

Integrability and Solutions of the (2+1)-Dimensional Broer–Kaup Equation with Variable Coefficients*

WANG Lu-Hua (王路华) and HE Jing-Song (贺劲松)[†]

Department of Mathematics, Ningbo University, Ningbo 315211, China

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Abstract The integrability of the (2+1)-dimensional Broer–Kaup equation with variable coefficients (VCBK) is verified by finding a transformation mapping it to the usual (2+1)-dimensional Broer–Kaup equation (BK). Thus the solutions of the (2+1)-dimensional VCBK are obtained by making full use of the known solutions of the usual (2+1)-dimensional BK. Two new integrable models are given by this transformation, their dromion-like solutions and rogue wave solutions are also obtained. Further, the velocity of the dromion-like solutions can be designed and the center of the rogue wave solutions can be controlled artificially because of the appearance of the four arbitrary functions in the transformation.

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1 Introduction

Nonlinear evolution equations (NEEs) have wide applications in describing diverse interesting nonlinear phenomena, which can be shown partially in books on soliton or integrable systems such as Ref. [1]. In recent years, a large number of effective methods to study nonlinear equations have been found, such as flowing well-known methods: the trigonometric function series method;^[2] the modified mapping method and the extended mapping method;^[3] the modified trigonometric function series method;^[4] the bifurcation method and qualitative theory of dynamical systems;^[5] the modified (G'/G) -expansion method;^[6] dynamical systems approach;^[7] Jacobi elliptic function expansion method,^[8] and so on.

Recently, from the point of view of the symmetry constraint associated with the Darboux transformation, one kind of (2+1)-dimensional extension of the Broer–Kaup (BK) equation as the following

$$U_{TY} - U_{XXY} + 2V_{XX} + 2(UU_X)_Y = 0, \quad (1)$$

$$V_T + 2(UV)_X + V_{XX} = 0, \quad (2)$$

has been introduced in Refs. [9–10], which has been studied extensively in Refs. [11–22]. Motivated by the applications in statistical physics, plasma physics and optical fiber communication, a (2+1)-dimensional Broer–Kaup equation with variable coefficients (VCBK),

$$H_{ty} - \beta(t)[H_{xy} - 2G_{xx} - 2(HH_x)_y] = 0, \quad (3)$$

$$G_t + \beta(t)[2(HG)_x + G_{xx}] = 0, \quad (4)$$

has been introduced in Ref. [23]. Here $\beta(t)$ is an arbitrary nonzero function of time t . The (2+1)-dimensional VCBK Eqs. (3) and (4) have been studied in literatures from following aspects: (i) Exact solutions are obtained by using homogeneous balance method^[23] and extended tanh-function method;^[24–25] (ii) Soliton-like solutions,^[26] phenomena of fission and annihilation of solitons^[27] are given by extended Riccati equation mapping method; (iii) New exact solutions and N -soliton solutions are given in Refs. [28–29] respectively by Exp-function method; (iv) Non-traveling wave solutions are derived by the (G'/G) -method in Ref. [30]; (v) The fission and fusion behaviors of the solitons are discussed by using an extended homogeneous balance approach and a linear variable separation method;^[31] (vi) New solitary solutions and non-elastic interactions are studied by means of symbolic computation,^[32] and so on. Although these results have shown very nice integrable properties of the (2+1)-dimensional VCBK, the integrability of the (2+1)-dimensional VCBK is not verified clearly, which is a little bit strange gap on this topic. So we shall fill this gap by constructing a more general (2+1)-dimensional integrable VCBK Eqs. (5) and (6) for special coefficients, which can be done by finding an explicit transformation mapping it to usual (2+1)-dimensional BK Eqs. (1) and (2). Furthermore, the expressions of the dromion-like solutions and rogue wave solutions are given for specific examples from known solutions of the usual (2+1)-dimensional BK equation. There are several arbitrary functions in the formula

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[†]Corresponding author, E-mail: hejingsong@nbu.edu.cn

so that we can further design different integrable models and control the solutions from different reality concerns in applied purposes. This result shows that the (2+1)-dimensional VCBK equation possesses designable integrability.^[33]

The organization of this paper is as follows. In Sec. 2, a transformation which maps (2+1)-dimensional VCBK equation to the (2+1)-dimensional BK equation is obtained. In Sec. 3, three examples are discussed meanwhile dromion-like solutions and rogue wave solutions for two new models are given explicitly. Further more, we will illustrate by examples how to choose appropriate functions to control the velocity of the dromion-like solutions and how to control the center of the rogue wave solutions. Such controllable property is not only seen clearly from the figures of the solutions but also analyzed thoroughly by means of the expressions. The conclusions will be given in Sec. 4.

