An Efficient Numerical Solution of Nonlinear Hunter–Saxton Equation

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(Received October 13, 2016; revised manuscript received February 20, 2017)

Abstract *In this paper, the nonlinear Hunter–Saxton equation, which is a famous partial differential equation, is solved by using a hybrid numerical method based on the quasilinearization method and the bivariate generalized fractional order of the Chebyshev functions (B-GFCF) collocation method. First, using the quasilinearization method, the equation is converted into a sequence of linear partial differential equations (LPD), and then these LPDs are solved using the B-GFCF collocation method. A very good approximation of solutions is obtained, and comparisons show that the obtained results are more accurate than the results of other researchers.*

PACS numbers: 02.30.Jr, 02.70.Jn **DOI:** 10.1088/0253-6102/67/5/483

Key words: Hunter–Saxton equation, fractional order of the Chebyshev functions, quasilinearization method, collocation method, nonlinear PDE

Nomenclature

$u(x,t)$	unknown function	t	time position coordinate
$\bar{u}(x,t)$	the approximate solution	x	scaled coordinate
${}_{\eta}FT_n^{\alpha}$	basis function	$\Phi(t)$	the vector of basis functions
α	the order of basis functions	η	length of the domain of function definition
A	unknown coefficients vector	$\boldsymbol{K}$	unknown coefficients matrix
m	the number of basis functions	E	the maximum of the absolute error

1 Introduction

The nonlinear Hunter–Saxton equation is one of the partial differential equations that by some researchers is studied:

$$(u_t + uu_x)_x - \frac{1}{2}u_x^2 = 0\,, \tag{1}$$

or, equivalently,

$$u_{xt} + uu_{xx} + \frac{1}{2}u_x^2 = 0\,, \tag{2}$$

where t and x are time position and scaled coordinates, respectively. The structures of the problem, for the first time, are described by Hunter and Saxton in 1991 in their paper entitle "Dynamics of Director Fields".[1] They have used this equation for studying a nonlinear instability in the direct field of a nematic liquid crystal, and have shown that smooth solutions of the asymptotic equation break down in finite time.

The Hunter–Saxton equation also arises as the short-wave limit of the Camassa–Holm equation,[2] an integrable model the unidirectional propagation of shallow water waves over a flat bottom,[3] and the geodesic flow on the diffeomorphism group of the circle with a bi-Hamiltonian structure,[4] which is completely integrable.[5]

Because of the many applications of this equation has been studied by some researchers, such as Beals *et al.*[6] have obtained the inverse scattering solutions to this equation, Penskoi[7] has studied Lagrangian time-discretizations of the Hunter-Saxton equation by using the Moser–Veselov approach, Yin[8] has proved the local existence of strong solutions of the periodic Hunter–Saxton equation and has shown that all strong solutions except space-independent solutions blow up in finite time, Lenells[9] has considered the Hunter–Saxton equation models the geodesic flow on a spherical manifold, Xu and Shu[10] have used the development of the local discontinuous Galerkin method and a new dissipative discontinuous Galerkin method for this equation, Wei and Yin[11] have considered the periodic Hunter–Saxton equation with weak dissipation, Wei[12] has obtained global weak solution for a periodic generalized Hunter–Saxton equation, Nadjafikhah and Ahangari[13] have studied a Lie group symmetry analysis of the equation and have obtained some exact solutions, Baxter *et al.*[14] have obtained the separable solutions and self-similar solutions of the equation, Arbabi *et al.*[15] have obtained a semi-

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analytical solution for the equation using the Haar wavelet quasilinearization method.

We know that, the solution of some equations is generated by fractional powers or the structure of the solution of some equations is not exactly known. For example, one of the famous equations that its solution is generated by fractional powers is Thomas-Fermi equation.[16–17] Baker[17] has proved that the solution of Thomas–Fermi equation is generated by the powers of $t^{1/2}$. For these reasons, in this paper, we decided that we solve the Hunter–Saxton equation using the fractional basis, namely the bivariate generalized fractional order of the Chebyshev function (B-GFCF), in order to obtain more information about the structure of the solution and obtaining acceptable results.

