

Two-Dimensional Breather Lattice Solutions and Compact-Like Discrete Breathers and Their Stability in Discrete Two-Dimensional Monatomic β -FPU Lattice*

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Abstract We restrict our attention to the discrete two-dimensional monatomic β -FPU lattice. We look for two-dimensional breather lattice solutions and two-dimensional compact-like discrete breathers by using trying method and analyze their stability by using Aubry's linearly stable theory. We obtain the conditions of existence and stability of two-dimensional breather lattice solutions and two-dimensional compact-like discrete breathers in the discrete two-dimensional monatomic β -FPU lattice.

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Key words: β -FPU lattice, two-dimensional breather lattice solution, two-dimensional compact-like discrete breather

1 Introduction

Since the discovery of intrinsic localized modes (ILMs) or discrete breathers (DBs) in FPU lattices in the late 1980s,^[1,2] they have been a topic of increasing focus in view of their significant role in energy localization and transport. Discrete breathers are time-periodic and spatially localized solutions of coupled chain of nonlinear oscillators. In 1994, Mackay and Aubry^[3] established the existence of stationary breathers in a broad range of lattice models. In their method breathers are obtained by continuation from the anti-continuum limit at which the network is reduced to an array of uncoupled oscillators. In this limit, breather are found trivially, since one may take breather for which only one oscillator is excited, and the others remain at rest.

This method is not applicable to FPU lattices of the type first investigated by Fermi *et al.*,^[4] since these do not possess an uncoupled limit in which trivial breathers exist. Nevertheless, early analytical and numerical work indicated that FPU lattices could indeed support breathers.^[5,6] Later, rigorous proofs for existence in particular FPU models were provided, such as breather or gap breather in diatomic FPU chain,^[7–11] the problem of β -FPU,^[12,13] asymptotic dynamics of breather in FPU chain,^[14] interaction of a kink soliton with a breather in a FPU chain,^[15] bright and dark breathers in FPU lattice,^[16] q -breathers and the FPU problem,^[17] etc. It indicates that discrete breather can indeed exist in the FPU system, and most of the studies on the FPU system consider only discrete breather as localized excitations.

Recently, the breather lattice solutions (BLSs), a new periodic solution, in the continuous nonlinear partial differential equation, such as the sine-Gordon equation and the mKDV equation, were presented in explicit analytic form in Refs. [18] ~ [22]. BLSs are a kind of solution in the form of a spatially periodic array of single breathers. Can

the discrete system also support these periodic solutions? The key to this problem is in the affirmative. Because the transformation from the compactons^[23–25] into the compact-like discrete breathers (CDBs)^[26–29] has demonstrated that the discrete system can also support the localized solutions with exact compact structure supported by the continuous nonlinear partial differential equation.

All papers mentioned above focus on one-dimensional (1D) nonlinear system. However, the properties of 2D nonlinear models are gradually catching peoples' attention.^[30–33] In this paper, we consider a discrete two-dimensional (2D) monatomic β -FPU lattice. We look for the 2DBLSs and 2DCDBs in this system by using the trying method,^[34–36] and analyze their stability by using Aubry's linearly stable theory.^[3,37,38]

The plan of the paper is as follows: in Sec. 2 we derive the equations of motion governing the system. In Sec. 3 we look for the 2DBLSs and 2DCDBs in this system. In Sec. 4 we discuss the linearly stable conditions of 2DBLSs and 2DCDBs, and finally we conclude in Sec. 5.

2 Equations of Motion

We restrict our attention to the discrete 2D monatomic β -FPU lattice. In such a lattice, atoms with unit mass interact with their nearest-neighbors. The Hamiltonian of the system is given by

$$H = \sum_{n,m} \left\{ \frac{1}{2} \dot{u}_{n,m}^2 + \frac{1}{2} \omega_0^2 [(u_{n+1,m} - u_{n,m})^2 + (u_{n,m+1} - u_{n,m})^2] + \frac{1}{4} \beta [(u_{n+1,m} - u_{n,m})^4 + (u_{n,m+1} - u_{n,m})^4] \right\}, \quad (1)$$

where $u_{n,m}$ denotes the displacement of the (n,m) -th atom from its equilibrium position, ω_0^2 and β are the parameter controlling the strength of the nonlinear coupling.

