Exact Polynomial Solutions of Schrödinger Equation with Various Hyperbolic Potentials*

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Abstract *The Schrödinger equation with hyperbolic potential $V(x) = -V_0 \sinh^{2q}(x/d)/\cosh^6(x/d)$ $(q = 0, 1, 2, 3)$ is studied by transforming it into the confluent Heun equation. We obtain general symmetric and antisymmetric polynomial solutions of the Schrödinger equation in a unified form via the Functional Bethe ansatz method. Furthermore, we discuss the characteristic of wavefunction of bound state with varying potential strengths. Particularly, the number of wavefunction's nodes decreases with the increase of potential strengths, and the particle tends to the bottom of the potential well correspondingly.*

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1 Introduction

It is well known that the exact solutions of the Schrödinger equation with different potentials play an important role in mathematics and physics.[1−6] For example, Zhang studied exact polynomial solutions of second order differential equations and their applications;[1] Lee *et al.* applied some polynomial algebras to get exact solutions of general quantum nonlinear optical models;[2−3] Azad investigated the polynomial solutions of differential equations by the second order operators.[4] The exact solutions of the Schrödinger equation are very important in quantum mechanics since they contain all the necessary information of the quantum system, and some experimental alterable parameters, which can be used to check the numerical analysis of the equation.[7−8] Recently, the hyperbolic potentials have attracted great attention due to their wide range of applications in physics.[9−16] For instance, Oyewumi *et al.* investigated the bound-state solutions of the Rosen–Morse potential;[9] Wei *et al.* researched the Dirac equation with hyperbolic like potential;[11] Xie studied the energy spectra of a two-dimensional two-electron quantum dot with Pöschl–Teller confining potential;[13] Zhang *et al.* considered bound states of the Dirac equation with the Scarf-type potential.[15]

In general the stationary one-dimensional Schrödinger equation for a non-relativistic particle of mass m and energy E in a hyperbolic potential $V(x)$ can be written as follows:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2} + V(x)\psi(x) = E\psi(x)\,, \tag{1}$$

where

$$V(x) = -V_0\frac{\sinh^{2q}(x/d)}{\cosh^6(x/d)}\,, \quad (q = 0, 1, 2, 3)\,,$$

V_0 and d are the depth (or height) and width of $V(x)$ respectively. If the parameters $V_0 > 0$, $V(x)$ represents a potential well, see Fig. 1; if the parameter $V_0 < 0$, $V(x)$ depicts a potential barrier, see Fig. 2. Downing presented the polynomial solutions of the case with $V_0 > 0$ and $q = 2$ via reducing confluent Heun function to Heun polynomials.[6] In this work, we aim to present the general symmetric and antisymmetric polynomial solutions of all cases and discuss their applications in physics.

This paper is organized as follows. In Sec. 2, we transform the Schrödinger equation into the confluent Heun equation by suitable transformations. Then, the general symmetric and antisymmetric polynomial solutions can be obtained via an effective technique and the Functional Bethe ansatz method. We find that all expressions for energy eigenvalues, wavefunctions and constraints can be unified in one form for symmetric and antisymmetric solutions, respectively. Moreover, in order to ensure the energy to be real, we need to choose $q = 0$, 2 when $V_0 > 0$, and $q = 1$, 3 when $V_0 < 0$. In Sec. 3, we present the wavefunction of the first state, and discuss the wavefunctions of bound state. Finally, we draw conclusions in Sec. 4.