The B-GFCFs are introduced as a new basis for Spectral methods and this basis can be used to develop a framework or theory in Spectral methods. In this research, the fractional basis was used for solving a partial differential equation (Hunter–Saxton equation) and it provided insight into an important issue. The B-GFCF collocation method is combined with the quasilinearization method (QLM) to calculate a more accurate and faster result.

The organization of the paper is expressed as follows: in Sec. 2, the generalized fractional order of the Chebyshev functions (GFCFs) and their properties are expressed. In Sec. 3, the work method is explained. In Sec. 4, the numerical examples are presented to show the efficiency of the method. Finally, a conclusion is provided.

2 Generalized Fractional Order of the Chebyshev Functions

The Chebyshev polynomials have many properties, for example orthogonal, recursive, simple real roots, complete in the space of polynomials. For these reasons, many authors have used these functions in their works.[18–21]

Using some transformations, some researchers extended Chebyshev polynomials to infinite or semi-infinite domains. For example, by using $x = (t - L)/(t + L)$, $L > 0$ the rational Chebyshev functions on semi-infinite interval,[22–25] by using $x = t/\sqrt{t^2 + L}$, $L > 0$ the rational Chebyshev functions on infinite interval,[26] and by using $x = 1 - 2(t/\eta)^\alpha$, $\alpha, \eta > 0$ the generalized fractional order of the Chebyshev functions (GFCF) on the finite interval $[0, \eta]$[27] are introduced.

In the present work, the transformation $x = 1 - 2(t/\eta)^\alpha$, $\alpha, \eta > 0$ on the Chebyshev polynomials of the first kind is used, that was introduced in Ref. [27] and can use to solve differential equations.

The GFCFs are defined on the interval $[0, \eta]$ and are denoted by ${}_\eta FT_n^\alpha(t) = T_n(1 - 2(t/\eta)^\alpha)$.

The analytical form of ${}_\eta FT_n^\alpha(t)$ of degree $n\alpha$ is given by[27]

$$ {}_\eta FT_n^\alpha(t) = \sum_{k=0}^{n} \beta_{n,k,\eta,\alpha} \cdot t^{\alpha k}, \quad t \in [0, \eta], \tag{3} $$

where

$$ \beta_{n,k,\eta,\alpha} = (-1)^k \frac{n 2^{2k}(n + k - 1)!}{(n - k)!(2k)!\eta^{\alpha k}} $$
$$ \text{and } \beta_{0,k,\eta,\alpha} = 1 . $$

The GFCFs are orthogonal with respect to the weight function $w(t) = t^{\alpha/2-1}/\sqrt{\eta^\alpha - t^\alpha}$ on the interval $(0, \eta)$:

$$ \int_0^\eta {}_\eta FT_n^\alpha(t) {}_\eta FT_m^\alpha(t) w(t) \mathrm{d}t = \frac{\pi}{2\alpha} c_n \delta_{mn} , \tag{4} $$

where δ_{mn} is Kronecker delta, $c_0 = 2$, and $c_n = 1$ for $n \geq 1$.

Any function of continuous and differentiable $y(t)$, $t \in [0, \eta]$ can be expanded as follows:

$$ y(t) = \sum_{n=0}^{\infty} a_n {}_\eta FT_n^\alpha(t) , $$

and using the property of orthogonality in the GFCFs:

$$ a_n = \frac{2\alpha}{\pi c_n} \int_0^\eta {}_\eta FT_n^\alpha(t)\ y(t)\ w(t)\ \mathrm{d}t, \quad n = 0, 1, 2, \ldots $$

But in the numerical methods, we have to use first $(m + 1)$-terms of the GFCFs and approximate $y(t)$:

$$ y(t) \approx \hat{y}_m(t) = \sum_{n=0}^{m} a_n {}_\eta FT_n^\alpha(t) = A^{\mathrm{T}} \Phi(t) , \tag{5} $$

with

$$ A = [a_0, a_1, \ldots, a_m]^{\mathrm{T}}, \tag{6} $$
$$ \Phi(t) = [{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \ldots, {}_\eta FT_m^\alpha(t)]^{\mathrm{T}}. \tag{7} $$

The following theorem shows that by increasing m, the approximation solution $f_m(t)$ is convergent to $f(t)$ exponentially.

