Equation (44) takes the following form in the two regions (i.e. the inside and the outside of the solenoid),

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Big) + \hbar^2\frac{(m + \nu/\hbar)^2}{r^2} + C\ r^2\Big] f^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m} f^{1,2}_{\gamma,m}(r), \qquad r > R\,, \tag{45}$$

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r\frac{\partial}{\partial r}\Big) + \hbar^2\frac{m^2}{r^2} + Cr^2\Big] f^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m} f^{1,2}_{\gamma,m}(r), \qquad r < R\,. \tag{46}$$

Since equations (45) and (46) have a regular and irregular solution, it is necessary to give a boundary condition

$$f_{\gamma,m}(R - \epsilon) = f_{\gamma,m}(R + \epsilon)\,, \tag{47}$$

$$\Big[\frac{df_{\gamma,m}(r)}{dr}\Big]_{R-\epsilon}^{R+\epsilon} = \frac{s\nu}{\hbar R} f_{\gamma,m}(R)\,, \tag{48}$$

which allows a unique result to be obtained.

Case 1 $C>0$

This case corresponds exactly to that treated by Bouguerra *et al.*[34] (for $C=1$). The solutions of the Eqs. (45) and (46) can be obtained in the usual way in the two regions ($r>R$ and $r<R$) in terms of the regular and irregular confluent hypergeometric functions. Using (47)and (48), the eigensolution can be written as (we refer the reader to [34] for details)

$$\begin{aligned} f^{1,2}_{\gamma,m}(\xi) &= D^{1,2}_m R^{|m|}\exp\Big[-\frac{\sqrt{C}\xi^2}{2\hbar}\Big]\Big\{R^{-|m+\nu/\hbar|}\Big(\frac{1}{2}+\frac{|m|+\nu s/\hbar}{2|m+\nu/\hbar|}\Big) \\ &\times \xi^{|m+\nu/\hbar|}\,{}_1F_1\Big(\frac{1}{2}\Big[1+\Big|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],\Big|m+\frac{\nu}{\hbar}\Big|+1;\frac{\sqrt{C}\xi^2}{\hbar}\Big)+R^{|m+\nu/\hbar|}\Big[\frac{1}{2}-\frac{|m|+\frac{\nu s}{\hbar}}{2|m+\nu/\hbar|}\Big] \\ &\times \xi^{-t|m+\nu/\hbar|}\,{}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1-|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],1-\Big|m+\frac{\nu}{\hbar}\Big|;\frac{\sqrt{C}\xi^2}{\hbar}\Big)\Big\}\,, \end{aligned} \tag{49}$$

the $R\to 0$, ($R^2\approx 0$) limit implies that the irregular solution contributes if the following condition

$$\Big|m+\frac{\nu}{\hbar}\Big| = -|m|-\frac{\nu s\rho}{\hbar} \tag{50}$$

is satisfied. In that case one must include the next higher power of R in the coefficient of the regular solution term in (49). This yields

$$\begin{aligned} f^{1,2}_{\gamma,m} &= R^{|m|}e^{-\sqrt{C}\xi^2/2\hbar}\Big\{C^{1,2}_m R^{|m|+(\nu s/\hbar)+2}\xi^{|m+\nu/\hbar|}\,{}_1F_1\Big(\frac{1}{2}\Big[1+|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],\Big|m+\frac{\nu}{\hbar}\Big|+1;\frac{\sqrt{C}\xi^2}{\hbar}\Big) \\ &+ R^{-|mt|-(\nu s/\hbar)}\xi^{-|m+(\nu/\hbar)|}\,{}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1-\Big|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],1-\Big|m+\frac{\nu}{\hbar}\Big|;\frac{\sqrt{C}\xi^2}{\hbar}\Big)\Big\}\,, \end{aligned} \tag{51}$$

where $C^{1,2}_m$ is a nonvanishing constant.