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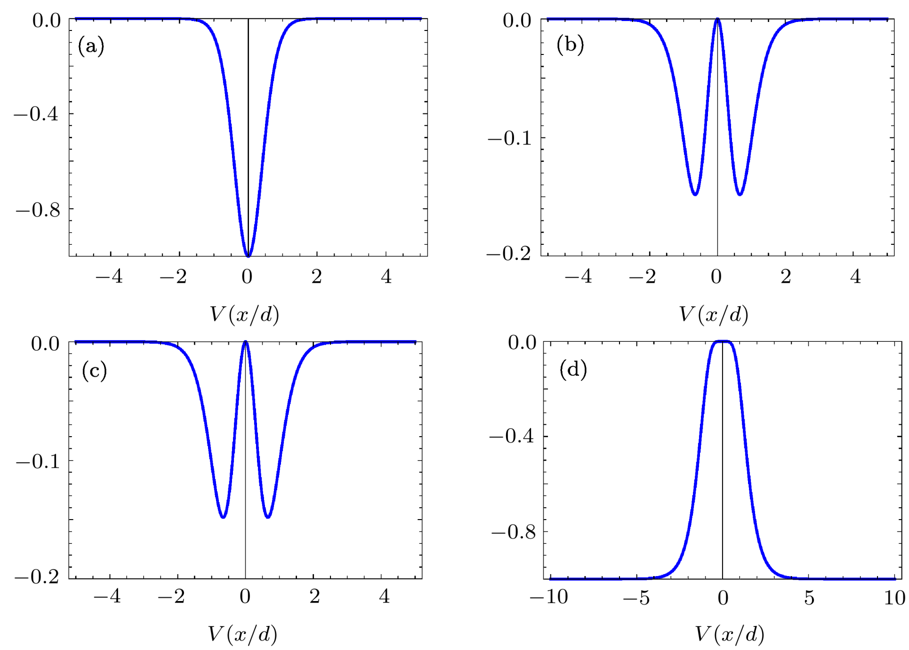

Fig. 1 (Color online) The plot of the hyperbolic double-well potentials when $V_0 > 0$, (a) For $q = 0$, (b) For $q = 1$, (c) For $q = 2$, (d) For $q = 3$.

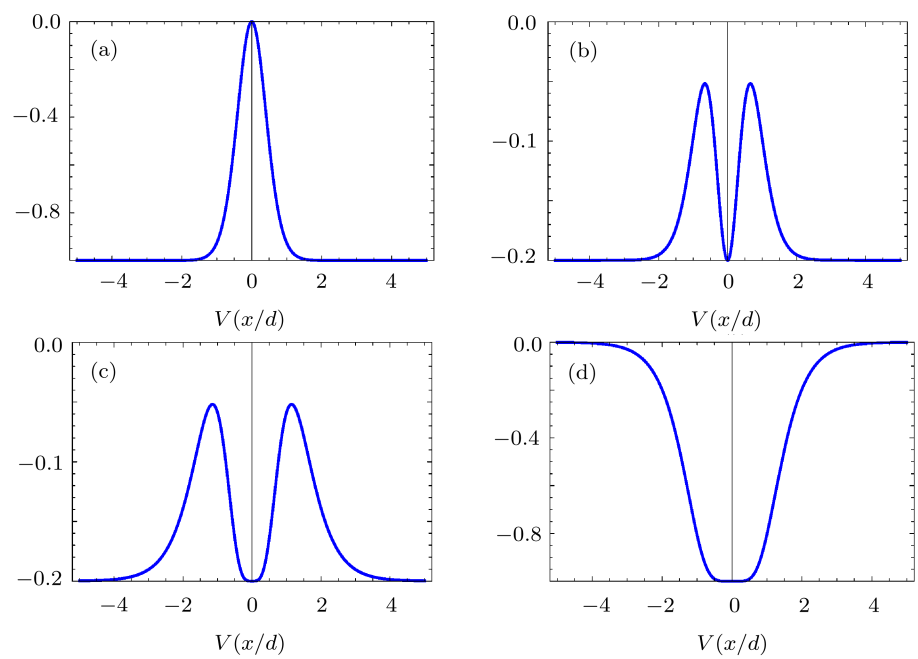

Fig. 2 (Color online) The plot of the hyperbolic double-barrier potentials when $V_0 < 0$, (a) For $q = 0$, (b) For $q = 1$, (c) For $q = 2$, (d) For $q = 3$.

2 Reduction to a Confluent Heun Equation and General Exact Polynomial Solutions

Most of the theoretical physics known today is described by using a small number of differential equations, such as the hypergeometric equation, the Heun equation, etc.[17−21] In this section, we will transform the Schrödinger equation into confluent Heun equation by appropriate transformations. Heun equation is a second-order linear differential with four regular singular points, which was initially studied by Heun.[17] It has several special cases of great importance in mathematical physics, namely the Lamé, Mathieu, and Spheriodal differential equations. Recently, many researchers start focusing on Heun equation, for it has become increasingly widespread application in physics, such as quantum ring, black holes etc.