Theorem 1 Suppose that $D^{k\alpha} f(t) \in C[0, \eta]$ for $k = 0, 1, \ldots, m$, and ${}_\eta F_m^\alpha$ is the subspace generated by $\{{}_\eta FT_0^\alpha(t), {}_\eta FT_1^\alpha(t), \ldots, {}_\eta FT_{m-1}^\alpha(t)\}$. If $f_m(t) = A^{\mathrm{T}}\Phi(t)$ (in Eq. (5)) is the best approximation to $f(t)$ from ${}_\eta F_m^\alpha$, then the error bound is presented as follows

$$ \| f(t) - f_m(t) \|_w \leq \frac{\eta^{m\alpha} M_\alpha}{2^m \Gamma(m\alpha + 1)} \sqrt{\frac{\pi}{\alpha\ m!}} , $$

where $M_\alpha \geq |D^{m\alpha} f(t)|$, $t \in [0, \eta]$.

Proof See Ref. [27]. *

Theorem 2 The generalized fractional order of the Chebyshev function ${}_\eta FT_n^\alpha(t)$, has precisely n real zeros on interval $(0, \eta)$ in the form

$$ t_k = \eta\Big(\frac{1 - \cos((2k - 1)\pi/2n)}{2}\Big)^{1/\alpha}, \quad k = 1, 2, \ldots, n . $$

Moreover, $(\mathrm{d}/\mathrm{d}t) {}_\eta FT_n^\alpha(t)$ has precisely $n - 1$ real zeros on interval $(0, \eta)$ in the following points:

$$ t'_k = \eta\Big(\frac{1 - \cos(k\pi/n)}{2}\Big)^{1/\alpha} , $$
$$ k = 1, 2, \ldots, n - 1 . $$

Proof See Ref. [27]. *

3 Methodology

In this section, the quasi-linearization method is introduced and is used for solving nonlinear Hunter–Saxton equation.

3.1 *The Quasilinearization Method*

The quasi-linearization method (QLM), based on the Newton–Raphson method,[28–29] by Bellman and Kalaba have introduced.[30] This method is used for solving the nonlinear differential equations (NDEs) of n-th order in p dimensions. In this method, the NDEs convert to a sequence of linear differential equations, and the solution of this sequence of linear differential equations is convergence to the solution of the NDEs.[31–33] Some researchers have used this method in their papers.[34–37]

Occasionally the linear differential equation that gets from the QLM at each iteration does not solve analytically. Hence we can use the Spectral methods to approximate the solution.

We consider nonlinear PDEs of the form

$$\bar{\Gamma}\Big(\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial \xi\partial t},\frac{\partial^2 u}{\partial t^2},\ldots,\frac{\partial^m u}{\partial t^m}\Big) = \bar{\Psi}\Big(u,\frac{\partial u}{\partial \xi},\frac{\partial^2 u}{\partial \xi^2},\ldots,\frac{\partial^n u}{\partial \xi^n}\Big)\,, \tag{8}$$

where n and m are the orders of differentiation for ξ and t, respectively, $t\in[0,T]$, $\xi\in[a,b]$, $u(\xi,t)$ is the unknown function, $\bar{\Psi}$ is a nonlinear operator that contains all the partial derivatives of $u(\xi,t)$ to ξ , and $\bar{\Gamma}$ is a linear operator of both variables ξ and t that contains all the partial derivatives of $u(\xi,t)$ to t.

By using the transformation $\xi=[(b-a)/\eta]x+a$, the interval $\xi\in[a,b]$ can be converted into the interval $x\in[0,\eta]$, thus Eq. (8) can be written as follows

$$\Gamma\Big(\frac{\partial u}{\partial t},\frac{\partial^2 u}{\partial \xi\partial t},\frac{\partial^2 u}{\partial t^2},\ldots,\frac{\partial^m u}{\partial t^m}\Big) = \Psi\Big(u,\frac{\partial u}{\partial x},\frac{\partial^2 u}{\partial x^2},\ldots,\frac{\partial^n u}{\partial x^n}\Big)\,, \tag{9}$$

where T and η are real positive constants, $\Gamma=\bar{\Gamma}|_{\xi=[(b-a)/\eta]x+a}$, and $\Psi=\bar{\Psi}|_{\xi=[(b-a)/\eta]x+a}$.

