Remove Degeneracy in Relativistic Symmetries for Manning–Rosen Plus Quasi-Hellman Potentials by Tensor Interaction

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(Received June 7, 2016; revised manuscript received August 11, 2016)

Abstract *The relativistic Dirac equation under spin and pseudo-spin symmetries is investigated for Manning–Rosen plus quasi-Hellman potentials with tensor interaction. For the first time we consider the Hulthen plus Yukawa for tensor interaction. The Formula method is used to obtain the energy eigen-values and wave functions. We also discuss about the energy eigen-values and the Dirac spinors for the Manning–Rosen plus quasi-Hellman potentials for the spin and pseudo-spin symmetry with Formula method. To show the accuracy of the present model, some numerical results are shown in both pseudo-spin and spin symmetry limits.*

PACS numbers: 03.65.Ge, 03.65.Pm, 02.30.Gp

Key words: Dirac equation, Manning–Rosen potential, quasi-Hellman potential, tensor interaction, formula method

1 Introduction

Since the early years of quantum mechanics the study of analytically solvable problems for some special potentials of physical interest has attracted much attention in theoretical physics. Obtaining analytical solutions of the Klein–Gordon, Dirac and other wave equations is one of the interesting problems in high energy and nuclear physics. These wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics.[1] It is well known that the Klein–Gordon equation and Dirac equation can always be reduced to a Schrödinger-type equation specially when the Lorentz scalar and vector potential are equal. The study of relativistic effects is always useful in some quantum mechanical systems.[2–3] Therefore, the Dirac equation has become the most appealing relativistic wave equation for spin-1/2 particles. For example, in the relativistic treatment of nuclear phenomena the Dirac equation is used to describe the behavior of the nuclei in nucleus and also in solving many problems of high-energy physics and chemistry. For this reason, it has been used extensively to study the relativistic heavy ion collisions, heavy ion spectroscopy and more recently in laser-matter interaction (for a review, see Ref. [4] and references there in) and condensed matter physics.[5–6] The idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model has been introduced in 1969 by Arima *et al.* (1969), Hecht and Adler (1969).[7–8] Spin and pseudo-spin symmetries are SU(2) symmetries of a Dirac Hamiltonian with vector and scalar potentials. They are realized when the difference, $\Delta(r) = V(r) - S(r)$, or the sum, $\Sigma(r) = V(r) + S(r)$, are constants. The near realization of these symmetries may explain degeneracy in some heavy meson spectra (spin symmetry) or in single-particle energy levels in nuclei (pseudo-spin symmetry), when these physical systems are described by relativistic mean-field theories (RMF) with scalar and vector potentials.[9] Recently, some authors have studied various-type potentials with a tensor potential, under the conditions of pseudo-spin and spin symmetry.[10–11] They have found out that the tensor interaction removes the degeneracy between two states in the pseudo-spin and spin doublet.[12–13] The pseudo-spin and spin symmetry appeared in nuclear physics refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n, l + 2, j = l + 3/2)$, where n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively.[14] In recent time, the study of Dirac equation and Klein–Gordon equation with exponential-type potential models has attracted the attention of many researchers in the field.[15–17] The kind of various analytical techniques have been employed to find the solution of the Klein–Gordon equation and Dirac equation such as the super symmetric quantum mechanics,[18–19] asymptotic iteration method (AIM),[20–21] factorization method,[22–23] formula method,[24] GPS Method[25–26] and the path integral method,[27–28] Nikiforov–Uvarov method[27–29] and others. The Klein–Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics with some typical potential by using different methods.[30] For example, Kratzer potential,[31–32] Woods–Saxon potential,[33–34] Scarf potential,[35–36] Hartmann potential,[37–38] Rosen Morse potential,[39–40] quasi-Hellman potentials[41] and Manning–Rosen potential.[42–44]

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In this paper, we attempt to investigate analytically degeneracy in Dirac wave equation for Manning–Rosen plus quasi-Hellman potentials in the spin and pseudo-spin symmetry with a tensor potential by using the formula method. For the first time we consider the Hulthen plus Yukawa for tensor interaction and we investigate the energy eigen-values and wave functions. The organization of this paper is as follows: in Sec. 2, the formula method is reviewed. In Sec. 3 we review basic Dirac equations briefly. In Secs. 4 and 5, Dirac wave equation for the spin and pseudo-spin symmetry of these potentials in the presence of Hulthen plus Yukawa-like tensor interaction are presented, respectively. In Sec. 6 we provide results and discussion. The conclusion is given in Sec. 7.