From Eq. (1), we have

$$\frac{\mathrm{d}^2\psi(z)}{\mathrm{d}z^2}+\Big(\varepsilon d^2+U_0 d^2\frac{(\cosh^2(z)-1)^q}{\cosh^6(z)}\Big)\psi(z)=0\,, \tag{2}$$

where $\varepsilon=2mE/\hbar^2$, $U_0=2mV_0/\hbar^2$, $z=x/d$.

Upon making the change of variable $\eta=1/\cosh^2(z)$, such that the domain $-\infty<z<\infty$ maps to $0<\eta<1$, we obtain

$$\eta^2(1-\eta)\frac{\mathrm{d}^2\psi(\eta)}{\mathrm{d}\eta^2}+\Big(\eta-\frac{3}{2}\eta^2\Big)\frac{\mathrm{d}\psi(\eta)}{\mathrm{d}\eta}+\frac{1}{4}[\varepsilon d^2+U_0 d^2\eta^{3-q}(1-\eta)^q]\psi(\eta)=0\,. \tag{3}$$

The last item contains the third power of the η, which brings great difficulties to solve the equation. Therefore, we make the following transformation

$$\psi(\eta)=\exp(A\eta)\phi(\eta)\,, \tag{4}$$

where A is a parameter to be determined.

Then, we obtain

$$\eta^2(1-\eta)\frac{\mathrm{d}^2\phi(\eta)}{\mathrm{d}\eta^2}+\Big[2A\eta^2(1-\eta)+\Big(\eta-\frac{3}{2}\eta^2\Big)\Big]\frac{\mathrm{d}\phi(\eta)}{\mathrm{d}\eta}+\Big\{A^2\eta^2(1-\eta)+A\Big(\eta-\frac{3}{2}\eta^2\Big)+\frac{1}{4}[\varepsilon d^2+U_0 d^2\eta^{3-q}(1-\eta)^q]\Big\}\phi(\eta)=0. \tag{5}$$

Using the identical equation

$$\eta^{3-q}(1-\eta)^q=\frac{q(q-1)(q-2)}{6}+\frac{(-1)^q}{2}q(q-1)\eta-(-1)^q q\eta^2+(-1)^q\eta^3\,, \tag{6}$$

we obtain

$$\begin{aligned}&\eta^2(1-\eta)\frac{\mathrm{d}^2\phi(\eta)}{\mathrm{d}\eta^2}+\Big[2A\eta^2(1-\eta)+\Big(\eta-\frac{3}{2}\eta^2\Big)\Big]\frac{\mathrm{d}\phi(\eta)}{\mathrm{d}\eta}+\Big\{A^2\eta^2+A\Big(\eta-\frac{3}{2}\eta^2\Big)\\&+\frac{1}{4}\Big[\varepsilon d^2+U_0 d^2\Big(\frac{q(q-1)(q-2)}{6}+\frac{(-1)^q}{2}q(q-1)\eta-(-1)^q q\eta^2\Big)\Big]\Big\}\phi(\eta)+\Big[-A^2+\frac{1}{4}(-1)^q U_0 d^2\Big]\eta^3\phi(\eta)=0\,.\end{aligned} \tag{7}$$

Choosing A to make the last item to be zero, i.e.

$$-A^2+\frac{1}{4}(-1)^q U_0 d^2=0\,. \tag{8}$$

Therefore, Eq. (7) can be rewritten as follows

$$\begin{aligned}&\eta^2(1-\eta)\frac{\mathrm{d}^2\phi(\eta)}{\mathrm{d}\eta^2}+\Big[2A\eta^2(1-\eta)+\Big(\eta-\frac{3}{2}\eta^2\Big)\Big]\frac{\mathrm{d}\phi(\eta)}{\mathrm{d}\eta}+\Big\{A^2\eta^2+A\Big(\eta-\frac{3}{2}\eta^2\Big)\\&+\frac{1}{4}\Big[\varepsilon d^2+U_0 d^2\Big(\frac{q(q-1)(q-2)}{6}+\frac{(-1)^q}{2}q(q-1)\eta-(-1)^q q\eta^2\Big)\Big]\Big\}\phi(\eta)=0\,.\end{aligned} \tag{9}$$

Applying the transformation

$$\phi=\eta^\alpha\varphi(\eta) \tag{10}$$

to transform Eq. (9) into the confluent Heun equation, where α is a parameter to be determined.

