Commun. Theor. Phys. (Beijing, China) **53** (2010) pp. 440–442
© Chinese Physical Society and IOP Publishing Ltd Vol. 53, No. 3, March 15, 2010

Generalized Variational Iteration Solution of Soliton for Disturbed KdV Equation*

MO Jia-Qi (莫嘉琪)†

Department of Mathematics, Anhui Normal University, Wuhu 241000, China

Division of Computational Science, E-Institutes of Shanghai Universities at SJTU, Shanghai 200240, China

(Received March 24, 2009; revised manuscript received October 12, 2009)

Abstract *The corresponding solution for a class of disturbed KdV equation is considered using the analytic method. From the generalized variational iteration theory, the problem of solving soliton for the corresponding equation translates into the problem of variational iteration. And then the approximate solution of the soliton for the equation is obtained.*

PACS numbers: 02.30.Jr
Key words: soliton, disturbed, variational iteration

1 Introduction

Soliton is an important notion of nonlinear science, which has been widely applied in natural sciences such as chemistry, biology, mathematics, communication, and particularly in almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, condensed matter physics etc.[1−8] Recently, more new methods are presented, for examples, the method of hyperbolic tangent function and the homogeneous equilibrium method, the method of the Jacobi elliptic function, the method of auxiliary equation.[9−10] And many scholars have done a great deal of work, such as the shock wave, the scattering light wave, the quantum mechanics, the atmospheric physics, the network of neurons and so on had studied for the theorem of solitary wave.[1−2,11] The asymptotic method for the nonlinear theory of solitary wave is a new study. The main essential of this method is that the nonlinear problem is treated with linear methods by using the asymptotic expansion. The method of generalized variational iteration[12] is namely such a new method.

During the past decade many approximate methods for the nonlinear problem have been developed and refined, including the method of averaging, the boundary layer method, the methods of matched asymptotic expansion and the multiple scales. Recently, many scholars such as Ni and Wei,[13] Bartier,[14] Libre, Silva and Teixeira,[15] and Guarguaglini and Natalini[16] have done a great deal of work. Using the method of differential inequalities and others Mo *et al.* considered also a class of reaction diffusion problems,[17] the activator inhibitor systems,[18] the ecological environment,[19] the shock wave,[20] the soliton wave,[21−22,30] the laser pulse,[23] the ocean science,[24−25] and the atmospheric physics,[26−29] etc. We consider a class of generalized nonlinear Korteweg de Vries (KdV) equation and obtain approximate solution of the solitary wave.

2 Generalized Variational Iteration

Consider the following generalized nonlinear KdV equation:[30−32]

$$u_t + 6uu_{xx} + u_{xxx} = f(t, x, u)\,, \tag{1}$$

where f is a disturbed term, which is a sufficiently smooth bounded function with regard to their variables in corresponding domains.

We first consider the typical KdV equation

$$v_t + v_{xxx} = -6uu_{xx}\,. \tag{2}$$

It is easy to see that there is the following soliton wave for Eq. (2):[30−32]

$$v(t, x) = 2a \operatorname{sech} a(x - x_0 - 4a^2 t)\,, \tag{3}$$

where a, x_0 are arbitrary constants, which can decide by conditions of KdV equation. As $a = 1, x_0 = 0$, surface of the soliton wave $v(t, x)$ in o-(t, x, u) see Fig. 1.

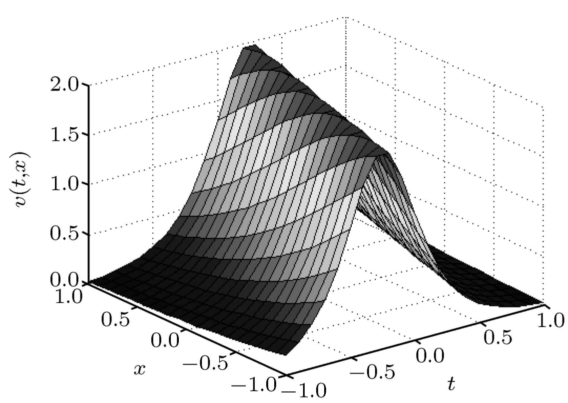

Fig. 1 Surface of the soliton wave $v(t, x)$.