Before applying the QLM, the operator Ψ is split into its linear and nonlinear parts and rewrite Eq. (9) as follows:

$$L[u,u',u'',\ldots,u^{(n)}]+N[u,u',u'',\ldots,u^{(n)}] - \Gamma(\dot{u},\dot{u}',\ddot{u},\ldots,\dot{u}^{(m)})=0\,, \tag{10}$$

where the dots and primes denote the derivative with respect to t and x, respectively, and L and N are the linear and nonlinear operators of Ψ, respectively.

Now, the QLM is used for the nonlinear operator N as follows (similar to Taylor's series):[38]

$$N[u,u',\ldots,u^{(n)}]\approx N[u_r,u'_r,\ldots,u_r^{(n)}] +\sum_{k=0}^{n}\frac{\partial N}{\partial u^{(k)}}\big(u_{r+1}^{(k)}-u_r^{(k)}\big)\,, \tag{11}$$

where r and $r+1$ denote previous and current iterations, respectively, and the functions $\partial N/\partial u^{(k)}$ are functional derivatives with respect to $u^{(k)}$ from the $N[u,u',\ldots,u^{(n)}]$.

By substituting Eq. (11) into Eq. (10), and using the QLM, we have

$$L[u_{r+1},u'_{r+1},\ldots,u_{r+1}^{(n)}]+\sum_{k=0}^{n}\frac{\partial N}{\partial u^{(k)}}u_{r+1}^{(k)} - \Gamma(\dot{u}_{r+1},\dot{u}'_{r+1},\ddot{u}_{r+1},\ldots,\dot{u}_{r+1}^{(m)}) = H_r[u_r,u'_r,\ldots,u_r^{(n)}]\,, \tag{12}$$

where

$$H_r[u_r,u'_r,\ldots,u_r^{(n)}]=\sum_{k=0}^{n}\frac{\partial N}{\partial u^{(k)}}u_r^{(k)}-N[u_r,u'_r,\ldots,u_r^{(n)}]\,,$$

and $r=0,1,2,3,\ldots$

By using the QLM, the solution of Eq. (9) determines the $(r+1)$-th iterative approximation $u_{r+1}(x,t)$ as a solution of the linear partial differential equation (12) with their initial and boundary conditions.

The QLM iteration requires an initialization or "initial guess" $u_0(x,t)$, that it is usually selected based on the initial and boundary conditions.

3.2 *The B-GFCFs Collocation Method*

It is assumed that the solution can be approximated by using the bivariate generalized fractional order of the Chebyshev functions (B-GFCFs) in the form

$$u(x,t)\approx\hat{u}(x,t)=\sum_{i=0}^{m_1}\sum_{j=0}^{m_2}k_{i,j\eta}FT_i^{\alpha}(x)_{\eta}FT_j^{\alpha}(t)\,, \tag{13}$$

where m_1 is the number of collocation points in the t space and m_2 is the number of collocation points in the x space. Equation (13) can be written in the following matrix form:

$$u(x,t)\approx\Phi^{\mathrm{T}}(x)\boldsymbol{K}\Phi(t)\,, \tag{14}$$

where $\boldsymbol{K}=[k_{i,j}]_{i,j=0}^{m_1,m_2}$ is an $(m_1+1)\times(m_2+1)$ matrix, and $\Phi(x)=[FT_0^{\alpha}(x),FT_1^{\alpha}(x),\ldots,FT_{m_1}^{\alpha}(x)]^{\mathrm{T}}$, $\Phi(t)=[FT_0^{\alpha}(t),FT_1^{\alpha}(t),\ldots,FT_{m_2}^{\alpha}(t)]^{\mathrm{T}}$ are basis vectors.

We apply the B-GFCFs collocation method to solve the linear partial differential equations at each iteration Eq. (12) with their initial and boundary conditions.