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The Hamiltonian (1) yields the equation of motion

$$\ddot{u}_{n,m} = \omega_0^2(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}) + \beta[(u_{n+1,m} - u_{n,m})^3 - (u_{n,m} - u_{n-1,m})^3 + (u_{n,m+1} - u_{n,m})^3 - (u_{n,m} - u_{n,m-1})^3]. \quad (2)$$

This model appears in many realistic physical systems, and we will discuss it by using the trying method.

3 Proofs of Existence of 2DBLSs and 2DCDBs

In this section we first look for the 2DBLSs of Eq. (2), and we choose variable transformation,

$$u_{n,m}(t) = (-1)^{n+m} \Phi_{n,m} G(t), \quad (3)$$

where $\Phi_{n,m} = \cos[q_n(n - n_0) + q_m(m - m_0)]$. Substituting Eq. (3) into Eq. (2), Eq. (2) can be rewritten as

$$\ddot{G} + 2\omega_0^2(\cos q_n + \cos q_m + 2)G + CG^3 = 0, \quad (4a)$$

$$4\beta[(2\cos^3 q_n + 3\cos^2 q_n - 1) + (2\cos^3 q_m + 3\cos^2 q_m - 1)]\Phi_{n,m}^2 + 6\beta[(\cos q_n + 1)\sin^2 q_n + (\cos q_m + 1)\sin^2 q_m] - C = 0. \quad (4b)$$

where C is a determined constant.

When $C > 0$, Eq. (4a) has solution

$$G = G_0 \text{cn}(\omega_b t, l), \quad (5)$$

where cn is a Jacobian elliptic function of modulus

$$l^2 = \frac{CG_0^2}{2[2\omega_0^2(\cos q_n + \cos q_m + 1) + CG_0^2]}$$

and pulsation

$$\omega_b = \sqrt{2\omega_0^2(\cos q_n + \cos q_m + 1) + CG_0^2}.$$

When $C < 0$, Eq. (4a) has solution

$$G = G_0 \text{sn}(\omega_b t, l), \quad (6)$$

where sn is a Jacobian elliptic function of modulus

$$l^2 = -\frac{CG_0^2}{2[2\omega_0^2(\cos q_n + \cos q_m + 1) + CG_0^2]}$$

and pulsation

$$\omega_b = \sqrt{2\omega_0^2(\cos q_n + \cos q_m + 1) + \frac{1}{2}CG_0^2}.$$

Thus Eq. (4b) requires

$$2\cos^3 q_n + 3\cos^2 q_n + 2\cos^3 q_m + 3\cos^2 q_m - 2 = 0, \quad (7a)$$

$$C = 6\beta[(\cos q_n + 1)\sin^2 q_n + (\cos q_m + 1)\sin^2 q_m]. \quad (7b)$$

Therefore, when $\beta > 0$, $C > 0$, Eq. (2) has the following solution:

$$u_{n,m} = (-1)^{n+m} G_0 \cos[q_n(n - n_0) + q_m(m - m_0)] \times \text{cn}(\omega_b t, l), \quad (8)$$

and when $\beta < 0$, $C < 0$, Eq. (2) has the following solution:

$$u_{n,m} = (-1)^{n+m} G_0 \cos[q_n(n - n_0) + q_m(m - m_0)] \times \text{sn}(\omega_b t, l). \quad (9)$$

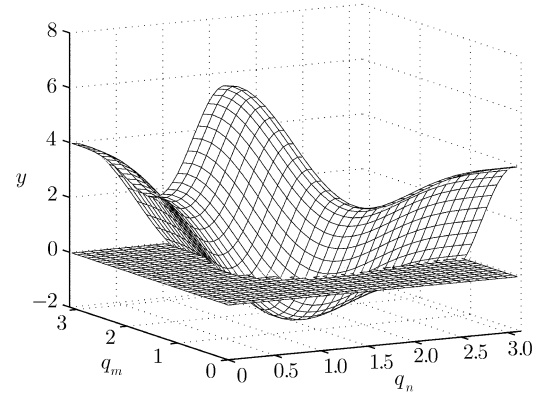


Fig. 1 The graphical presentation shows the relation of the function of Eq. (7a).