*Supported by the National Natural Science Foundation of China under Grant No. 40876010, the Knowledge Innovation Project of Chinese Academy of Sciences under Grant No. KZCX2-YW-Q03-08, the R & D Special Fund for Public Welfare Industry (meteorology) under Grant No. GYHY200806010, the LASG State Key Laboratory Special Fund, the E-Institutes of Shanghai Municipal Education Commission under Grant No. E03004 and the Natural Science Foundation of Zhejiang Province under Grant No. Y6090164

†E-mail: mojiaqi@mail.ahnu.edu.cn

In order to obtain solution of the generalized nonlinear KdV equation (1), introducing the following functional $F[u]$:

$$F[u] = u - \int_0^t \lambda\Big(\frac{\partial u}{\partial \tau} + 6\bar{u}\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^3 \bar{u}}{\partial x^3} - f(\tau, x, \bar{u})\Big) d\tau\,, \quad (4)$$

where $\bar{u}$ is the restricted variable of u,[12] λ is the Lagrange multiplier. Compute the variation δF of functional (4),

$$\delta F = \delta u - (\lambda \delta u)|_{\tau=t} + \int_0^t \frac{\partial \lambda}{\partial \tau}\delta u d\tau\,, \quad (5)$$

let $\delta F = 0$, we have

$$\frac{\partial \lambda}{\partial \tau} = 0, \quad (\tau < t)\,. \quad (6)$$

From Eqs. (5)–(6), we have

$$\lambda(t) = 1\,. \quad (7)$$

From Eqs. (4) and (7), we construct the generalized variational iteration:

$$u_{n+1} = u_n - \int_0^t \Big(\frac{\partial u_n}{\partial \tau} + 6u_n\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^3 u_n}{\partial x^3} - f(\tau, x, u_n)\Big) d\tau\,. \quad (8)$$

From analytic behavior for the nonlinear terms of generalized disturbed KdV equation, we have

$$u(t, x) = \lim_{n\to\infty} u_n(t, x)\,. \quad (9)$$

It is the solution of original KdV equation (1).

Selecting the initial variational iteration u_0 of Eq. (8) as v:

$$u_0(t, x) = 2a \operatorname{sech} a(x - x_0 - 4a^2 t)\,, \quad (10)$$

which is soliton wave (3) of the non-disturbed typical KdV equation (2).

Substituting Eq. (9) into Eq. (8) as $n = 0$, we have

$$u_1(t, x) = 2a \operatorname{sech} a(x - x_0 - 4a^2 t) + F_0(t, x)\,,$$

where

$$F_0(t, x) = \int_0^t f(\tau, x, 2a \operatorname{sech} a(x - x_0 - 4a^2\tau)) d\tau\,. \quad (11)$$

And substituting Eqs. (9)–(10) into Eq. (8) as $n = 1$, we have also

$$\begin{aligned} u_2(t, x) &= 2a \operatorname{sech} a(x - x_0 - 4a^2 t) \\ &- \int_0^t \frac{\partial F_0(\tau, x)}{\partial \tau} + \Big[6\Big(F_0(\tau, x)\frac{\partial^2 u_0}{\partial x^2} + u_0\frac{\partial^2 F_0(\tau, x)}{\partial x^2} \\ &+ F_0(\tau, x)\frac{\partial^2 F_0(\tau, x)}{\partial x^2}\Big) + \frac{\partial^3 F_0(\tau, x)}{\partial x^3}\Big] d\tau \\ &+ \int_0^t [f(\tau, x, u_0) - (f(\tau, x, (u_0 + F_0(\tau, x)))] d\tau\,. \end{aligned} \quad (12)$$

Analogously, we have the figures of curve $u_2(t, x)$. From generalized variational iteration (8), in the same method, we can obtain $u_n(t, x)$, $n = 2, 3, \ldots$, respectively. Thus we have the n-th order approximate solution $u_n(t, x)$ of the soliton wave for the disturbed KdV equation (1). But we do not discuss here.