2 Review of Formula Method

The Formula method has been used to solve the Schrödinger, Dirac and Klein–Gordon wave equations for a certain kind of potentials. In this method the differential equation can be written as follows:[21]

$$\Psi_n''(s)+\frac{(k_1-k_2 s)}{s(1-k_3 s)}\Psi_n'(s)+\frac{(\zeta_2 s^2+\zeta_1 s+\zeta_0)}{s^2(1-k_3 s)^2}\Psi_n(s)=0\,. \tag{1}$$

For a given Schrödinger-like equation in the presence of any potential model which can be written in the form of Eq. (1), the energy eigen-values and the corresponding wave function can be obtained by using the following formulas, respectively.[21]

$$\left[\frac{k_4^2-k_5^2-[\frac{1-2n}{2}-\frac{1}{2k_3}(k_2-\sqrt{(k_3-k_2)^2-4\zeta_2})]^2}{2[\frac{1-2n}{2}-\frac{1}{2k_3}(k_2-\sqrt{(k_3-k_2)^2-4\zeta_2})]}\right]^2-k_5^2=0,\quad k_3\neq 0\,, \tag{2}$$

$$\Psi_n(s)=N_n s^{k_4}(1-k_3 s)^{k_5}{}_2\mathrm{F}_1\Bigl(-n,n+2(k_4+k_5)+\frac{k_2}{k_3}-1;2k_4+k_1,k_3 s\Bigr)\,, \tag{3}$$

where,

$$k_4=\frac{(1-k_1)+\sqrt{(1-k_1)^2-4\zeta_0}}{2}\,,$$

$$k_5=\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}+\sqrt{\Bigl[\frac{1}{2}+\frac{k_1}{2}-\frac{k_2}{2k_3}\Bigr]^2-\Bigl[\frac{\zeta_2}{k_3^2}+\frac{\zeta_1}{k_3}+\zeta_0\Bigr]}\,. \tag{4}$$

And N_n is the normalization constant. In the special case where $k_3\to 0$ the energy eigen-values and the corresponding wave function can be obtained as:[21]

$$\Bigl[\frac{\zeta_1-k_4 k_2-nk_2}{2k_4+k_1+2n}\Bigr]^2-k_5^2=0\,, \tag{5}$$

$$\Psi_n(s)=N_n s^{k_4}\exp(-k_5 s)\times{}_1\mathrm{F}_1(-n;2k_4+k_1;(2k_5+k_2)s)\,. \tag{6}$$

The solutions provide a valuable means for checking and improving models and numerical methods introduced for solving complicated quantum systems.

3 Basic Dirac Equations

In the relativistic description, the Dirac equation of a single-nucleon with the mass moving in an attractive scalar potential $S(r)$ and a repulsive vector potential $V(r)$ can be written as:[44]

$$[-\mathrm{i}\hbar c\hat{\alpha}\cdot\hat{\nabla}+\hat{\beta}(Mc^2+S(r))]\Psi_{n_r,k}=[E-V(r)]\Psi_{n_r,k}\,, \tag{7}$$

where E is the relativistic energy, M is the mass of a single particle and α and β are the 4×4 Dirac matrices. For a particle in a central field, the total angular momentum J and $\hat{K}=-\hat{\beta}(\hat{\alpha}\cdot\hat{L}+\hbar)$ commute with the Dirac Hamiltonian where L is the orbital angular momentum. For a given total angular momentum j, the eigen-values of the $\hat{K}$ are $k=\pm(j+1/2)$ where negative sign is for aligned spin and positive sign is for unaligned spin. The wave-functions can be classified according to their angular momentum j and spin-orbit quantum number k as follows:

$$\Psi_{n_r,k}(r,\theta,\phi)=\frac{1}{r}\begin{bmatrix}F_{n_r,k}(r)\mathrm{Y}_{jm}^{l}(\theta,\phi)\\ \mathrm{i}G_{n_r,k}(r)\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\phi)\end{bmatrix}\,, \tag{8}$$

where $F_{n_r,k}(r)$ and $G_{n_r,k}(r)$ are upper and lower components, $\mathrm{Y}_{jm}^{l}(\theta,\phi)$ and $\mathrm{Y}_{jm}^{\tilde{l}}(\theta,\phi)$ are the spherical harmonic functions. n_r is the radial quantum number and m is the projection of the angular momentum on the z axis. The orbital angular momentum quantum numbers l and $\tilde{l}$ represent to the spin and pseudo-spin quantum numbers. Substituting Eq. (8) into Eq. (7), we obtain couple equations for the radial part of the Dirac equation as follows by $h=c=1$.

$$\begin{cases}(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}-U(r))F_{n_r,k}(r)=[M+E_{n,k}-\Delta(r)]G_{n_r,k}(r)\,,\\ (\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}+U(r))G_{n_r,k}(r)=[M-E_{n,k}+\sum(r)]F_{n_r,k}(r)\,,\end{cases} \tag{9}$$

where $\Delta(r)=V(r)-S(r)$ and $\sum(r)=V(r)+S(r)$ are the difference and the sum of the potentials $V(r)$ and $S(r)$, respectively and $U(r)$ is a tensor potential From Eq. (9), we obtain the second-order Schrödinger-like equation as:

$$\Bigl\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{k(k+1)}{r^2}+\frac{2kU(r)}{r}-\frac{\mathrm{d}U(r)}{r}-U^2(r)-[M+E_{n,k}-\Delta(r)]\Bigl[M-E_{n,k}+\sum(r)\Bigr]+\frac{\frac{\mathrm{d}\Delta(r)}{\mathrm{d}r}(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{k}{r}-U(r))}{(M+E_{n,k}-\Delta(r))}\Bigr\}F_{n_r,k}(r)=0\,, \tag{10}$$