It follows that the irregular solution dominates

$$f^{1,2}_{\gamma,m}\to \xi^{-|m+\nu/\hbar|}\,{}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1-\Big|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],1-\Big|m+\frac{\nu}{\hbar}\Big|;\frac{\sqrt{C}\xi^2}{\hbar}\Big)\,, \tag{52}$$

provided that

$$|m|+\frac{\nu s}{\hbar}+1>0\,. \tag{53}$$

By using relation (50), the last condition (53) is written in the form

$$\Big|m+\frac{\nu}{\hbar}\Big|<1\,. \tag{54}$$

At the limit $R\to 0$, it is noticed that the solution (49) is always regular except if the two relations (50) and (54) are simultaneously verified. In order to give a signification of (50) and (54), it is convenient to write

$$\frac{\nu}{\hbar}=N+\eta,\quad \text{where } N \text{ is an integer and}\quad 0\le\eta<1\,. \tag{55}$$

Consequently, the solution $f^{1,2}_{\gamma,m}$ is always a regular solution except for the two cases

$$m=-N,\qquad N\ge 0,\qquad s=-1\,, \tag{56}$$

or

$$m=-N-1,\qquad N+1\le 0,\qquad s=+1\,, \tag{57}$$

where the irregular solution at the origin occurs.

Then, the eigenfunction $\chi^{1,2}_{\gamma,m}$ can be represented by

$$\begin{aligned} \chi^{1,2}_{\gamma,m}(r,\varphi) &= D^{1,2}_m\Big[1-\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}-\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}\Big]r^{|m+\nu/\hbar|}e^{-\sqrt{C}r^2/2\hbar}e^{im\varphi} \\ &\times {}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1+\Big|m+\frac{\nu}{\hbar}\Big|-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],\Big|m+\frac{\nu}{\hbar}\Big|+1,\frac{\sqrt{C}r^2}{\hbar}\Big) \\ &+\theta(-s)\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}D^{1,2}_m r^{-\eta}\,{}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1-\eta-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],1-\eta,\frac{\sqrt{C}r^2}{\hbar}\Big)e^{-\sqrt{C}r^2/2\hbar}e^{-iN\varphi} \\ &+\theta(s)\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}D^{1,2}_m r^{\eta-1}\,{}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[\eta-\frac{\lambda_{\gamma,m}}{\hbar\sqrt{C}}\Big],\eta,\frac{\sqrt{C}r^2}{\hbar}\Big)e^{-\sqrt{C}r^2/2\hbar}e^{-i(N+1)\varphi}\,, \end{aligned} \tag{58}$$

where $\theta(x)$ is the usual step function and δ_{ij} is the Kronecker symbol.

The asymptotic behavior ${}_1\mathrm{F}_1(\alpha,\gamma,z)\to[\Gamma(\gamma)/\Gamma(\alpha)]e^z z^{\alpha-\gamma}$ of the confluent series for large values of its argument shown that the function $\chi^{1,2}_{\gamma,m}$ is exponentially divergent. This divergence cannot be avoided except that by putting[37] $\alpha=-n$, where $n=0,1,2,\ldots$, thus transforming series into a polynomial of degree n (Laguerre polynomials). Hence

$$\chi^{1,2}_{n,m}(r,\varphi)=D^{1,2}_{m,n}\Big[1-\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}-\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}\Big]r^{|m+\nu/\hbar|}\mathrm{L}^{|m+\nu/\hbar|}_n\Big(\frac{\sqrt{C}r^2}{\hbar}\Big)\exp\Big(-\frac{\sqrt{C}r^2}{2\hbar}\Big)e^{im\varphi}$$

$$+\theta(-s)\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}D^{1,2}_{m,n}r^{-\eta}\mathrm{L}_n^{-\eta}\Big(\frac{\sqrt{C}r^2}{\hbar}\Big)\exp\Big(-\frac{\sqrt{C}r^2}{2\hbar}\Big)e^{-iN\varphi}$$
$$+\theta(s)\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}D^{1,2}_{m,n}r^{\eta-1}\mathrm{L}_n^{\eta-1}\Big(\frac{\sqrt{C}r^2}{\hbar}\Big)\exp\Big(-\frac{\sqrt{C}r^2}{2\hbar}\Big)e^{-i(N+1)\varphi}\,, \tag{59}$$

where the constant
$$D_{m,n}=\frac{n!\Gamma(|m+\nu/\hbar|+1)}{\Gamma(|m+\nu/\hbar|+1+n)}D_m\,.$$

If the conditions (56) and (57) are not satisfied (contribution of the regular solution), the eigenvalue of the regular solution are obtained starting from the condition
$$\frac{1}{2}\Big[\Big|m+\frac{\nu}{\hbar}\Big|+1-\frac{\lambda_{m,n}}{\hbar\sqrt{C}}\Big]=-n\,,$$
it follows that
$$\lambda_{m,n}=\hbar\sqrt{C}\Big(2n+|m+\frac{\nu}{\hbar}|+1\Big)\,. \tag{60}$$

If the conditions (56) and (57) are satisfied (contribution of the irregular solution), the eigenvalue takes different values for the two following cases.