2 General Method

In the following, to find more larger universality of the (2+1)-dimensional VCBK equation, we shall extend Eqs. (3) and (4) to the following more general one

$$u_{ty} - \alpha_1(x, y, t)u_{xxy} + 2\beta_1(x, y, t)v_{xx} + 2\gamma(x, y, t)(uu_x)_y = 0, \quad (5)$$

$$v_t + 2\alpha_2(x, y, t)(uv)_x + \beta_2(x, y, t)v_{xx} = 0. \quad (6)$$

Here $\alpha_i(x, y, t)$, $\beta_i(x, y, t)$, $i = 1, 2$, $\gamma(x, y, t)$ are real arbitrary functions.

Our target is to find a transformation mapping Eqs. (5) and (6) to the usual (2+1)-dimensional BK Eqs. (1) and

(2), and to fix the coefficients simultaneously. If such a transformation can be found, we can claim that Eqs. (5) and (6) are integrable associated with given coefficients.

Similar to the case of variable coefficient nonlinear Schrödinger equation,^[33] this procedure shall be split into three steps:

- set up a trial transformation and get the transformed equation;
- simplify the transformed equation and set several coefficients of it(k_i) to be zero such that the transformed equation becomes a usual (2+1)-dimensional BK equation;
- solve the system of equations with k_i equal to zero ($i = 4 \cdots 14$ and $i = 17, 18, 19$) or equal to certain constants ($i = 1 \cdots 3$ and $i = 15, 16$) to fix the transformation.

We first set two trial transformations in the form of

$$u = p_1(x, y, t)U(X, Y, T), \quad (7)$$

$$v = p_2(x, y, t)V(X, Y, T), \quad (8)$$

where $X = X(x, y, t)$, $Y = Y(x, y, t)$, $T = T(x, y, t)$. Then the significant task is to find the concrete expressions of $\alpha_i(x, y, t)$, $\beta_i(x, y, t)$ ($i = 1, 2$), $\gamma(x, y, t)$, X , Y , and T , by letting $U(X, Y, T)$, $V(X, Y, T)$ to satisfy Eqs. (1) and (2).

Substituting Eqs. (7) and (8), then we get the following equation

$$\begin{aligned} & p_1 T_t Y_y U_{TY} - \alpha_1 p_1 X_x^2 Y_y U_{XXY} + 2\beta_1 p_2 X_x^2 V_{XX} + 2\gamma p_1^2 X_x Y_y (UU_X)_Y + p_1 Y_t Y_y U_{YY} \\ & - 2\alpha_1 p_1 X_x Y_y U_{XY} - 2\alpha_1 p_1 X_x T_x Y_y U_{XYT} + 2\gamma p_1^2 X_x Y_y UU_{XX} + 2\gamma p_1^2 X_x T_y UU_{XT} \\ & + p_1 X_t Y_y U_{XY} + 4\gamma p_1 p_{1x} X_y UU_X + 4\gamma p_1 p_{1x} Y_y UU_Y + 4\gamma p_1 p_{1x} T_y UU_T + \cdots = 0. \end{aligned} \quad (9)$$

In Eq. (9), U_{TY} denotes $\partial U / \partial T \partial Y$ and so on. Through comparing Eq. (9) with Eq. (1), if Eq. (9) becomes Eq. (1), it has to preserve the terms $\partial^2 U / \partial T \partial Y$ and $\partial^2 V / \partial X^2$ meanwhile eliminate the terms of $\partial^3 U / \partial X \partial Y \partial T$, $\partial^3 U / \partial X \partial X \partial T$, $\partial^3 U / \partial X \partial Y \partial Y$, $\partial^3 U / \partial X \partial T \partial T$, $\partial^3 U / \partial Y \partial Y \partial T$, and $\partial^3 U / \partial Y \partial T \partial T$ in the transformed equations, then we get $\partial X / \partial y = 0$, $\partial X / \partial t = 0$, $\partial Y / \partial x = 0$, $\partial Y / \partial t = 0$, $\partial T / \partial x = 0$, and $\partial T / \partial y = 0$. Thus we should set $X = X(x)$, $Y = Y(y)$, $T = T(t)$ in the transformation and further we can find that Eq. (9) becomes