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{k(k-1)}{r^2}+\frac{2kU(r)}{r}+\frac{\mathrm{d}U(r)}{r}-U^2(r)-[M+E_{n,k}-\Delta(r)][M-E+\Sigma(r)]$$
$$+\frac{\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k}{r}+U(r))}{(M-E_{n,k}+\Sigma(r))}\Big\}G_{n_r,k}(r)=0\,. \tag{11}$$

We consider bound state solutions that demand the radial components satisfying $F_{n_r,k}(0)=G_{n_r,k}(0)=0$, and $F_{n_r,k}(\infty)=G_{n_r,k}(\infty)=0$.[45]

4 Spin Symmetry with Tensor Interaction

Under the condition of the spin symmetry, i.e. $\mathrm{d}\Delta(r)/\mathrm{d}r=0$ or $\Delta(r)=C_s=\text{const.}$, the upper component Dirac equation can be written as:[45]

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{k(k+1)}{r^2}+\frac{2kU(r)}{r}-\frac{\mathrm{d}U(r)}{r}-U^2(r)-[M+E_{n,k}-C_s][M-E_{n,k}+\sum(r)]\Big\}F_{n_r,k}(r)=0\,. \tag{12}$$

The potential $\sum(r)$ is taken as the Manning–Rosen[42–43] plus quasi-Hellman potentials.[41]

$$\sum(r)=-A\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})}+B\Big(\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\Big)^2+b\frac{\mathrm{e}^{-\alpha r}}{r^2}-\frac{a}{r}\ \text{and}\ A=\alpha^2\frac{z}{q},\quad B=\alpha^2\frac{\nu(\nu-1)}{q}\,, \tag{13}$$

where a, b, z, q, and ν are real parameters, these parameters describe the depth of the potential well, and the parameter α is related to the range of the potential.

For the tensor term, we consider the Hulthen[46] plus Yukawa potentials,[27]

$$U(r)=-\frac{v_0}{(1-\mathrm{e}^{-\alpha r})}-\frac{v_1}{r}\mathrm{e}^{-\alpha r}\,, \tag{14}$$

where v_0 and v_1 are real parameters, these parameters describe the depth of the potential well, and the parameter α is related to the range of the potential.

By substituting Eqs. (13) and (14) into Eq. (12), we obtain the upper radial equation of Dirac equation as:

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{1}{r^2}(k(k+1)+2kv_1\mathrm{e}^{-\alpha r}+v_1\mathrm{e}^{-\alpha r}+v_1^2\mathrm{e}^{-2\alpha r})-\frac{1}{r}\Big(\frac{2kv_0}{(1-\mathrm{e}^{-\alpha r})}+\alpha v_1\mathrm{e}^{-\alpha r}+2v_0v_1\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})}\Big)$$
$$-\frac{v_0}{(1-\mathrm{e}^{-\alpha r})^2}(v_0+\alpha\,\mathrm{e}^{-\alpha r})-\gamma-\delta\Big(-A\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})}+B\Big(\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\Big)^2+b\frac{\mathrm{e}^{-\alpha r}}{r^2}-a\frac{1}{r}\Big)\Big\}F_{n_r,k}(r)=0\,, \tag{15}$$

where $\gamma=(M+E_{n,k}-C_s)(M-E_{n,k})$ and $\delta=(M+E_{n,k}-C_s)$.

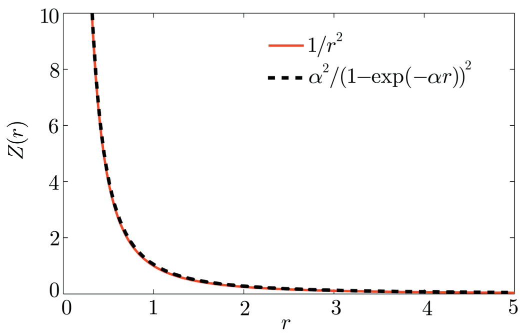

Fig. 1 The behavior approximation for $\alpha=0.07\ \mathrm{fm}^{-1}$.

Equation (15) is exactly solvable only for the case of $k=0,-1$. In order to obtain the analytical solution of Eq. (15), we employ the improved approximation proposed by Greene and Aldrich[47] and replace the spin-orbit coupling term with the expression that is valid for $\alpha\le 1$.[48–49]

$$\frac{1}{r^2}\approx\frac{\alpha^2}{(1-\mathrm{e}^{-\alpha r})^2}\,. \tag{16}$$

The behavior of the improved approximation is plotted in Fig. 1. We can see the good agreement for small α values.