Sub-case 1.1 ($m=-N$, $N\geq 0$, $s=-1$): From
$$\frac{1}{2}\Big[1-\eta-\frac{\lambda_{m,n}}{\hbar\sqrt{C}}\Big]=-n\,,$$
the eigenvalue takes the value
$$\lambda_{m,n}=\hbar\sqrt{C}(2n+1-\eta)\,. \tag{61}$$

Sub-case 1.2 ($m=-N-1$, $N+1\leq 0$, $s=+1$): From
$$\frac{1}{2}\Big[\eta-\frac{\lambda_{m,n}}{\hbar\sqrt{C}}\Big]=-n\,,$$
the eigenvalue takes the value
$$\lambda_{m,n}=\hbar\sqrt{C}(2n+\eta)\,. \tag{62}$$

Then, one deduces that the eigenvalue $\lambda_{m,n}$ of the invariant is given by

$$\lambda_{n,m}=\hbar\sqrt{C}\Big[1-\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}-\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}\Big]\Big(2n+\Big|m+\frac{\nu}{\hbar}\Big|+1\Big)$$
$$+\hbar\sqrt{C}\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}(2n+1-\eta)+\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}(2n+\eta)\,, \tag{63}$$

and consequently the phase (43) is given by

$$\alpha_{n,m}(t)=-\hbar\sqrt{C}\Big[1-\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}-\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}\Big]\Big[2n+\Big|m+\frac{\nu}{\hbar}\Big|+1\Big]\int_0^t\frac{dt'}{M\rho^2}$$
$$-\hbar\sqrt{C}\Big[\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N}(2n+1-\eta)+\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}(2n+\eta)\Big]\int_0^t\frac{dt'}{M\rho^2}\,. \tag{64}$$

It is worth noting that the phase depends on the spin and the magnetic flux. The first term is due to the contribution of the regular solution while the second term is due to the irregulars solutions.

In the particular case of spinless particle, one component of the general solution is reduced to the solution of a two dimensional time-dependent harmonic oscillator in presence of the AB effect if one takes only the contribution of this component in the phase (64).[29] In the absence of the AB effect ($\nu=0$), we find also that one component of the general solution is reduced to the solution of a 2D dimensional time-dependent harmonic oscillator if one takes only the contribution of this component in the phase (64).[26,29]

Case 2 $C=0$

Let us note that for $C=0$ the wave equations defined in (45) and (46) represent a free particle in AB effect analyzed by Hagen.[3–5] In particular, it was shown[3–5] that the solutions in the two regions $r<R$ and $r>R$ are respectively

$$f^{1,2}_{k,m}(\xi)=\begin{cases}A^{1,2}_m\mathrm{J}_{|m+\nu/\hbar|}(kr)+B^{1,2}_m\mathrm{J}_{-|m+\nu/\hbar|}(kr), & r>R\,,\\ C^{1,2}_m\mathrm{J}_{|m|}(kr), & r<R\,,\end{cases} \tag{65}$$

where
$$k^2=\frac{2\lambda_{k,m}}{\hbar^2}\,, \tag{66}$$

$A^{1,2}_m$, $B^{1,2}_m$, and $D^{1,2}_m$ are constants and J is the usual Bessel function.