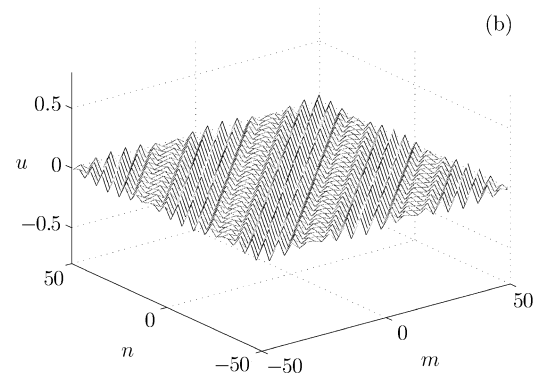
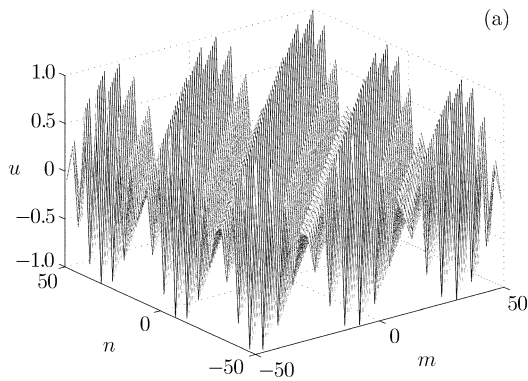


Fig. 2 The graphical presentation shows the space-time evolution of the 2DBLSs equations (8) and (9). (a) $t = 0$ and $t = t_b$; (b) $t = t_b/4$ and $t = 3t_b/4$.

The relation of the function of Eq. (7a) is shown in Fig. 1. From Fig. 1 we can know Eq. (7a) has real solutions, indeed.

The existence of real solution of Eq. (7a) indicates that Eq. (2) has the solutions in the forms of Eqs. (8) and (9). They are shown in Fig. 2. From Fig. 2 we know that the solutions in the forms of Eqs. (8) and (9) are the 2DBLSs.

If solutions Eqs. (8) and (9) are restricted to be in following forms,

$$u_{n,m} = (-1)^{n+m} G_0 \cos[q_n(n - n_0) + q_m(m - m_0)] \text{cn}(\omega_b t, l), \quad \text{if } |n - n_0| \leq \frac{\pi}{2q_n}, \quad |m - m_0| \leq \frac{\pi}{2q_m}, \quad (10a)$$

while $u_{n,m}(t) = 0$, otherwise, and

$$u_{n,m} = (-1)^{n+m} G_0 \cos[q_n(n - n_0) + q_m(m - m_0)] \operatorname{sn}(\omega_b t, l), \quad \text{if } |n - n_0| \leq \frac{\pi}{2q_n}, \quad |m - m_0| \leq \frac{\pi}{2q_m}, \quad (10b)$$

while $u_{n,m}(t) = 0$, otherwise, respectively, Eq. (2) has the 2DCDBs in the forms of Eqs. (10a) and (10b). They are shown in Fig. 3.