By using the transformation $s=\exp(-\alpha r)$, Eq. (15) brings into the form:

$$F''_{n,k}(s)+\frac{(1-s)}{s(1-s)}F'_{n,k}(s)$$
$$+\frac{1}{s^2(1-s)^2}[\zeta_2s^2+\zeta_1s+\zeta_0]F_{n,k}(s)=0\,, \tag{17}$$

where the parameters ζ_2, ζ_1, and ζ_0 are considered as follows:

$$\zeta_2=-\frac{1}{\alpha^2}[\gamma+\delta A+\delta B]-v_1(v_1-1)\,,$$
$$\zeta_1=\frac{1}{\alpha^2}[2\gamma+\delta A]-\frac{1}{\alpha}[v_0+\delta a]-\delta b-2v_1(\eta_k+1)\,,$$
$$\text{and}\ \eta_k=\Big(k+\frac{v_0}{\alpha}\Big)\,,$$
$$\zeta_0=-\frac{\gamma}{\alpha^2}+\frac{1}{\alpha}[v_0-\delta a]-\eta_k(\eta_k+1)\,. \tag{18}$$

By comparing Eq. (17) with Eq. (1), we can easily obtain the coefficients k_i $(i=1,2,3)$ as follows:

$$k_1=k_2=k_3=1\,. \tag{19}$$

The values of the coefficients k_i $(i=4,5)$ are also found from Eq. (4) as below:

$$k_4=\sqrt{-\zeta_0},\quad k_5=\frac{1}{2}+\sqrt{\frac{1}{4}-[\zeta_2+\zeta_1+\zeta_0]}\,. \tag{20}$$

With using the energy equation, Eq. (2) for energy eigenvalues we have:

$$\left[\frac{-\zeta_0-[\frac{1}{2}+\sqrt{\frac{1}{4}-(\zeta_2+\zeta_1+\zeta_0)}]^2-b[\frac{1-2n}{2}-\frac{1}{2}(1-\sqrt{-4\zeta_2})]^2}{2[\frac{1-2n}{2}-\frac{1}{2}(1-\sqrt{-4\zeta_2})]}\right]^2-\left[\frac{1}{2}+\sqrt{\frac{1}{4}-(\zeta_2+\zeta_1+\zeta_0)}\,\right]^2=0\,. \tag{21}$$

In Tables 1–3, we give the numerical results for the spin symmetric energy eigen-values (in units of fm^{-1}).

Table 1 Energies of the spin symmetry limit in the presence and absence of Hulthen plus Yukawa-like tensor interaction by parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_s = 5\ \mathrm{fm}^{-1}$.

l	$n, k > 0$	State (l, j)	$E^s_{n,k}r$ $(v_0 = v_1 = 0)$	$E^s_{n,k}$ $(v_0 = v_1 = 0.65)$	$n, k < 0$	State $(l, j + 1)$	$E^s_{n,k}$ $(v_0 = v_1 = 0)$	$E^s_{n,k}$ $(v_0 = v_1 = 0.65)$
1	$1, 1$	$1p_{1/2}$	10.420 749 91	10.477 204 23	$1, -2$	$1p_{3/2}$	10.420 749 91	10.477 204 23
2	$1, 2$	$1d_{3/2}$	10.461 200 66	10.540 918 56	$1, -3$	$1d_{5/2}$	10.461 200 66	10.284 251 38
3	$1, 3$	$1f_{5/2}$	10.518 134 99	10.614 177 99	$1, -4$	$1f_{7/2}$	10.518 134 99	10.417 006 48
4	$1, 4$	$1g_{7/2}$	10.587 849 81	10.693 328 44	$1, -5$	$1g_{9/2}$	10.587 849 81	10.465 345 94

Table 2 The energy eigen-values (in units of fm^{-1}) for the spin symmetry limit with parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_s = 5\ \mathrm{fm}^{-1}$.

α/fm^{-1}	$E^s_{n,k}$ $(v_0 = v_1 = 0)$		$E^s_{n,k}$ $(v_0 = v_1 = 0.7)$	
	$1f_{5/2}$	$1f_{7/2}$	$1f_{5/2}$	$1f_{7/2}$
0.05	9.868 444 057	9.868 444 057	9.970 462 079	9.910 649 918
0.15	10.087 926 39	10.087 926 139	10.201 824 56	10.070 900 83
0.25	10.280 186 48	10.280 186 48	10.397 859 49	10.214 180 08
0.35	10.445 158 51	10.445 158 51	10.559 883 09	10.345 404 76
0.45	10.585 323 38	10.585 323 38	10.681 993 77	10.473 262 85
0.60	10.804 802 06	10.804 802 06	10.826 955 60	10.640 836 33
0.70	10.848 872 09	10.848 872 09	10.900 282 90	10.737 360 92

Table 3 The energy eigen-values (in units of fm^{-1}) for the spin symmetry limit with parameters $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_s = 5\ \mathrm{fm}^{-1}$.