Then, we have

$$\frac{\mathrm{d}^2\varphi(\eta)}{\mathrm{d}\eta^2}+\Big(2A+\frac{2\alpha+1}{\eta}+\frac{\gamma+1}{\eta-1}\Big)\frac{\mathrm{d}\varphi(\eta)}{\mathrm{d}\eta}+\Big(\frac{\mu}{\eta}+\frac{\nu}{\eta-1}\Big)\varphi(\eta)+\Big[\alpha^2+\frac{d^2}{4}\Big(\varepsilon+U_0\frac{q(q-1)(q-2)}{6}\Big)\Big]\varphi(\eta)=0\,, \tag{11}$$

where

$$\gamma=-\frac{1}{2}\,,\quad \mu=(2\alpha+1)\Big(A-\frac{\alpha}{2}\Big)+\frac{1}{8}U_0 d^2(-1)^q q(q-1)\,,\quad -\nu=A^2-\frac{A}{2}-\frac{1}{8}(-1)^q qU_0 d^2(3-q)-\frac{\alpha}{2}(2\alpha+1).$$

Choosing α to make the last item to be zero, i.e.

$$\alpha^2+\frac{d^2}{4}\Big(\varepsilon+U_0\frac{q(q-1)(q-2)}{6}\Big)=0\,. \tag{12}$$

Finally, we get

$$\frac{\mathrm{d}^2\varphi(\eta)}{\mathrm{d}\eta^2}+\Big(2A+\frac{2\alpha+1}{\eta}+\frac{\gamma+1}{\eta-1}\Big)\frac{\mathrm{d}\varphi(\eta)}{\mathrm{d}\eta}+\Big(\frac{\mu}{\eta}+\frac{\nu}{\eta-1}\Big)\varphi(\eta)=0\,. \tag{13}$$

Equation (13) is a confluent Heun's differential equation, with regular singularities at $\eta=0,1,\infty$.

Furthermore, in order to obtain symmetric and antisymmetric solutions in a unified form, we make the following transformation

$$\varphi=(1-\eta)^{\beta/2}f(\eta)\,, \tag{14}$$

where $\beta=0$ or 1, and we obtain

$$\eta(\eta-1)\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2}+[2A\eta^2+(-2A+2\alpha+\gamma+2+\beta)\eta-(2\alpha+1)]\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}$$

$$+[(A\beta+\mu+\nu)\eta+\beta(\alpha+1/2)-\mu]f(\eta)=0\,. \tag{15}$$

Applying the procedure of Ref. [1], Eq. (15) has polynomial solutions of degree $n=1,2,3,\ldots$

$$f(\eta)=\prod_{i=1}^{n}(\eta-\eta_i)\,, \tag{16}$$

where the roots η_1, η_2, $\eta_3,\ldots,\eta_n$ obey the Bethe ansatz equations

$$\sum_{j\neq i}^{n}\frac{2}{\eta_i-\eta_j}+\frac{2A\eta_i^2+(-2A+2\alpha+\gamma+2+\beta)\eta_i-2\alpha-1}{\eta_i^2-\eta_i}=0\,,\quad i=1,2,3,\ldots,n\,, \tag{17}$$

$$\sum_{i=1}^{n}\eta_i=\frac{(2\alpha+1)(A-\alpha/2-\beta/2)+(1/8)U_0d^2(-1)^q q(q-1)-n(n-1)-(-2A+2\alpha+\gamma+2+\beta)n}{2A}\,. \tag{18}$$

Moreover, the parameters α and A obey the following relation

$$\alpha=-\frac{(q-1)A+\beta+3/2+2n}{2}\,. \tag{19}$$

Then, from Eqs. (12), (19), we obtain

$$\varepsilon=-\frac{1}{4d^2}[2(q-1)A+2\beta+3+4n]^2-\frac{U_0}{6}q(q-1)(q-2)\,. \tag{20}$$