3 Example

Now we consider an infinitesimal disturbance $f(t, x, u) = \varepsilon \sin u$ for the KdV equation (1), where ε is a positive parameter. Thus disturbed KdV equation is

$$u_t + 6uu_{xx} + u_{xxx} = \varepsilon \sin u\,. \quad (13)$$

From Eq. (10), it is easy to see that the first order approximate expansion u_1 of the soliton wave for the infinitesimal disturbance KdV equation (13) taking $a = 1$, $x_0 = 0$ is

$$u_1(z) = 2 \operatorname{sech} z + \varepsilon f_0(Z),$$

where $z = x - 4t$ and

$$f_0(z) = -\frac{1}{4}\int_0^z \sin(2 \operatorname{sech} \tau) d\tau.$$

Approximate curves of the soliton wave $u_0(z)$ and $u_1(z)$ on o-(z, u) as $\varepsilon = 0.05$ and $\varepsilon = 0.01$, respectively, see Figs. 2 and 3.

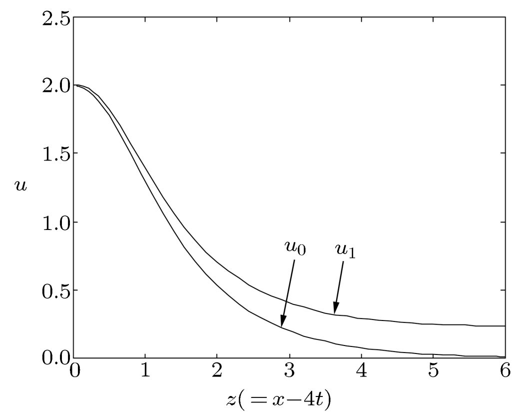

Fig. 2 Approximate curves of the soliton wave $u(z)$ ($\varepsilon = 0.05$).

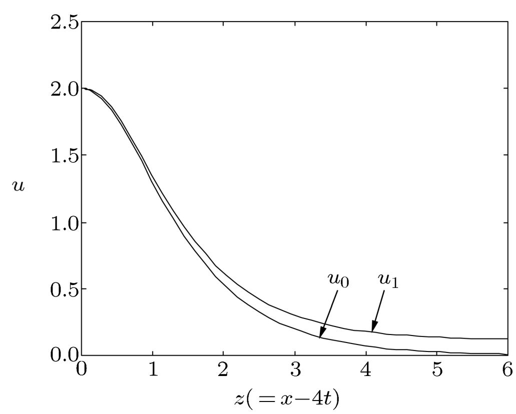

Fig. 3 Approximate curves of the soliton wave $u(z)$ ($\varepsilon = 0.01$).

From Figs. 2 and 3, we can know that the first approximate solution u_1 of travelling wave $u(x - 4t)$ changes and tends to stable as larger $z = x - 4t$.

From Eq. (12), the second order approximate expansion $u_2(z)$ of the soliton wave for the infinitesimal disturbance KdV equation (13) taking $a = 1$, $x_0 = 0$ is

$$\begin{aligned} u_2(z) = 2\operatorname{sech} z + \varepsilon f_0(z) + \frac{1}{4}\varepsilon\int_0^z \Big[-\frac{1}{4}\frac{d}{d\tau}(\sin(2\operatorname{sech}\tau)) + 6f_0(\tau)\frac{\partial^2}{\partial\tau^2}(2\operatorname{sech}\tau) \\ + 6(2\operatorname{sech}\tau)\frac{d^2 f_0(\tau)}{d\tau^2} + \frac{d^3 f_0(\tau)}{d\tau^3}\Big]d\tau + \frac{1}{4}\varepsilon^2\int_0^z \Big[6\frac{d^2 f_0(\tau)}{d\tau^2}\Big] f_0(\tau)d\tau \\ - \frac{\varepsilon}{4}\int_0^z [\sin(2\operatorname{sech}\tau) - (\sin(2\operatorname{sech}\tau + \varepsilon f_0(\tau)))]d\tau. \end{aligned}$$

From generalized variational iteration (8), in the same method, we can obtain $u_n(z)$, $n = 2, 3, \ldots$, respectively.