$$\begin{aligned} & U_{TY} + k_1 U_{XXY} + k_2 V_{XX} + k_3 (UU_X)_Y + k_4 U + k_5 U_X + k_6 U_Y + k_7 U_T + k_8 U_{XX} \\ & + k_9 U_{XY} + k_{10} U^2 + k_{11} UU_X + k_{12} UU_Y + k_{13} V + k_{14} V_X = 0, \end{aligned} \quad (10)$$

after a tedious simplification. Note that k_i ($i = 1, 2, \dots, 14$) will be listed later. Similarly, setting $X = X(x)$, $Y = Y(y)$, $T = T(t)$, then bringing Eqs. (7) and (8) into Eq. (6) will imply

$$V_T + k_{15} (UV)_X + k_{16} V_{XX} + k_{17} UV + k_{18} V + k_{19} V_X = 0. \quad (11)$$

In Eqs. (10) and (11), k_i ($i = 1, \dots, 19$) are given by

$$\begin{aligned} k_1 &= -\frac{\alpha_1 X_x^2}{T_t}, \quad k_2 = \frac{2\beta_1 p_2 X_x^2}{p_1 Y_y T_t}, \quad k_3 = \frac{2p_1 \gamma X_x}{T_t}, \quad k_4 = \frac{p_{1yt} - \alpha_1 p_{1xxy}}{p_1 Y_y T_t}, \quad k_5 = -\frac{\alpha_1 (2p_{1xy} X_x + p_{1y} X_{xx})}{p_1 T_t}, \\ k_6 &= \frac{p_{1t} - \alpha_1 p_{1xx}}{p_1 T_t}, \quad k_7 = \frac{p_{1y}}{p_1 Y_y}, \quad k_8 = \frac{-\alpha_1 p_{1y} X_x^2}{p_1 Y_y T_t}, \quad k_9 = \frac{-\alpha_1 (2p_{1x} X_x + p_1 X_{xx})}{p_1 T_t}, \quad k_{10} = \frac{2\gamma (p_{1xy} + p_{1x} p_{1y})}{p_1 Y_y T_t}, \end{aligned}$$

$$k_{11} = \frac{4\gamma p_{1y} X_x}{Y_y T_t}, \quad k_{12} = \frac{4\gamma p_{1x}}{T_t}, \quad k_{13} = \frac{2\beta_1 p_{2xx}}{p_1 Y_y T_t}, \quad k_{14} = \frac{2\beta_1 (2p_{2x} X_x + p_2 X_{xx})}{p_1 Y_y T_t}, \quad k_{15} = \frac{2\alpha_2 p_1 X_x}{T_t},$$

$$k_{16} = \frac{\beta_2 X_x^2}{T_t}, \quad k_{17} = \frac{2\alpha_2 (p_1 p_{2x} + p_2 p_{1x})}{p_2 T_t}, \quad k_{18} = \frac{p_{2t} + \beta_2 p_{2xx}}{p_2 T_t}, \quad k_{19} = \frac{\beta_2 (2p_{2x} X_x + p_2 X_{xx})}{p_2 T_t}.$$

In the above expressions, $\alpha_i = \alpha_i(x, y, t)$, $\beta_i = \beta_i(x, y, t)$ ($i = 1, 2$), $\gamma = \gamma(x, y, t)$, and X_x means $\partial X / \partial x$.

In order to get the usual (2+1)-dimensional BK equations for U and V in Eqs. (10) and (11), we have to set

$$k_4 = k_5 = k_6 = k_7 = k_8 = k_9 = k_{10} = k_{11} = k_{12} = k_{13} = k_{14} = k_{17} = k_{18} = k_{19} = 0,$$