M/fm^{-1}	$E^s_{n,k}$ $(v_0 = v_1 = 0)$		$E^s_{n,k}$ $(v_0 = v_1 = 0.75)$	
	$1f_{5/2}$	$1f_{7/2}$	$1f_{5/2}$	$1f_{7/2}$
5	5.726 086 516	5.726 086 516	5.925 780 211	5.402 801 439
5.5	6.173 902 518	6.173 902 518	6.368 978 458	5.904 500 494
6	6.644 952 191	6.644 952 191	6.822 456 012	6.405 401 297
6.5	6.936 698 707	6.936 698 707	7.283 783 216	6.905 853 985
7	7.431 185 789	7.431 185 789	7.751 206 229	7.406 040 986
8	8.422 512 127	8.422 512 127	8.699 518 849	8.405 984 934
9	9.416 028 092	9.416 028 092	9.660 485 249	9.405 840 187
10	10.411 017 69	10.411 017 69	10.473 996 49	10.405 208 67

Let us find the corresponding wave functions. In reference to Eq. (3) and Eq. (20), we can obtain the upper wave function as:

$$\begin{aligned}F^s_{n,k}(r) &= N(\mathrm{e}^{-2\alpha r})^{(\sqrt{-c})}(1-\mathrm{e}^{-2\alpha r})^{(1/2+\sqrt{1/4+A+B+C})}\\ &\times {}_2\mathrm{F}_1\biggl(-n, n+2\biggl(\sqrt{-c}+\frac{1}{2}+\sqrt{\frac{1}{4}+A+B+C}\,\biggr); 2\sqrt{-c}+1,\, \mathrm{e}^{-2\alpha r}\biggr)\,,\end{aligned} \tag{22}$$

where N is the normalization constant, on the other hand, the lower component of the Dirac spinor can be calculated from Eq. (23) as:

$$G^s_{n,k}(r) = \frac{1}{M + E^s_{n,k} - C_s}\Bigl(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{k}{r} - U(r)\Bigr)F^s_{n,k}(r)\,. \tag{23}$$

The effects of the Hulthen plus Yukawa-like tensor interactions on the upper and lower components for the spin symmetry are shown in Figs. 2.

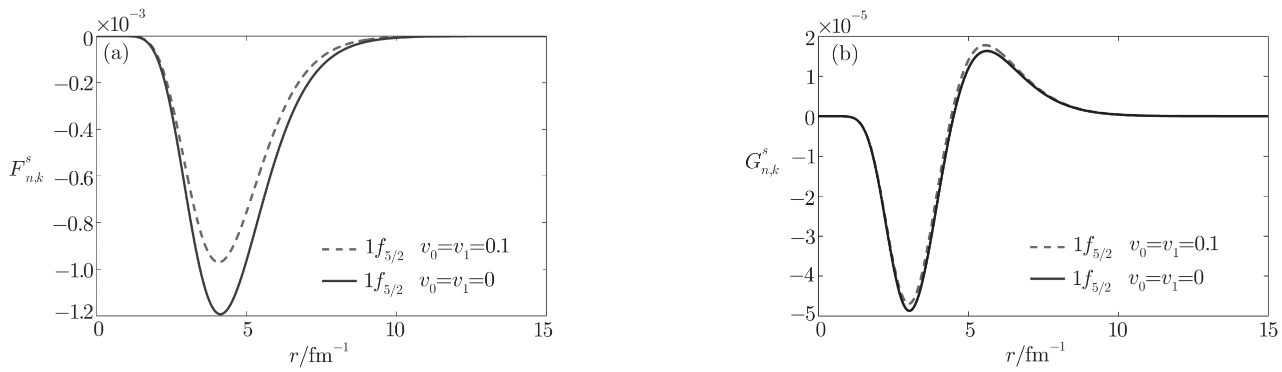

Fig. 2 Wave functions of $1f_{5/2}$ in the spin symmetry in the presence and absence of Hulthen plus Yukawa-like tensor interaction by parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $v_0 = v_1 = 0.1$, $C_s = 5\ \mathrm{fm}^{-1}$.

We have obtained the energy eigen-values and the wave function of the radial Dirac equation for Manning–Rosen plus quasi-Hellman potentials with Hulthen plus Yukawa-like tensor interaction in the presence of the spin symmetry for $k \neq 0$.

5 Pseudo-Spin Symmetry with Tensor Interaction

For the pseudo-spin symmetry, i.e., $\mathrm{d}\Sigma(r)/\mathrm{d}r = 0$ or $\sum(r) = C_{\mathrm{ps}} = \text{const.}$ the lower component Dirac equation can be written as:[45]

$$\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{k(k-1)}{r^2} + \frac{2kU(r)}{r} + \frac{\mathrm{d}U(r)}{r} - U^2(r) - [M + E_{n,k} - \Delta(r)][M - E + \Sigma(r)]\Big\}G_{n_r,k}(r) = 0\,. \tag{24}$$

We consider the scalar, vector, and tensor potentials as the following:

$$\Delta(r) = -A\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})} + B\Big(\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\Big)^2 + b\frac{\mathrm{e}^{-\alpha r}}{r^2} - \frac{a}{r}\,, \tag{25}$$

$$U(r) = -\frac{v_0}{(1-\mathrm{e}^{-\alpha r})} - \frac{v_1}{r}\,\mathrm{e}^{-\alpha r}. \tag{26}$$