Therefore, the energy eigenvalues can be obtained as follows

$$E=-\frac{\hbar^2}{8md^2}[2(q-1)A+2\beta+3+4n]^2-\frac{U_0\hbar^2}{12m}q(q-1)(q-2)\,, \tag{21}$$

where $A=\pm(d/2)\sqrt{(-1)^qU_0}$. In order to ensure the energy to be real, we only choose $q=0,2$ when $U_0>0$ and $q=1,3$ when $U_0<0$.

Furthermore, we obtain the wavefunctions as follows

$$\psi=\tanh^{\beta}(z)\exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}\times\prod_{i=1}^{n}\Big(\frac{1}{\cosh^2(z)}-\eta_i\Big)\,. \tag{22}$$

Thus, we have obtained the explicit expressions of energy eigenvalues, wavefunctions, and constraint conditions in unified forms respectively.

For $\beta=0$, we obtain the energy eigenvalues and wavefunctions of symmetric state

$$E_s=-\frac{\hbar^2}{8md^2}[2(q-1)A+3+4n]^2-\frac{U_0\hbar^2}{12m}q(q-1)(q-2), \tag{23}$$

$$\psi_s=\exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}\prod_{i=1}^{n}\Big(\frac{1}{\cosh^2(z)}-\eta_i\Big)\,. \tag{24}$$

For $\beta=1$, we have the energy eigenvalues and wavefunctions of antisymmetric state

$$E_a=-\frac{\hbar^2}{8md^2}[2(q-1)A+5+4n]^2-\frac{U_0\hbar^2}{12m}q(q-1)(q-2), \tag{25}$$

$$\psi_a=\tanh(z)\exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}\times\prod_{i=1}^{n}\Big(\frac{1}{\cosh^2(z)}-\eta_i\Big). \tag{26}$$

Obviously, Eq. (24) indicates that the symmetric wavefunctions have even nodes and the most nodes is $2n$ for the n state. Similarly, Eq. (26) shows that the antisymmetric wavefunctions have odd nodes and the most nodes is $2n+1$ for the n state. It is worth pointing out that symmetric and antisymmetric solutions were obtained by two initial variable transformations in Ref. [6], while we find that the symmetric and antisymmetric solutions can be unified effectively. Although our results are different from Downing's results when $q=2$,[6] we have verified that our polynomial solutions do indeed satisfy Eq. (1) as it is required.

3 Discussion on Wave Function with $n=1$ State

In Sec. 2, we obtain the general polynomial solutions of wavefunction for the symmetric state and the antisymmetric state. In this section, for simplicity, we only consider the first state (i.e. $n=1$), and higher states $n=2,3,\ldots$ can be obtained by the same recipe.

3.1 *The First Symmetric State*

The energy eigenvalue is

$$E_{s1}=-\frac{\hbar^2}{2m}\Big\{\frac{1}{4d^2}[2(q-1)A+7]^2+\frac{U_0}{6}q(q-1)(q-2)\Big\}, \tag{27}$$

and the wavefunction is

$$\psi_{s1}=\exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}\Big(\frac{1}{\cosh^2(z)}-\eta_1\Big)\,. \tag{28}$$

There is a constraint between U_0 and d from Eqs. (17) and (18) as follows,

$$\pm\sqrt{(-2A+2\alpha+\gamma+2)^2+8A(2\alpha+1)}=(2A-\alpha)(2\alpha+1)+\frac{1}{4}U_0d^2(-1)^q q(q-1)-(-2A+2\alpha+\gamma+2)\,, \tag{29}$$

where

$$A=\pm\frac{d}{2}\sqrt{(-1)^qU_0}\,,\quad \alpha=-\frac{(q-1)A+3/2+2}{2}\,.$$

For simplicity, we fix the width of potential $d = 1$, and then obtain

Case 1 $U_0 > 0$, $V(x)$ is potential well

(i) $q = 0$, $A = \pm\sqrt{U_0}/2$, $\alpha = -(-A + 7/2)/2$, the potential $U_0 = 0.0225\ldots, 9.0, 25.0$, and $26.8663\ldots$;