Using the continuity relations (47) and (48), and the following Bessel's proprieties

$$\mathrm{J}_{|m|}(kR)\approx\frac{(kR)^{|m|}}{2^{|m|}\Gamma(|m|+1)}\,, \tag{67}$$
$$\frac{d}{dR}\mathrm{J}_{|m|}(kR)\approx\frac{k|m|(kR)^{|m|-1}}{2^{|m|}\Gamma(|m|+1)}\,, \tag{68}$$

and taking the limit $R\rightarrow 0$, ($R^2\approx 0$), we find

$$f^{1,2}_{k,m}(\xi)=\Big\{(kR)^{|m|}(kR)^{-|m+\nu/\hbar|}\Big[\frac{1}{2}+\frac{|m|+s\nu/\hbar}{2|m+\nu/\hbar|}\Big]\mathrm{J}_{|m+\nu/\hbar|}(k\xi)+(kR)^{|m+\nu/\hbar|}\Big[\frac{1}{2}-\frac{|m|+s\nu/\hbar}{2|m+\nu/\hbar|}\Big]\mathrm{J}_{-|m+\nu/\hbar|}(k\xi)\Big\}\,. \tag{69}$$

As for the precedent case, the $R \to 0$, $(R^2 \approx 0)$ limit implies that the irregular solution contributes if the condition (50) is satisfied. In that case one must include the next higher power of R in the coefficient of the regular solution term in (69). This yields

$$f^{1,2}_{k,m} = (kR)^{|m|}\{C^{1,2}_m (kR)^{|m|+s\nu/\hbar+2} \mathrm{J}_{|m+\nu/\hbar|}(k\xi) + (kR)^{-|m|-s\nu/\hbar} \mathrm{J}_{-|m+\nu/\hbar|}(k\xi)\}\,, \tag{70}$$

where $C^{1,2}_m$ is a nonvanishing constant.

It may be noted that equations (53)–(57) imply that

$$\begin{aligned}\chi^{1,2}_{k,m}(r,\varphi) = \Big\{{\sum}' e^{-i\pi/2|m+\nu/\hbar|} \mathrm{J}_{|m+\nu/\hbar|}\Big(k\frac{r}{\rho}\Big) e^{im\varphi} + \theta(-s)\theta\Big(\frac{\nu}{\hbar}\Big) e^{-iN\varphi} e^{-i(\pi/2)(N-\nu/\hbar)} \mathrm{J}_{-\eta}\Big(k\frac{r}{\rho}\Big) \\ + \theta(+s)\theta\Big(-\frac{\nu}{\hbar}\Big) e^{-i(N+1)\varphi} e^{-i(\pi/2)(\nu/\hbar-N-1)} \mathrm{J}_{\eta-1}\Big(k\frac{r}{\rho}\Big)\Big\}\,,\end{aligned} \tag{71}$$

be an admissible eigenfunction solution.

Then, using (66) one deduces that the phase (43) is given by

$$\alpha_{k,m}(t) = -\frac{\hbar^2 k^2}{2}\int_0^t \frac{dt'}{M\rho^2}\,. \tag{72}$$

Case 3 $C < 0$

Let us note that for $C < 0$ the wave equations defined in (45) and (46) correspond to the Pauli equations of the nonrelativistic charged spin-1/2 particle confined to (x, y) plane in the presence of the AB effect and a 2D parabolic potential barrier known also as 2D inverted isotropic oscillator, with purely imaginary frequency $C = -\varpi^2 = (\pm i\varpi)^2$,

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Big) + \hbar^2 \frac{(m+\nu/\hbar)^2}{r^2} - \varpi^2\, r^2\Big] f^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m} f^{1,2}_{\gamma,m}(r), \quad r > R\,, \tag{73}$$

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Big) + \hbar^2 \frac{m^2}{r^2} - \varpi^2 r^2\Big] f^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m} f^{1,2}_{\gamma,m}(r), \quad r < R\,. \tag{74}$$

The connection with the Pauli equations of the nonrelativistic charged spin-1/2 particle in the presence of the AB effect and a 2D harmonic potential (45) and (46) may be established by the following scaling operator,[38]

$$\hat{V}_{\pi/4} = \exp\Big(\frac{\pi}{8\hbar}[r p_r + p_r r]\Big)\,.$$

Now, let us introduce

$$f'^{1,2}_{\gamma,m}(r) = \hat{V}_{\pi/4} f^{1,2}_{\gamma,m}(r)\,. \tag{75}$$

This yields for $f'^{1,2}_{\gamma,m}(r)$ the result

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Big) + \hbar^2 \frac{(m+\nu/\hbar)^2}{r^2} + \varpi^2\, r^2\Big] f'^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m}\, f'^{1,2}_{\gamma,m}(r), \quad r > R\,, \tag{76}$$

$$\frac{1}{2}\Big[-\hbar^2\Big(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Big) + \hbar^2 \frac{m^2}{r^2} + \varpi^2 r^2\Big] f'^{1,2}_{\gamma,m}(r) = \lambda_{\gamma,m} f'^{1,2}_{\gamma,m}(r), \quad r < R\,. \tag{77}$$