We assume that $u(x,0)=f(x)$ is an initial condition for Eq. (9). For satisfying the initial condition at each iteration, we define the approximate solution as follows

$$\bar{u}_{r+1}(x,t)=f(x)+t\;\hat{u}_{r+1}(x,t)\,, \tag{15}$$

where $\hat{u}_{r+1}(x,t)$ is defined in Eq. (13).

Now, to apply the collocation method, the residual function for Eq. (12) at each iteration is constructed by substituting $\bar{u}_{r+1}$ for u_{r+1}:

$$\mathrm{Res}_r(x,t)=L[\bar{u}_{r+1},\bar{u}'_{r+1},\ldots,\bar{u}_{r+1}^{(n)}] +\sum_{k=0}^{n}\frac{\partial N}{\partial u^{(k)}}\bar{u}_{r+1}^{(k)}-\Gamma(\dot{\bar{u}}_{r+1},\ldots,\dot{\bar{u}}_{r+1}^{(m)}) - H_r[\bar{u}_r,\bar{u}'_r,\ldots,\bar{u}_r^{(n)}]\,. \tag{16}$$

Now, by choice (m_1+1) arbitrary points $\{x_i\}$, $i=1,\ldots,m_1+1$ in the interval $[0,\eta]$, and (m_2+1) arbitrary points $\{t_j\}$, $j=1,\ldots,m_2+1$ in the interval $[0,T]$ as collocation points and substituting them in $\mathrm{Res}_r(x,t)$, and the use of their initial and boundary conditions, a set of $(m_1+1)(m_2+1)$ linear algebraic equations is generated as follows (Collocation method)

$$\mathrm{Res}_r(x_i,t_j)=0,\quad i=1,\ldots,m_1+1,\quad j=1,\ldots,m_2+1\,.$$

By solving this system using a suitable method such as Newton's method, the approximate solution of Eq. (9) according to Eqs. (13) and (15) is obtained.

In this study, the roots of the GFCFs in the intervals of $[0,T]$ and $[0,\eta]$ (Theorem 2) have been used as collocation points in the t and x spaces, respectively. Also consider that all of the computations have been done by *Maple* 2015.

3.3 *Solving Nonlinear Hunter–Saxton Equation*

We consider nonlinear Hunter–Saxton equation:

$$u_{xt}=-uu_{xx}-\frac{1}{2}u_x^2\,, \tag{17}$$

with the initial and boundary conditions

$$\begin{aligned}&u(x,0)=f(x),\quad \frac{\partial u}{\partial t}(x,0)=g(x)\,,\\ &u(0,t)=\phi(t),\quad \frac{\partial u}{\partial x}(0,t)=\theta(t)\,,\end{aligned} \tag{18}$$

where the functions $f(x)$, $g(x)$, $\phi(t)$, and $\theta(t)$ are sufficiently smooth.

By applying the technique described in the previous section, we have

$$\begin{aligned}&\Gamma[\dot{u},\dot{u}',\ddot{u},\ldots,\dot{u}^{(m)}]=\dot{u}',\quad L[u,u',u'',\ldots,u^{(n)}]=0\,,\\ &N[u,u',u'',\ldots,u^{(n)}]=-uu''-\frac{1}{2}u'^2,\\ &\frac{\partial N}{\partial u}=-u'',\quad \frac{\partial N}{\partial u'}=-u',\quad \frac{\partial N}{\partial u''}=-u\,,\\ &H[u,u',u'',\ldots,u^{(n)}]=uu''+\frac{1}{2}u'^2.\end{aligned}$$

For satisfying the initial conditions at each iteration, the approximate solution $\bar{u}_r(x,t)$ is defined as follows

$$\begin{aligned}&\bar{u}_{r+1}(x,t)=f(x)+tg(x)+t^2\hat{u}_{r+1}(x,t)\,,\\ &r=0,1,2,3,\ldots\,,\end{aligned} \tag{19}$$

and according to Eq. (16), we have

$$\begin{aligned}\mathrm{Res}_r(x,t)=&\;\bar{u}_r\bar{u}''_{r+1}+\bar{u}'_r\bar{u}'_{r+1}+\bar{u}''_r\bar{u}_{r+1}\\ &-\Big(\bar{u}_r\bar{u}''_r+\frac{1}{2}(\bar{u}'_r)^2\Big)+\dot{\bar{u}}'_{r+1}\,.\end{aligned} \tag{20}$$