(ii) $q = 2$, $A = \pm\sqrt{U_0}/2$, $\alpha = -(A + 7/2)/2$, the potential $U_0 = 0.0116\ldots, 10.5753\ldots, 149.5742\ldots$ and $595.8386\ldots$

Case 2 $U_0 < 0$, $V(x)$ is potential barrier

(i) $q = 1$, $A = \pm\sqrt{-U_0}/2$, $\alpha = -7/4$, the potential $U_0 = -0.0160\ldots$ and $-107.6089\ldots$;

(ii) $q = 3$, $A = \pm\sqrt{-U_0}/2$, $\alpha = -(2A + 7/2)/2$, the potential $U_0 = -0.0088\ldots$ and $-2.0349\ldots$

Here we find that the case with $q = 2$, $A = -\sqrt{U_0}/2$, $\alpha = -(A + 7/2)/2$ has symmetric wavefunctions of bound state. The energy is $E_{s1} = -(\hbar^2/8m)(-\sqrt{U_0} + 7)^2$, and the wavefunction is

$$\psi_{s1} = \exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}\Big(\frac{1}{\cosh^2(z)} - \eta_1\Big).$$

Figures 3(a) and 4(a) show that both wavefunctions are symmetric. As we increase the potential strengths, the wavefunctions become more tightly, namely the higher the potential strength the tighter the confinement, it is not hard to understand by the Heisenberg's uncertainty relation. Moreover, we can see that a drop from two nodes to zero node in Figs. 3(a) and 4(a), and the corresponding $|E_{s1} - V(x/d)_{\min}|$ become smaller see Figs. 3(c) and 4(c). The change of the nodes is similar with some simple quantum systems such as one-dimensional infinite potential well, one-dimensional harmonic oscillator and so on. Furthermore, Fig. 3(b) indicates that the probability density approximates to zero at the bottom of the potential well, while Fig. 4(b) shows that the probability density is the largest at the lowest point of the potential well. This indicates that the particle tends to the bottom of the potential well with the potential strengths increase.

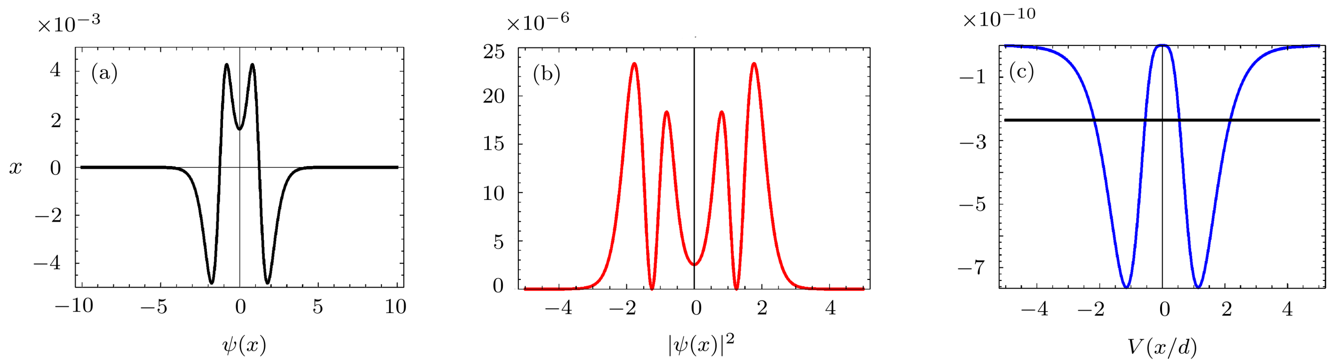

Fig. 3 (Color online) The plot of symmetric state, with $q = 2$, $n = 1$, $d = 1$, $U_0 = 149.574\,25\ldots$, and energy $E_{s1} = -2.353\,09 \times 10^{-10}$ neV. (a) For wavefunction, (b) For probability density, (c) For hyperbolic double-well potential and energy level.