By substituting Eqs. (25) and (26) into Eq. (24), we obtain the lower radial equation of Dirac equation as:

$$\begin{aligned}&\Big\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r^2}(k(k-1) + 2kv_1\,\mathrm{e}^{-\alpha r} - v_1\,\mathrm{e}^{-\alpha r} + v_1^2\,\mathrm{e}^{-2\alpha r}) - \frac{1}{r}\Big(\frac{2kv_0}{(1-\mathrm{e}^{-\alpha r})} - \alpha v_1\,\mathrm{e}^{-\alpha r} + 2v_0v_1\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})}\Big)\\ &\quad - \frac{v_0}{(1-\mathrm{e}^{-\alpha r})^2}(v_0 + \alpha\,\mathrm{e}^{-\alpha r}) - \gamma' - \delta'\Big(-A\frac{\mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-\alpha r})} + B\Big(\frac{\mathrm{e}^{-\alpha r}}{1-\mathrm{e}^{-\alpha r}}\Big)^2 + b\frac{\mathrm{e}^{-\alpha r}}{r^2} - a\frac{1}{r}\Big)\Big\}F_{n_r,k}(r) = 0\,,\end{aligned} \tag{27}$$

where $\gamma' = (M + E_{n,k})(M - E_{n,k} + C_{\mathrm{ps}})$ and $\delta' = (M - E_{n,k} + C_{\mathrm{ps}})$.

By using the transformation $s = \exp(-\alpha r)$ and employing the improved approximation Eq. (27) brings into the form:

$$G''_{n,k}(s) + \frac{(1-s)}{s(1-s)}G'_{n,k}(s) + \frac{1}{s^2(1-s)^2}(\zeta'_2 s^2 + \zeta'_1 s + \zeta'_0)G_{n,k}(s) = 0\,, \tag{28}$$

where the parameters ζ'_2, ζ'_1, and ζ'_0 are considered as follows:

$$\begin{aligned}&\zeta'_2 = -\frac{1}{\alpha^2}(\gamma' - \delta' A - \delta' B) - v_1(v_1+1), \quad \zeta'_1 = \frac{1}{\alpha^2}(2\gamma' - \delta' A) - \frac{1}{\alpha}(\delta' a - v_0) - \delta' b - 2v_1\eta_k\,,\\ &\zeta'_0 = -\frac{\gamma'}{\alpha^2} - \frac{1}{\alpha}(\delta' a + v_0) - \eta_k(\eta_k - 1)\,.\end{aligned} \tag{29}$$

By comparing Eq. (28) with Eq. (1), we can easily obtain the coefficients k'_i $(i = 1, 2, 3)$ as follows:

$$k'_1 = k'_2 = k'_3 = 1\,. \tag{30}$$

The values of the coefficients k'_i $(i = 4, 5)$ are also found from Eq. (4) as below:

$$k'_4 = \sqrt{-\zeta'_0}, \quad k'_5 = \frac{1}{2} + \sqrt{\frac{1}{4} - (\zeta'_2 + \zeta'_1 + \zeta'_0)}\,. \tag{31}$$

With using the energy equation, Eq. (2) for energy eigen-values we have:

$$\left[\frac{-C' - \left[\frac{1}{2} + \sqrt{\frac{1}{4} - (\zeta_2' + \zeta_1' + \zeta_0')}\,\right]^2 - [\frac{1-2n}{2} - \frac{1}{2}(1 - \sqrt{-4\zeta_2'})]^2}{2[\frac{1-2n}{2} - \frac{1}{2}(1 - \sqrt{-4\zeta_2'})]}\right]^2 - \left[\frac{1}{2} + \sqrt{\frac{1}{4} - (\zeta_2' + \zeta_1' + \zeta_0')}\right]^2 = 0\,. \tag{32}$$

In Tables 4–6, we give the numerical results for the pseudo-spin symmetric energy eigen-values (in units of fm^{-1}).

Table 4 The energy eigen-values (in units of fm^{-1}) for the pseudo-spin symmetry limit in the presence and absence of Hulthen plus Yukawa-like tensor by parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_{\rm ps} = -5\ \mathrm{fm}^{-1}$.

l	$n, k < 0$	State (l, j)	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0)$	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0.65)$	$n, k > 0$	State $(l+2, j+1)$	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0)$	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0.65)$
1	$1, -1$	$1s_{1/2}$	5.019 919 131	5.014 411 589	$1, 2$	$1d_{3/2}$	5.019 919 131	5.096 930 918
2	$1, -2$	$1p_{3/2}$	5.055 970 775	5.021 549 569	$1, 3$	$1f_{5/2}$	5.055 970 775	5.144 958 264
3	$1, -3$	$1d_{5/2}$	5.102 241 390	5.047 831 688	$1, 4$	$1g_{7/2}$	5.102 241 390	5.197 871 403
4	$1, -4$	$1f_{7/2}$	5.154 862 403	5.088 564 324	$1, 5$	$1h_{9/2}$	5.154 862 403	5.254 076 716

Table 5 The energy eigen-values (in units of fm^{-1}) for the pseudo-spin symmetry limit with parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_{\rm ps} = -5\ \mathrm{fm}^{-1}$.