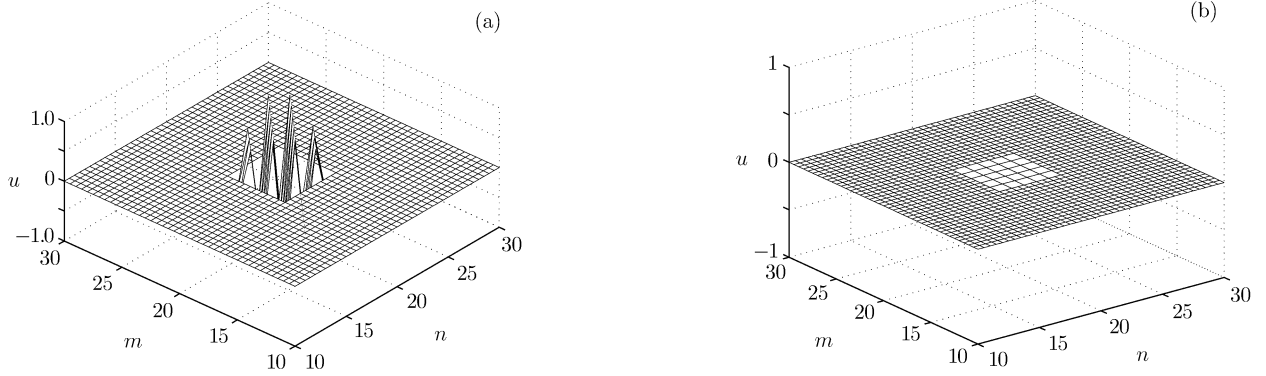


Fig. 3 The graphical presentation shows the space-time evolution of the 2DCDBs equations (11a) and (11b). (a) $t = 0$ and $t = t_b$; (b) $t = 4/t_b$ and $t = 3t_b/4$.

Thus we can conclude that the 2DBLSs and 2DCDBs indeed exist in the discrete two-dimensional monatomic β -FPU lattice.

4 Analyses of Linear Stability

Next, we discuss the stability of these solutions. According to Aubry's theory,^[14] we introduce the variable transformation $u_{n,m}(t) \rightarrow u_{n,m}(t) + \varepsilon_{n,m}(t)$, the linearized equations of Eq. (2) reads

$$\begin{aligned} \ddot{\varepsilon}_n = & \omega_0^2(\varepsilon_{n+1,m} + \varepsilon_{n-1,m} + \varepsilon_{n,m+1} + \varepsilon_{n,m-1} - 4\varepsilon_{n,m}) \\ & + 3\beta[(u_{n+1,m} - u_{n,m})^2(\varepsilon_{n+1,m} - \varepsilon_{n,m}) - (u_{n,m} - u_{n-1,m})^2(\varepsilon_{n,m} - \varepsilon_{n-1,m})] \\ & + 3\beta[(u_{n,m+1} - u_{n,m})^2(\varepsilon_{n,m+1} - \varepsilon_{n,m}) - (u_{n,m} - u_{n,m-1})^2(\varepsilon_{n,m} - \varepsilon_{n,m-1})], \end{aligned} \quad (11)$$

where $\varepsilon_{n,m}$ represents a small perturbation of the solution of the dynamical equations.

A solution $u_{n,m}(t)$ of Eq. (3) is said to be linearly stable when there is no solution $\varepsilon_{n,m}(t)$ of Eq. (11), which diverges exponentially at large time. This property does not imply that a perturbation on the initial conditions will remain small for all time for the real nonlinear dynamical system (1). However, this linear stability implies that this perturbation will not grow exponentially as a function of time. Such trajectories have physically a significantly longer lifetime than those which are linearly unstable.

In this case, $u_{n,m}(t)$ is a real and time-independent solution of Eq. (2). The general solution of Eq. (11) can be searched as following form:

$$\varepsilon_{n,m}(t) = a_{n,m} e^{i\theta_{n,m}(t)}, \quad (12)$$

where the $\theta_{n,m}(t)$ and eigenmodes $a_{n,m}$ are determined by following eigenequation:

$$\begin{aligned} & -\omega_0^2(a_{n+1,m} + a_{n-1,m} + a_{n,m+1} + a_{n,m-1} - 4a_{n,m}) \\ & - 3\beta[(u_{n+1,m} - u_{n,m})^2(a_{n+1,m} - a_{n,m}) - (u_{n,m} - u_{n-1,m})^2(a_{n,m} - a_{n-1,m})] \\ & - 3\beta[(u_{n,m+1} - u_{n,m})^2(a_{n,m+1} - a_{n,m}) - (u_{n,m} - u_{n,m-1})^2(a_{n,m} - a_{n,m-1})] \\ & = \left[\left(\frac{\partial \theta_{n,m}}{\partial t} \right)^2 - i \frac{\partial^2 \theta_{n,m}}{\partial t^2} \right] a_{n,m}. \end{aligned} \quad (13)$$