4 Discussion

Solitary wave denotes a class of complicated natural phenomenon. Hence we need reduced basic models. And we solve them using the approximate method. The method of generalized variational iteration is a simple and valid method.

The method of generalized variational iteration is an approximate analytic method, which differ from general numerical method. The expansions of solution using the method of generalized variational iteration can be continuously analytic operation. Thus, from Eq. (12), we can study further that the qualitative and quantitative behaviors of the solitary wave.

References

[1] M.J. McPhaden and D. Zhang, Nature (London) **415** (2002) 60.

[2] D.F. Gu and S.G.H. Philander, Science **275** (1997) 805.

[3] S.H. Ma, J.Y. Qiang, and J.P. Fang, Acta Phys. Sin. **56** (2007) 620 (in Chinese).

[4] S.H. Ma, J.Y. Qiang, and J.P. Fang, Commun. Theor. Phys. **48** (2007) 662.

[5] I. Loutsenko, Commun. Math. Phys. **268** (2006) 465.

[6] M. Gedalin, Phys. Plasmas **5** (1998) 127.

[7] E.J. Parkes, Chaos, Solitons & Fractals **38** (2008) 154.

[8] X.Z. Li and M.L. Wang, Phys. Lett. A **361** (2007) 115.

[9] M.L. Wang, Phys. Lett. A **199** (1995) 169.

[10] J.S. Sirendaoreji, Phys. Lett. A **309** (2003) 387.

[11] L.X. Pan, W.M. Zuo, and J.R. Yan, Acta Phys. Sin. **54** (2005) 1 (in Chinese).

[12] J.H. He and X.H. Wu, Chaos, Solitions & Fractals **29** (2006) 108.

[13] W.M. Ni and J.C. Wei, J. Differ. Equations **221** (2006) 158.

[14] J.P. Bartier, Asymptotic Anal. **46** (2006) 325.

[15] J. Libre, P.R. da Silva, and M.A. Teixeira, J. Dyn. Differ. Equations **19** (2007) 309.

[16] F.R. Guarguaglini and R. Natalini, Commun. Partial Differ. Equations **32** (2007) 163.

[17] J.Q. Mo, Science in China Ser A **32** (1989) 1306.

[18] J.Q. Mo and W.T. Lin, J. Sys. Sci. & Complexity **20** (2008) 119.

[19] J.Q. Mo and H. Wang, Acta Ecologica Sin. **27** (2007) 4366.

[20] J.Q. Mo, J. Zhu, and H. Wang, Prog. Natl. Sci. **13** (2003) 768.

[21] J.Q. Mo, W.J. Zhang, and M. He, Acta Phys. Sin. **56** (2007) 1843 (in Chinese).

[22] J.Q. Mo, W.J. Zhang, and X.F. Chen, Acta Phys. Sin. **56** (2007) 6169 (in Chinese).

[23] J.Q. Mo, W.J. Zhang, and M. He, Acta Phys. Sin. **55** (2006) 3233 (in Chinese).

[24] J.Q. Mo, W.T. Lin, and H. Wang, Chin. Geographical Sci. **18** (2008) 193.

[25] J.Q. Mo, W.Y. Lin, and H. Wang, Prog. Natl. Sci. **17** (2007) 230.

[26] J.Q. Mo, W.T. Lin, and Y.H. Lin, Acta Phys. Sin. **56** (2007) 3127 (in Chinese).

[27] J.Q. Mo, H. Wang, W.T. Lin, and Y.H. Lin, Chin. Phys. **15** (2006) 671.

[28] J.Q. Mo, W.T. Lin, and H. Wang, Chin. Phys. **16** (2007) 951.

[29] J.Q. Mo and W.T. Lin, Chin. Phys. **17** (2008) 370.

[30] J.R. Yan and Y. Tang, Phys. Rev. E **54** (1998) 6818.

[31] L.X. Pan, H.Y. Yu, and J.R. Yan, Acta Phys. Sin. **57** (2008) 1316 (in Chinese).

[32] S.H. Ma, J.P. Fang, and C.L. Zheng, Chin. Phys. **17** (2008) 2767.