α/fm^{-1}	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0)$		$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0.7)$	
	$1d_{5/2}$	$1g_{7/2}$	$1d_{5/2}$	$1g_{7/2}$
0.05	4.965 887 554	4.965 887 554	4.671 148 958	5.016 536 852
0.15	4.593 508 465	4.593 508 465	4.866 912 603	5.056 260 336
0.25	4.974 989 489	4.974 989 489	4.940 231 068	5.107 246 518
0.35	5.078 558 988	5.078 558 988	4.186 373 171	5.169 808 128
0.45	5.128 914 342	5.128 914 342	5.006 920 934	5.243 726 91
0.60	5.226 382 418	5.226 382 418	5.010 492 44	5.375 103 194
0.70	5.305 372 845	5.305 372 845	5.013 289 767	5.475 739 681

Table 6 The energy eigen-values (in units of fm^{-1}) for the pseudo-spin symmetry limit with parameters $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_{\rm ps} = -5\ \mathrm{fm}^{-1}$.

M/fm^{-1}	$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0)$		$E^{\rm ps}_{n,k}$ $(v_0 = v_1 = 0.75)$	
	$1d_{5/2}$	$1g_{7/2}$	$1d_{5/2}$	$1g_{7/2}$
5	0.299 583 014	0.299 583 014	0.122 238 961	0.613 218 316
5.5	0.752 667 809	0.752 667 809	0.521 107 678	1.021 322 750
6	1.217 934 79	1.217 934 79	1.017 487 164	1.451 690 478
6.5	1.458 897 949	1.458 897 949	1.514 921 629	1.897 557 639
7	2.170 364 141	2.170 364 141	2.013 010 111	2.354 494 845
8	3.139 545 324	3.139 545 324	3.057 391 086	3.290 672 446
9	4.118 051 359	4.118 051 359	4.048 659 693	4.245 896 947
10	5.102 241 390	5.102 241 390	5.007 349 118	5.212 876 954

By using Eq. (3) and Eq. (31) we can finally obtain the lower component of the Dirac spinor as below:

$$\begin{aligned} G^{\rm ps}_{n,k}(r) &= N(\mathrm{e}^{-2\alpha r})^{(\sqrt{-c'})}(1 - \mathrm{e}^{-2\alpha r})^{(1/2+\sqrt{1/4+A'+B'+C'})} \\ &\times {}_2\mathrm{F}_1\left(-n, n + 2\left(\sqrt{-c'} + \frac{1}{2} + \sqrt{\frac{1}{4} + A' + B' + C'}\,\right); 2\sqrt{-c'} + 1,\, \mathrm{e}^{-2\alpha r}\right)\,, \end{aligned} \tag{33}$$

where N' is the normalization constant, on the other hand, the upper component of the Dirac spinor can be calculated from Eq. (33) as:

$$F^{\rm ps}_{n,k}(r) = \frac{1}{M - E^{\rm ps}_{n,k} + C_{\rm ps}}\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k}{r} + U(r)\right)G^{\rm ps}_{n,k}(r)\,. \tag{34}$$

The effects of the Hulthen plus Yukawa-like tensor interactions on the upper and lower components for the pseudo-spin symmetry are shown in Figs. 3.

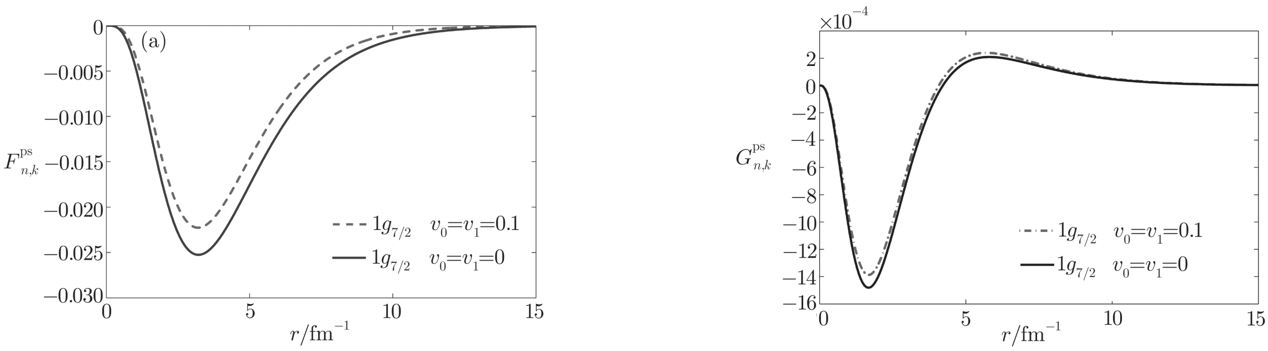

Fig. 3 (a)-Upper and (b)-lower components of $1g_{7/2}$ in the pseudo-spin symmetry in the presence and absence of Hulthen plus Yukawa-like tensor interaction by parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $v_0 = v_1 = 0.1$, $C_{\mathrm{ps}} = -5\ \mathrm{fm}^{-1}$.