A set of $(m_1+1)(m_2+1)$ linear algebraic equations is generated as follows:

$$\mathrm{Res}_r(x_i,t_j)=0,\quad i=1,\ldots,m_1+1,\quad j=1,\ldots,m_2+1. \tag{21}$$

By using the initial and boundary conditions (18), for satisfying the initial conditions at the first, it is assumed that the initial guess $u_0(x,t)=f(x)+tg(x)$, and the boundary conditions are implemented in the first and last rows (i.e. $i=1$ and $i=m_1+1$) in Eqs. (21). By solving the linear algebraic equations, the approximate solution of Eq. (17) according to Eqs. (13) and (19) is obtained.

Now, we must try to select an appropriate value for the parameter of α. To achieve this goal, we can use the maximum of the absolute error or the residual error. That is, we will solve the problem for various values of α, and then based on the maximum of the absolute error or the residual error, an appropriate value for α is selected.

We define the maximum of the absolute error and the maximum of the residual error as follows

$$\begin{aligned}E_1=\max\{&|u(x_i,t_j)-\bar{u}(x_i,t_j)|:\\ &i=1,\ldots,m_1+1,\ j=1,\ldots,m_2+1\}\,,\end{aligned} \tag{22}$$

or

$$\begin{aligned}E_2=\max\{&|Res_r(x_i,t_j)|:\\ &i=1,\ldots,m_1+1,\quad j=1,\ldots,m_2+1\}\,,\end{aligned} \tag{23}$$

where $\bar{u}(x,t)$ is the approximate solution and $u(x,t)$ is the exact solution.

4 The Numerical Examples

In this section, by using the present method, some examples of the Hunter–Saxton equation are solved. To show the efficiency and capability of the present method, the obtained results with the corresponding analytical or numerical solutions are compared.

4.1 *Example 1*

In Eq. (18), it is assumed that:[15]

$$\begin{aligned}&u(x,0)=2x,\quad u_t(x,0)=-2x,\quad u(0,t)=0\,,\\ &u_x(0,t)=\frac{2}{1+t}\,.\end{aligned}$$

Baxter *et al.*[14] have proved that the exact solution is as follows:

$$u(x,t)=\frac{2x}{1+t}\,. \tag{24}$$

By applying the technique described in the previous section, we have

$$\begin{aligned}&\bar{u}_{r+1}(x,t)=2x-2xt+xt^2\hat{u}_{r+1}(x,t)\,,\\ &r=0,1,2,3,\ldots\,,\end{aligned} \tag{25}$$

and the initial guess $u_0(x,t)=2x-2xt+xt^2$. It can be seen that they are satisfied in the initial conditions and one of the boundary conditions.

Figure 1 shows the graph of the maximum of the absolute errors for various values of α. We can see that an appropriate value for the parameter of α is 1.

Tables 1–3 show the obtained results of the present method for various values of x, 225 nodes ($m_1=m_2=14$), 5th iterations, and $t=0.1$, $t=0.01$ and $t=0.001$ respectively, and comparing them with the obtained results by Arbabi *et al.*[15] using the Haar wavelet quasilinearization approach (HWQA) and the exact solution. It is seen that the obtained results of the present method are more accurate than the previous results.

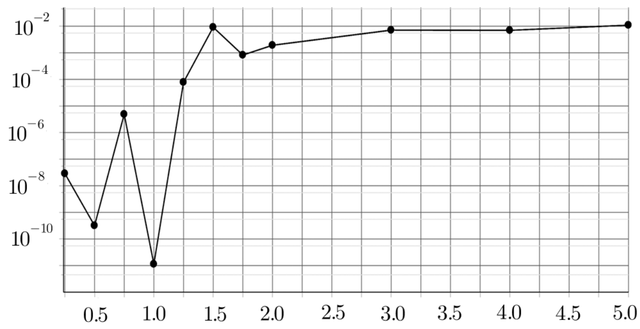

Fig. 1 Graph of the maximum of the absolute errors for various values of α, for example 1.