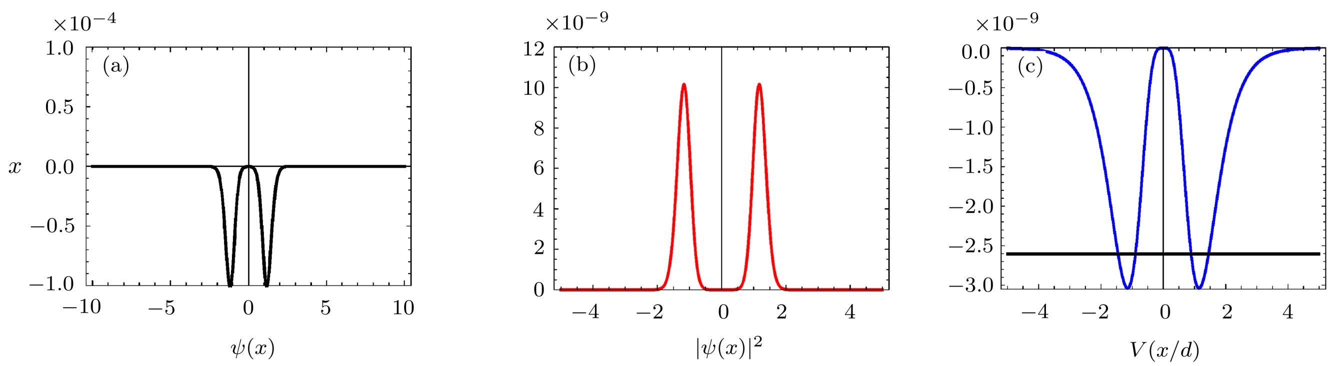

Fig. 4 (Color online) The plot of symmetric state, with $q = 2$, $n = 1$, $d = 1$, $U_0 = 595.838\,65\ldots$, and energy $E_{s1} = -2.607\,44 \times 10^{-9}$ neV. (a) For wavefunction, (b) For probability density, (c) For hyperbolic double-well potential and energy level.

3.2 *The First Anti-Symmetric State*

The energy eigenvalue is

$$E_{a1} = -\frac{\hbar^2}{2m}\Big\{\frac{1}{4d^2}[2(q-1)A+9]^2 + \frac{U_0}{6}q(q-1)(q-2)\Big\}, \tag{30}$$

and the wavefunction is

$$\psi_{a1} = \tanh(z)\exp\Big(\frac{A}{\cosh^2(z)}\Big)\frac{1}{\cosh^{2\alpha}(z)}$$

$$\times\left(\frac{1}{\cosh^2(z)}-\eta_1\right). \tag{31}$$

There is a constraint between U_0 and d from Eqs. (17) and (18) as follows,

$$\begin{aligned}&\pm\sqrt{(-2A+2\alpha+\gamma+3)^2+8A(2\alpha+1)}\\ &\quad=(2A-\alpha-1)(2\alpha+1)+\frac{1}{4}U_0d^2(-1)^q q(q-1)\\ &\qquad-(-2A+2\alpha+\gamma+3)\,,\end{aligned} \tag{32}$$

where

$$A=\pm\frac{d}{2}\sqrt{(-1)^qU_0}\,,\quad \alpha=-\frac{(q-1)A+9/2}{2}\,.$$

For simplicity, we let the width of potential $d=1$, and obtain

Case 1 $U_0>0$, $V(x)$ is potential well

(i) $q=0$, $A=\pm\sqrt{U_0}/2$, $\alpha=-(-A+9/2)/2$, the potential $U_0=0.0072\ldots$, $15.3260\ldots$, 25.0 and 49.0;

(ii) $q=2$, $A=\pm\sqrt{U_0}/2$, $\alpha=-(A+9/2)/2$, the potential $U_0=0.0047\ldots$, $4.9642\ldots$, $426.2320\ldots$, and $1092.7989\ldots$

Case 2 $U_0<0$, $V(x)$ is potential barrier

(i) $q=1$, $A=\pm\sqrt{-U_0}/2$, $\alpha=-9/4$, the potential $U_0=-0.0058\ldots$ and $-16.7302\ldots$;