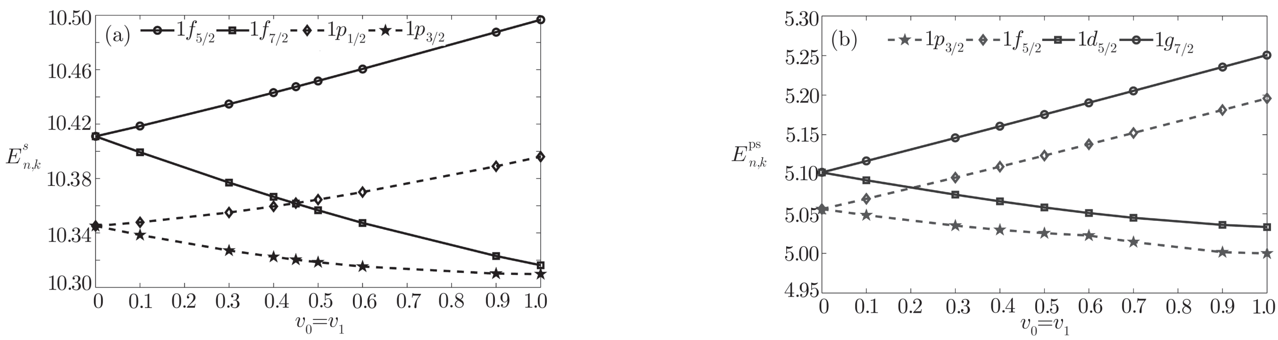

Fig. 4 Energy spectra in the (a)-spin and (b)-pseudo-spin symmetries versus $v_0 = v_1$ for Hulthen plus Yukawa like tensor interaction with parameters $M = 10\ \mathrm{fm}^{-1}$, $c = 1$, $h = 1$, $\alpha = 0.4\ \mathrm{fm}^{-1}$, $a = 1\ \mathrm{fm}^{-1}$, $b = 1\ \mathrm{fm}^{-1}$, $A = 1\ \mathrm{fm}^{-1}$, $B = 1\ \mathrm{fm}^{-1}$, $C_s = 5\ \mathrm{fm}^{-1}$, $C_{\mathrm{ps}} = -5\ \mathrm{fm}^{-1}$.

The sensitiveness of the pseudo-spin doublets ($1p_{3/2}$, $1f_{5/2}$), ($1d_{5/2}$, $1g_{7/2}$) and spin doublets ($1f_{5/2}$, $1f_{7/2}$), ($1p_{1/2}$, $1p_{3/2}$) for the parameters $v_0 = v_1$ are given in Fig. 4.

6 Results and Discussion

We obtain the energy eigen-values in the absence and the presence of the Coulomb-like tensor potential for various values of the quantum numbers n and k. In Tables 1 and 4 in the absence of the tensor interaction ($v_0 = v_1 = 0$), the degeneracy between spin doublets and pseudo-spin doublets are observed. For example, we observe the degeneracy in ($1p_{1/2}$, $1p_{3/2}$), ($1d_{3/2}$, $1d_{5/2}$), ... etc. in the spin symmetry, and we observe the degeneracy in ($1s_{1/2}$, $1d_{3/2}$), ($1p_{3/2}$, $1f_{5/2}$), ... etc. in the pseudo-spin symmetry. When we consider the tensor interaction for example by parameters $v_0 = v_1 = 0.65$, the degeneracy is removed. In Tables 2 and 3 for the spin symmetry, also in Tables 5 and 6 for the pseudo-spin symmetry we show that exist degeneracy between spin doublets for several of parameters α and M, and we show that degeneracy is removed in the present of tensor interaction. The effects of the Hulthen plus Yukawa-like tensor interactions on the upper and lower components of radial Dirac equation for the symmetries are shown in Figs. 2 and 3. We have showen in Fig. 4 behavior energy for various $v_0 = v_1$ in the spin and pseudo-spin symmetry. The degeneracy is removed by tensor interaction effect. Also, the amount of the energy difference between the two states in the doublets increases with increasing v_0 and v_1.

7 Conclusions

In this paper, we have discussed approximately the solutions of the Dirac equation for Manning–Rosen plus quasi-Hellman potentials with Hulthen plus Yukawa-like in Spin Symmetry and Pseudo-spin Symmetry for $k \neq 0$. We obtained the energy Eigen-values and the wave function in terms of the generalized polynomials functions via the formula method. To show the accuracy of the present model, some numerical values of the energy levels are shown in figures 2, 3, and 4. We have shown that the energy degeneracy in pseudo-spin and spin doublets is removed by the tensor interaction effect.

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