Table 1 Comparing the present method with the obtained results by Ref. [15], for example 1 with $t = 0.1$.

$x/128$	u_{Exact}	HWQA (with 256 nodes)		Present Method (with 225 nodes)	
		u_{Haar}	Abs. Err.	u_{Present}	Abs. Err.
1	0.014 204 545 454 545	0.014 209 157 32	4.61×10^{-6}	0.014 204 545 454 546	7.75×10^{-16}
3	0.042 613 636 363 636	0.042 613 636 36	3.43×10^{-5}	0.042 613 636 363 633	2.93×10^{-15}
5	0.071 022 727 272 727	0.071 076 718 70	5.40×10^{-5}	0.071 022 727 272 728	1.45×10^{-15}
7	0.099 431 818 181 818	0.099 507 044 98	7.52×10^{-5}	0.099 431 818 181 829	1.10×10^{-14}
9	0.127 840 909 090 909	0.127 937 743 9	9.68×10^{-5}	0.127 840 909 090 927	1.82×10^{-14}
59	0.838 068 181 818 181	0.838 703 081 9	6.35×10^{-4}	0.838 068 181 8182 73	9.20×10^{-14}
61	0.866 477 272 727 272	0.867 133 694 9	6.56×10^{-4}	0.866 477 272 727 360	8.73×10^{-14}
63	0.894 886 363 636 363	0.895 564 307 8	6.78×10^{-4}	0.894 886 363 636 442	7.89×10^{-14}
65	0.923 295 454 545 454	0.923 994 920 8	6.99×10^{-4}	0.923 295 454 545 523	6.86×10^{-14}
67	0.951 704 545 454 545	0.952 425 533 8	7.21×10^{-4}	0.951 704 545 454 604	5.86×10^{-14}
69	0.980 113 636 363 636	0.980 856 146 7	7.42×10^{-4}	0.980 113 636 363 687	5.09×10^{-14}
119	1.690 340 909 090 909	1.691 621 470	1.28×10^{-3}	1.690 340 909 091 035	1.26×10^{-13}
121	1.718 750 000 000 000	1.720 052 083	1.30×10^{-3}	1.718 750 000 000 120	1.20×10^{-13}
123	1.747 159 090 909 090	1.748 482 696	1.32×10^{-3}	1.747 159 090 909 222	1.31×10^{-13}
125	1.775 568 181 818 181	1.776 913 309	1.34×10^{-3}	1.775 568 181 818 334	1.52×10^{-13}
127	1.803 977 272 727 272	1.805 343 922	1.37×10^{-3}	1.803 977 272 727 423	1.51×10^{-13}

Table 2 Comparing the present method with the obtained results by Ref. [15], for example 1 with $t = 0.01$.