$k_1 = -1$, $k_2 = k_3 = k_{15} = 2$, and $k_{16} = 1$ by comparing Eq. (10) with Eq. (1), and Eq. (11) with Eq. (2). Furthermore, we can get

$$p_1 = c, \quad p_2 = p_2(y), \quad X = ax + b, \\ Y = Y(y), \quad T = T(t), \quad (12)$$

by solving above equations given by $k_i = 0$ ($i = 4, \dots, 14$ and $i = 17, 18, 19$). The coefficients of the (2+1)-dimensional VCBK Eqs. (5) and (6) are determined by

$$\alpha_1 = \beta_2 = \frac{T_t}{a^2}, \quad \alpha_2 = \gamma = \frac{T_t}{ac}, \quad \beta_1 = \frac{cY_y T_t}{a^2 p_2}, \quad (13)$$

from conditions $k_1 = -1$, $k_2 = k_3 = k_{15} = 2$, and $k_{16} = 1$. Here a , b , c are real constants with $a \neq 0$, $c \neq 0$, $p_2 = p_2(y)$, and $Y = Y(y)$ are arbitrary smooth functions of y , $T = T(t)$ is an arbitrary smooth function of t , under $p_2 \neq 0$, $Y_y \neq 0$ and $T_t \neq 0$.

So far, a general transformation defined by Eqs. (7), (8), (12), and (13) that maps (2+1)-dimensional VCBK Eqs. (5) and (6) to the usual (2+1)-dimensional BK equation as we expected. There are three arbitrary functions $p_2(y)$, $Y(y)$, $T(t)$ and a first order polynomial of x for X . Therefore, the (2+1)-dimensional VCBK equation associated with this transformation is a complete integrable model. Moreover, we can design more integrable models by choosing appropriate functions in the transformation and get its solutions from known ones of the usual (2+1)-dimensional BK equation. This transformation provides clearly the integrability and the solvability of the (2+1)-dimensional VCBK Eqs. (5) and (6) associated with given coefficients α_i, β_i ($i = 1, 2$), γ .

3 Examples

In this section, several examples will be given to show the application of the transformation on constructing integrable models and obtaining the solutions of the (2+1)-dimensional VCBK equation. For instance, setting $a = c = 1$, $Y_y = 1$, $p_2 = 1$ in transformation defined by Eqs. (7), (8), (12), and (13), then known example of (2+1)-dimensional VCBK Eqs. (3) and (4) is given by the general (2+1)-dimensional VCBK equation. Two new models will be given in the following context. Furthermore, several kinds of solutions of the new models are obtained from known ones of the (2+1)-dimensional BK

equation, which include dromion-like solutions and rogue wave solutions. Due to the existence of the arbitrary functions in the transformation, the profile and velocity of the dromion-like solutions and the center of the rogue wave solutions for the new models can be controllable.

Example 1 Firstly, a (2+1)-dimensional VCBK equation is brought forward as the following

$$u_{ty} - T_t u_{xxy} + \frac{(y^2 + 1)T_t}{5} v_{xx} + 2T_t (uu_x)_y = 0, \quad (14)$$

$$v_t + 2T_t (uv)_x + T_t v_{xx} = 0, \quad (15)$$

which is obtained from Eqs. (5) and (6) with $\alpha_1 = \alpha_2 = \beta_2 = \gamma = T_t$ and $\beta_1 = (y^2 + 1)T_t/10$ by setting $p_1 = c = 1$, $p_2 = 10/(y^2 + 1)$, $X = x$, $Y = y$, $T = T(t)$ in the transformation. According to the transformation defined in the last section, its dromion-like solutions are

$$u_1 = \frac{1}{2} + \frac{1}{2} \frac{\sinh(x + y - T)}{\cosh(x + y - T) + 1}, \quad (16)$$

$$v_1 = \frac{5}{(y^2 + 1)(\cosh(x + y - T) + 1)}, \quad (17)$$

which is constructed from following solution^[15]

$$U_1 = \frac{1}{2} + \frac{1}{2} \frac{\sinh(X + Y - T)}{\cosh(X + Y - T) + 1}, \quad (18)$$

$$V_1 = \frac{1}{2} \frac{1}{\cosh(X + Y - T) + 1}, \quad (19)$$

of the usual (2+1)-dimensional BK equation. Further, setting $T = t$, then Eqs. (14) and (15) become

$$u_{ty} - u_{xxy} + \frac{y^2 + 1}{5} v_{xx} + 2(uu_x)_y = 0, \quad (20)$$

$$v_t + 2(uv)_x + v_{xx} = 0, \quad (21)$$

and the corresponding solutions are

$$u_2 = \frac{1}{2} + \frac{1}{2} \frac{\sinh(x + y - t)}{\cosh(x + y - t) + 1}, \quad (22)$$