Then, we note that linear stability is achieved if and only if all the $\theta_{n,m}(t)$'s are real, because this perturbation cannot grow exponentially as a function of time. The eigenequation (13) is easily solved by introducing a wave vector k and writing

$$a_{n,m} = e^{i(k_n n + k_m m)}. \quad (14)$$

Substituting Eq. (15), Eq. (11a), or Eq. (8) into Eq. (12), we have

$$\begin{aligned} \left(\frac{\partial \theta_{n,m}}{\partial t} \right)^2 = & 2\omega_0^2(2 - \cos k_n - \cos k_m) + 6\beta G^2 \{ \Phi_{n,m}^2 (\cos q_n - 1)^2 [(1 - \cos k_n) + (1 - \cos k_m)] \\ & + (1 - \Phi_{n,m}^2) [\sin^2 q_n (1 - \cos k_n) + \sin^2 q_m (1 - \cos k_m)] \}, \end{aligned} \quad (15a)$$

$$\frac{\partial^2 \theta_{n,m}}{\partial t^2} = 8\beta G^2 \Phi_{n,m} \sin[q_n(n - n_0) + q_m(m - m_0)] [(\cos q_n - 1) \sin q_n \sin k_n + (\cos q_m - 1) \sin q_m \sin k_m]. \quad (15b)$$

We integrate Eqs. (15a) and (15b) from $t = 0$ to $t = t_b$, here t_b is the period of the 2DBLSs and 2DCDBs. The $\theta_{n,m}(t)$'s are given by

$$\theta_{n,m}(t_b) = \frac{6a}{\omega_b} \left(\frac{1}{4} \omega_b t_b^2 - \frac{1}{4\omega} \cos^2 \omega_b t_b \right), \quad (16a)$$

$$\theta_n(t_b) = -c \sqrt{1 - \cos^2 \omega_b t_b} \cdot \sqrt{\frac{c + b \cos^2 \omega_b t_b}{c}} + \frac{\text{Elliptic } E(\cos \omega_b t_b, \sqrt{-b/\omega_0^2})}{\omega_b \sin \omega_b t_b \sqrt{c + b \cos^2 \omega_b t_b}}, \quad (16b)$$

where

$$a = 8\beta G_0^2 \Phi_{n,m} \sin[q_n(n - n_0) + q_m(m - m_0)][(\cos q_n - 1) \sin q_n \sin k_n + (\cos q_m - 1) \sin q_m \sin k_m],$$

$$b = 6\beta G_0^2 \{ \Phi_{n,m}^2 (\cos q_n - 1)^2 [(1 - \cos k_n) + (1 - \cos k_m)] + (1 - \Phi_{n,m}^2) [\sin^2 q_n (1 - \cos k_n) + \sin^2 q_m (1 - \cos k_m)] \},$$

$c = 2\omega_0^2(2 - \cos k_n - \cos k_m)$ and $\text{Elliptic } E(\cos \omega_b t_b, \sqrt{-b/\omega_0^2})$ is the second Jacobian elliptic integration. From Eqs. (16a) and (16b), we know that real $\theta_{n,m}(t)$ requires $c + b \cos^2 \omega_b t_b > 0$ and $b < 0$. Therefore $\beta < 0$, we obtain the linearly stable condition of 2DBLSs and 2DCDBs.

When $\beta < 0$, we have

$$|\beta| < \frac{2\omega_0^2(2 - \cos k_n - \cos k_m)}{3\beta G_0^2 \{ \Phi_{n,m}^2 (\cos q_n - 1)^2 [(1 - \cos k_n) + (1 - \cos k_m)] + (1 - \Phi_{n,m}^2) [\sin^2 q_n (1 - \cos k_n) + \sin^2 q_m (1 - \cos k_m)] \}}. \quad (17)$$

So if β satisfies the relation of Eq. (17), the 2DBLSs and 2DCDBs of Eq. (2) can be said to be linearly stable. Substituting Eq. (14), Eq. (10b), or Eq. (9) into Eq. (11), we obtain the same results.

5 Conclusion

In this paper, we have investigated the existence and the linear stability of 2DBLSs and 2DCDBs in a discrete two-dimensional monatomic β -FPU lattice, separately. We obtain the conclusion that the discrete two-dimensional monatomic β -FPU lattice has the 2DBLSs and the 2DCDBs, indeed. Because they are linearly stable when β satisfies the condition of Eq. (17).

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