It is evident that these two last equations are equivalent to the case $C > 0$. Following the same steps as for $C > 0$ and applying $\hat{V}^{-1}_{\pi/4}$ at the end, the regular solution takes the form,[39]

$$\begin{aligned}\chi^{1,2}_{\mu,m}(r,\varphi) = D^{1,2}_m\Big[1 - \theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N} - \theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1}\Big]\xi^{|m+\nu/\hbar|} e^{-i\varpi\xi^2/2\hbar} e^{im\varphi} \\ \times {}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1 + \Big|m+\frac{\nu}{\hbar}\Big| + i\frac{\lambda_{\gamma,m}}{\hbar\varpi}\Big], \Big|m+\frac{\nu}{\hbar}\Big| + 1, \frac{i\varpi\xi^2}{\hbar}\Big) \\ + \theta(-s)\theta\Big(\frac{\nu}{\hbar}\Big)\delta_{m,-N} D^{1,2}_m \xi^{-\eta}\ {}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[1 - \eta + i\frac{\lambda_{\gamma,m}}{\hbar\varpi}\Big], 1-\eta, \frac{i\varpi\xi^2}{\hbar}\Big) e^{-i\varpi\xi^2/2\hbar} e^{-iN\varphi} \\ + \theta(s)\theta\Big(-\frac{\nu}{\hbar}\Big)\delta_{m,-N-1} D^{1,2}_m \xi^{\eta-1}\, {}_1\mathrm{F}_1\Big(\frac{1}{2}\Big[\eta - \frac{i\lambda_{\gamma,m}}{\hbar\varpi}\Big], \eta, \frac{i\varpi\xi^2}{\hbar}\Big) e^{-i\varpi\xi^2/2\hbar} e^{-i(N+1)\varphi}\,,\end{aligned} \tag{78}$$

where $\lambda_{\gamma,m}$ are the eigenvalues. Hence, as it was for the previous case, i.e. $C = 0$, the solution (78) belongs to a continuous spectrum.

4 Conclusion

We investigated Pauli solutions of a two-dimensional time-dependent Hamiltonian system involving a time-dependent Aharonov–Bohm effect (AB) and a time-dependent harmonic potential. To do this we employed invariant operator and unitary transformation methods together. The original invariant given in Eq. (27) is explicitly a function of t, though its time derivative vanishes: $dI/dt = 0$. However, the invariant, equation (38), which is transformed by $U(t)$ and $V_{\rho(t)}$, has a simple form and is no longer a function of t. Due to this fact, the management of transformed invariant in order to solve eigenvalue equation is much better than treating the original one. We found that the auxiliary equation is independent of the AB magnetic flux. We discussed the eigenvalue equation of the transformed invariant operator separately for the three cases, i.e., $C > 0$, $C = 0$, and $C < 0$. Following the C's values the spectrum of quantum states is discrete ($C > 0$) or continuous ($C \leq 0$). For particular cases, the irregular solution at the origin contributes to the wavefunctions and the phase depends on the spin and the magnetic flux.