$$v_2 = \frac{5}{(y^2 + 1)(\cosh(x + y - t) + 1)}. \quad (23)$$

These analytical and exact solutions of Eqs. (20) and (21) are useful to study the dynamical evolution of this system. A line-soliton V_1 of the usual (2+1)-dimensional BK equation is mapped to a dromion-like solution v_2 of the Eqs. (20) and (21), and the visual support of this analysis can be found in Figs. 1(a) and 1(c). For v_1 , there exists an extreme point at $(x = T(t), y = 0)$ for a given moment t , then this dromion-like solution evolves with respect to t along x -axis with a variable velocity T_t . From Figs. 1(c) to 1(f), we can find the uniform motion of the dromion-like solution v_2 along x -axis because $T_t = 1$. Of course, we can choose different $T(t)$ in v_1 such that dromion-like solution v_1 goes on (x, y) -plane with different velocity.

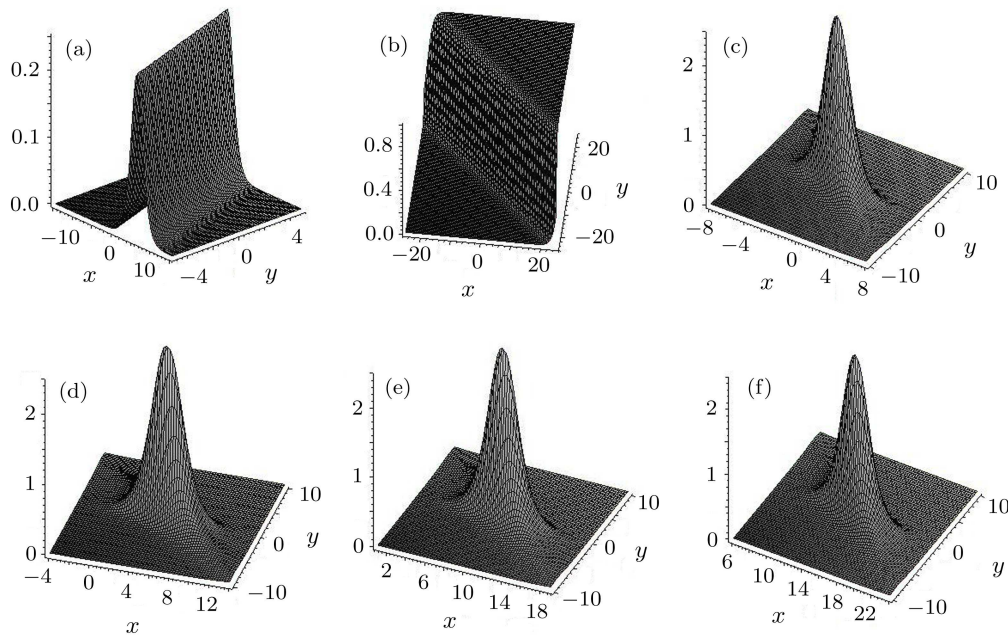


Fig. 1 (a) Depicts V_1 with $T = 1$; (b) Depicts u_2 ; (c) to (f) depict the evolution of v_2 along x -axis with times $t = 1, 5, 10, 15$ respectively.

To illustrate the applicability of this transformation, we would like to give the double dromion-like solution of the (2+1)-dimensional VCBK Eqs. (20) and (21) under $p_1 = c = 1$, $p_2 = 10/(y^2 + 1)$, $X = x$, $Y = y$, $T = t$ in the transformation. Similarly, the double dromion-like solution of Eqs. (20) and (21) is given by

$$u_3 = \frac{e^{x-y-3t} - e^{-x-y+t}}{e^{-x-y+t} + e^{x-y-3t} + 1} + 1, \quad (24)$$

$$v_3 = \frac{10(e^{-x-y+t} - e^{x-y-3t})}{(y^2 + 1)(e^{-x-y+t} + e^{x-y-3t} + 1)^2}, \quad (25)$$

from the double-soliton solution

$$U_2 = \frac{e^{X-Y-3T} - e^{-X-Y+T}}{e^{-X-Y+T} + e^{X-Y-3T} + 1} + 1, \quad (26)$$

$$V_2 = \frac{e^{-X-Y+T} - e^{X-Y-3T}}{(e^{-X-Y+T} + e^{X-Y-3T} + 1)^2}, \quad (27)$$

of the usual (2+1)-dimensional BK equation which is in-

cluded in Ref. [15]. It is easy to see from Figure 2 that a double-soliton (two crossing line-soliton) V_2 of the usual BK equation is mapped to a double dromion-like solution v_3 of the (2+1)-dimensional VCBK Eqs. (20) and (21), and later it splits into one bright dromion and one dark dromion during its evolution.