(ii) $q=3$, $A=\pm\sqrt{-U_0}/2$, $\alpha=-(2A+9/2)/2$, the potential $U_0=-0.0039\ldots$ and $-1.6615\ldots$

We obtain that the case with $q=2$, $A=-\sqrt{U_0}/2$, $\alpha=-(A+9/2)/2$ has antisymmetric wavefunctions of bound state. The energy is $E_{a1}=-(\hbar^2/8m)(-\sqrt{U_0}+9)^2$, and the wavefunction is

$$\psi_{a1}=\tanh(z)\exp\left(\frac{A}{\cosh^2(z)}\right)\frac{1}{\cosh^{2\alpha}(z)}\left(\frac{1}{\cosh^2(z)}-\eta_1\right).$$

The results for the first antisymmetric state have similar behavior with the first symmetric state, as can be seen in Figs. 5 and 6. We find that the higher the potential strength the tighter the confinement, but note that a decrease in nodes from three to one. In physics, the first antisymmetric state is similar with the first symmetric state.

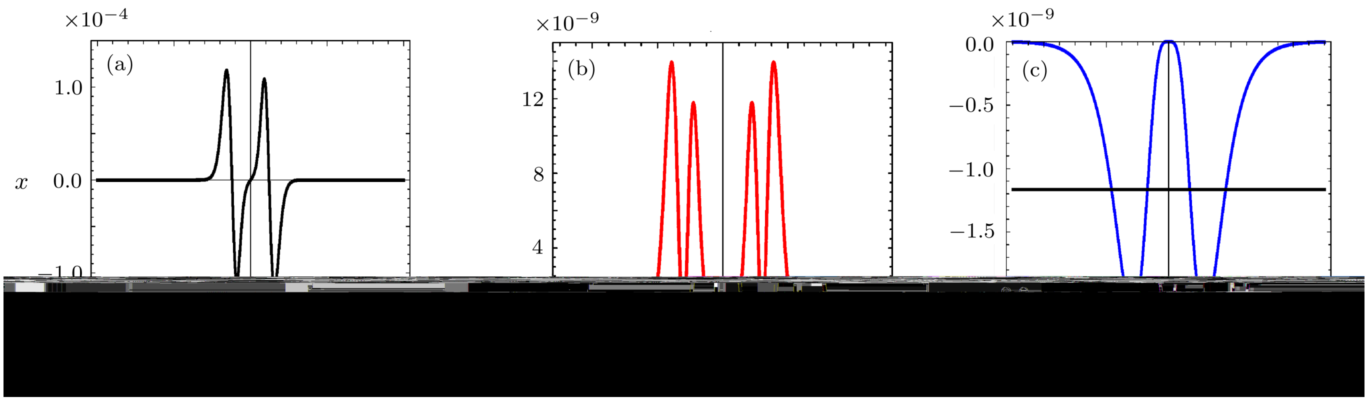

Fig. 5 (Color online) The plot of anti-symmetric state, with $q=2$, $n=1$, $d=1$, $U_0=426.232\,048\ldots$, and energy $E_{a1}=-1.166\,63\times10^{-9}$ neV. (a) For wavefunction, (b) For probability density, (c) For hyperbolic double-well potential and energy level.

Fig. 6 (Color online) The plot of antisymmetric state, with $q=2$, $n=1$, $d=1$, $U_0=1092.798\,97\ldots$, and energy $E_{a1}=-4.978\,83\times10^{-9}$ neV. (a) For wavefunction, (b) For probability density, (c) For hyperbolic double-well potential and energy level.

4 Conclusion

We obtain the general polynomial solutions of the Schrödinger equation with various hyperbolic potentials, which can be reduced to the symmetric wavefunctions with $\beta=0$ and antisymmetric wavefunctions with $\beta=1$. Additionally, we must set $q = 0, 2$ with $V_0 > 0$ and $q = 1, 3$ with $V_0 < 0$ for the hyperbolic potential to ensure that the energy eigenvalue be real, which can be derived from the analytical expression of energy eigenvalue. Furthermore, we find the number of wavefunction's nodes decrease with the increase of potential strengths, and the particle tends to the bottom of the potential well correspondingly, for the bound states.

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