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References

[1] Y. Aharonov and D. Bohm, Phys. Rev. **115** (1959) 4859.
[2] M. Peshkin and A. Tonomura, *The Aharonov-Bohm Effect*, Springer, Berlin (1989).
[3] C.R. Hagen, Phys. Rev. Lett. **64** (1990) 503.
[4] C.R. Hagen, Int. J. Mod. Phys. A **6** (1991) 3119.
[5] C.R. Hagen, Phys. Rev. D **48** (1993) 5935.
[6] X. Zhu and W.C. Henneberger, Nuovo Cimento B **108** (1993) 381.
[7] Y.S. Yi, T.Z. Qian, and Z.B. Su, Phys. Rev. B **55** (1997) 10632.
[8] C. Furtado and F. Moraes, J. Phys. A **33** (2000) 5513.
[9] S. Azevedo, Phys. Lett. A **293** (2002) 283.
[10] N. Ferkous and A. Bounames, Phys. Lett. A **325** (2004) 21.
[11] Qiong-Gui Lin, Phys. Rev. A **72** (2005) 042103.
[12] M.A. Markov (ed.), *Invariants and the Evolution of Nonstationary Quantum Systems*, Nova Science Publishers, Commack, New York (1989).
[13] M. Kleber, Phys. Rep. **236** (1994) 331.
[14] Chung-In Um, Kyu-Hwang Yeon, and Thomas F. George, Phys. Rep. **362** (2002) 63.
[15] M. Maamache, J. Phys. A **29** (1996) 2833; M. Maamache, J. Phys. A **31** (1998) 6849; M. Maamache, Ann. Phys. (NY) **254** (1997) 1; M. Maamache, J. Math. Phys. **39** (1998) 161; M. Maamache, J.P. Provost, and G. Vallée, Phys. Rev. A **59** (1999) 1777; M. Maamache and H. Choutri, J. Phys. A **33** (2000) 6203; M. Maamache, Phys. Rev. A **61** (2000) 026102; M. Maamache and H. Bekkar, J. Phys. **A 36** (2003) L359; M. Maamache, Phys. Rev. A **52** (1995) 936.
[16] I.A. Malkin, V.I. Man'ko, and D.A. Trifonov, Phys. Rev. D **2** (1970) 1371.
[17] I.A. Malkin and V.I. Man'ko, Physica **72** (1970) 597.
[18] V.V. Dodonov, I.A. Malkin, and V.I. Man'ko, J. Phys. A **9** (1976) 10.
[19] A.B. Nassar, Physica A **24** (1987) 24.
[20] M.S. Abdalla, Phys. Rev. A **37** (1988) 4026.
[21] J.M.F. Bassalo, L.C.L. Botelho, H.S.A. Neto, and P.T.S. Alencar, Rev. Bras. Fis. **19** (1989) 598.
[22] B. Baseia, S.S. Mizrahi, and M.H.Y. Moussa, Phys. Rev. A **46** (1992) 5885.
[23] C.A.S. Ferreira, P.T.S. Alencar, and J.M.F. Bassalo, Phys. Rev. A **66** (2002) 024103.
[24] C. Yüce, Ann. Phys. **308** (2003) 599.
[25] J.R. Choi, J. Phys. C **15** (2004) 823.
[26] M. Maamache, A. Bounames, and N. Ferkous, Phys. Rev A **73** (2006) 016101.
[27] Mai-Lin Liang and Fu-Lin Zhang, Phys. Scr. **73** (2006) 677.
[28] D. Laroze and R. Rivera, Phys. Lett. A **355** (2006) 348.
[29] Y. Bouguerra, M. Maamache, and A. Bounames, Int. J. Theor. Phys. **45** (2006) 1807.
[30] V.M. Villalba and A.R. Maggiolo, Eur. Phys. J. B **22** (2001) 31.
[31] M. Villalba and R. Pino, Mod. Phys. Lett. B **17** (2003) 1331.
[32] V. Alan Kostelecky, V.I. Man'ko, M.M. Nieto, and D.R. Truax, Phys. Rev. A **48** (1993) 951.
[33] V.M. Tkachuk, J. Phys. A **31** (1998) 1859.
[34] Y. Bouguerra, A. Bounames, M. Maamache, and Y. Saadi, J. Math. Phys. **49** (2008) 042107.
[35] H.R. Lewis, Jr. and W.B. Riesenfeld, J. Math. Phys. **10** (1969) 1458.
[36] M. Maamache and Y. Saadi, arXiv:0804.4289v1 [quant-ph].
[37] Z.X. Wang and D.R. Duo, *Special Functions*, World Scientific, Singapore (1989) p. 327.
[38] D. Chruscinski, Ann. Phys. (NY) **321** (2006) 840.
[39] A. Messiah, *Mécanique Quantique*, Dunod, Paris (1995).