Example 2 Another (2+1)-dimensional VCBK equation is presented as the following

$$u_{ty} - u_{xxy} + \frac{2e^{y+d+(y+d)^2}}{5}v_{xx} + 2(uu_x)_y = 0, \quad (28)$$

$$v_t + 2(uv)_x + v_{xx} = 0, \quad (29)$$

which is obtained from Eqs. (5) and (6) with $\alpha_1 = \alpha_2 = \beta_2 = \gamma = 1$ and $\beta_1 = 2e^{y+d+(y+d)^2}/5$ by setting $p_1 = c = 1$, $p_2 = 5/e^{(y+d)^2}$, $X = x + b$, $Y = e^{y+d}$, $T = t$ in the transformation. According to the transformation defined in the last section, its solutions are

$$u_4 = \frac{-1 - t + 2(x+b)^2 \coth((x+b)^2 + t^2 + e^{y+d}) + 2(x+b)^2 \operatorname{csch}((x+b)^2 + t^2 + e^{y+d})}{2(x+b)}, \quad (30)$$

$$v_4 = \frac{-5(x+b)}{e^{(y+d)^2}(\cosh((x+b)^2 + t^2 + e^{y+d}) - 1)}, \quad (31)$$

which is constructed from following solution of the usual (2+1)-dimensional BK equation

$$U_3 = \frac{-1 - T + 2X^2 \coth(X^2 + T^2 + Y) + 2X^2 \operatorname{csch}(X^2 + T^2 + Y)}{2X}, \quad (32)$$

$$V_3 = \frac{-X}{\cosh(X^2 + T^2 + Y) - 1}. \quad (33)$$

Note U_3 and V_3 are given from u_2 and v_2 under $\chi = (X^2 + T^2)$, $\varphi = Y$, $\sigma = -1$, and $B(T) = 1$ in Ref. [27]. From the expression of v_4 , we can claim that it is a short lived large amplitude wave on small domain of (x, y) -plane, i.e. rogue wave solution. Note that this is a non-rational form of the rogue wave solution in contrast with the usual rational one. Figures 3(a)–3(d) verify the very quick decaying for t and the well local properties of v_4 .

Further, v_4 has a maximum value and a minimum value around the point $(x, y, t) = (b, d, 0)$, so the point defined by $(x, y) = (b, d)$ is called the center of the rogue wave solution v_4 in the (x, y) -plane. Thus the center of rogue wave solution v_4 is controllable by taking different values of b and d , which is also shown in Figs. 3(a), 3(e), and 3(f).

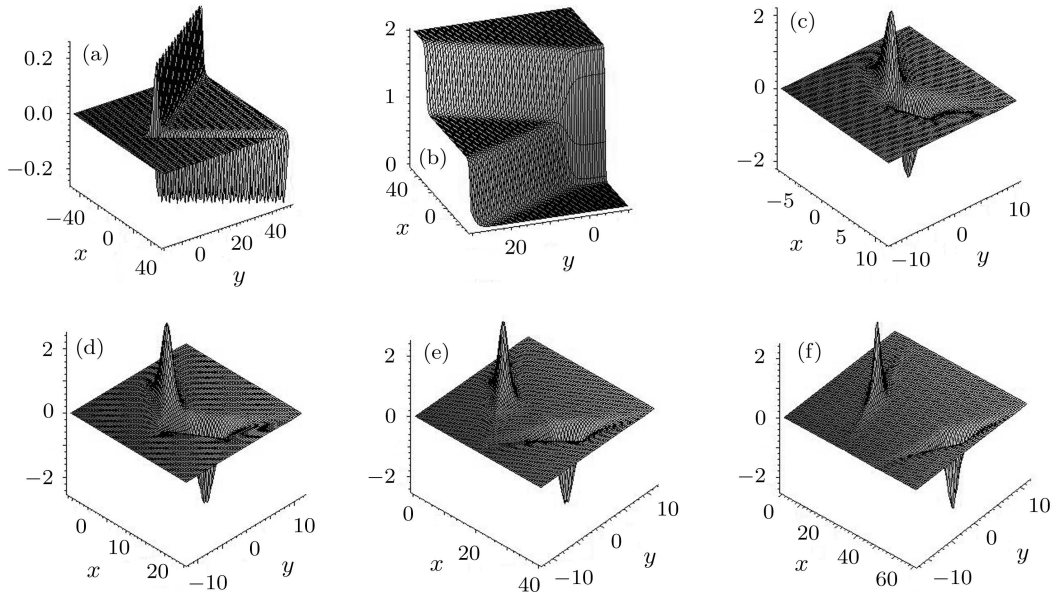


Fig. 2 (a) Depicts V_2 with $T = 1$, (b) Depicts u_3 , (c) to (f) depict the evolution of v_3 with times $t = 1, 5, 10, 20$ respectively. The split of the two-dromion solution is verified by latter four pictures.

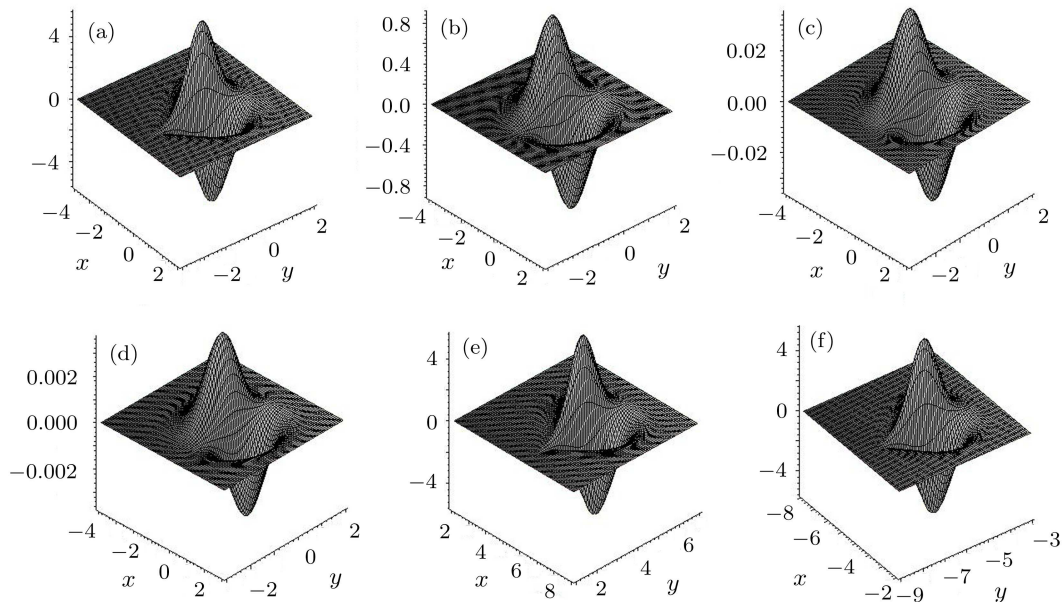


Fig. 3 (a) to (e) depict v_4 with $b = d = 0$ at different times $t = 0, 1, 2, 2.5$ respectively, (e) and (f) depict v_4 at the same moment $t = 0$ with different centers given by $b = d = -5$ and $b = d = 5$.

4 Conclusions

In this paper, we have found a transformation with four arbitrary functions to map the (2+1)-dimensional VCBK Eqs. (5) and (6) to the usual (2+1)-dimensional BK Eqs. (1) and (2), and thus the integrability of the former with given forms of the coefficients is verified. And to illustrate this method, two new integrable models of (2+1)-dimensional VCBK equation are constructed and their interesting solutions including dromion-like solutions and rogue wave solutions are shown analytically. In particular, a non-rational form of the rogue wave solution

v_4 is first constructed in Eq. (31) for example 2. Due to the appearance of the arbitrary functions in the transformation, the integrable model and the velocity or center of its solutions are controllable by appropriate choice of these functions. Moreover, our method is more convenient and simpler than others mentioned in Sec. 1, which solves (2+1)-dimensional VCBK equation directly. It is also possible to solve other higher dimensional equations with variable coefficients and to design other new integrable models according to different applicable purposes by this method.

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