$x/128$	u_{Exact}	HWQA (with 256 nodes)		Present Method (with 225 nodes)	
		u_{Haar}	Abs. Err.	u_{Present}	Abs. Err.
1	0.015 470 297 029 702 97	0.01547030439	7.36×10^{-9}	0.01547029702970303	$6.76e-17$
3	0.046 410 891 089 108 91	0.046 410 943 18	5.21×10^{-8}	0.046 410 891 089 109 30	3.93×10^{-16}
5	0.077 351 485 148 514 85	0.077 351 554 27	6.91×10^{-8}	0.077 351 485 148 515 17	3.19×10^{-16}
7	0.108 292 079 207 920 79	0.108 292 189 0	1.10×10^{-7}	0.108 292 079 207 920 87	8.30×10^{-17}
9	0.139 232 673 267 326 73	0.139 232 805 1	1.32×10^{-7}	0.139 232 673 267 326 77	3.83×10^{-17}
59	0.912 747 524 752 475 24	0.912 748 412 7	8.88×10^{-7}	0.912 747 524 752 477 75	2.50×10^{-15}
61	0.943 688 118 811 881 18	0.943 689 034 8	9.16×10^{-7}	0.943 688 118 811 884 35	3.17×10^{-15}
63	0.974 628 712 871 287 12	0.974 629 659 0	9.46×10^{-7}	0.974 628 712 871 290 87	3.74×10^{-15}
65	1.005 569 306 930 693 06	1.005 570 283	9.76×10^{-7}	1.005 569 306 930 697 18	4.11×10^{-15}
67	1.036 509 900 990 099 00	1.036 510 907	1.01×10^{-6}	1.036 509 900 990 103 22	4.21×10^{-15}
69	1.067 450 495 049 504 95	1.067 451 531	1.04×10^{-6}	1.067 450 495 049 508 98	4.03×10^{-15}
119	1.840 965 346 534 653 46	1.840 967 134	1.79×10^{-6}	1.840 965 346 534 658 17	4.71×10^{-15}
121	1.871 905 940 594 059 40	1.871 907 758	1.82×10^{-6}	1.871 905 940 594 062 70	3.29×10^{-15}
123	1.902 846 534 653 465 34	1.902 848 382	1.85×10^{-6}	1.902 846 534 653 468 91	3.56×10^{-15}
125	1.933 787 128 712 871 28	1.933 789 006	1.88×10^{-6}	1.933 787 128 712 877 37	6.09×10^{-15}
127	1.964 727 722 772 277 22	1.964 729 630	1.91×10^{-6}	1.964 727 722 772 283 72	6.49×10^{-15}

Table 3 Comparing the present method with the obtained results by Ref. [15], for example 1 with $t = 0.001$.

$x/128$	u_{Exact}	HWQA (with 256 nodes)		Present Method (with 225 nodes)	
		u_{Haar}	Abs. Err.	u_{Present}	Abs. Err.
1	0.015 609 390 609 390 609	0.015 609 390 62	1.00×10^{-11}	0.015 609 390 609 390 608	7.19×10^{-19}
3	0.046 828 171 828 171 828	0.046 828 171 88	5.00×10^{-11}	0.046 828 171 828 171 824	4.10×10^{-18}
5	0.078 046 953 046 953 046	0.078 046 953 12	7.00×10^{-11}	0.078 046 953 046 953 043	3.38×10^{-18}
7	0.109 265 734 265 734 265	0.109 265 734 4	1.00×10^{-10}	0.109 265 734 265 734 264	9.97×10^{-19}
9	0.140 484 515 484 515 484	0.140 484 515 6	1.00×10^{-10}	0.140 484 515 484 515 483	6.05×10^{-19}
59	0.920 954 045 954 045 954	0.920 954 044 9	1.10×10^{-9}	0.920 954 045 954 045 926	2.74×10^{-17}
61	0.952 172 827 172 827 172	0.952 172 827 4	2.00×10^{-10}	0.952 172 827 172 827 138	3.42×10^{-17}
63	0.983 391 608 391 608 391	0.983 391 608 8	4.00×10^{-10}	0.983 391 608 391 608 351	4.01×10^{-17}
65	1.014 610 389 610 389 610	1.014 610 391	1.00×10^{-9}	1.014 610 389 610 389 566	4.38×10^{-17}
67	1.045 829 170 829 170 829	1.045 829 172	1.00×10^{-9}	1.045 829 170 829 170 784	4.48×10^{-17}
69	1.077 047 952 047 952 048	1.077 047 954	2.00×10^{-9}	1.077 047 952 047 952 005	4.28×10^{-17}
119	1.857 517 482 517 482 517	1.857 517 487	4.00×10^{-9}	1.857 517 482 517 482 467	4.99×10^{-17}
121	1.888 736 263 736 263 736	1.888 736 268	4.00×10^{-9}	1.888 736 263 736 263 700	3.55×10^{-17}
123	1.919 955 044 955 044 955	1.919 955 049	4.00×10^{-9}	1.919 955 044 955 044 915	3.97×10^{-17}
125	1.951 173 826 173 826 173	1.951 173 830	4.00×10^{-9}	1.951 173 826 173 826 106	6.72×10^{-17}
127	1.982 392 607 392 607 392	1.982 392 611	4.00×10^{-9}	1.982 392 607 392 607 324	6.82×10^{-17}

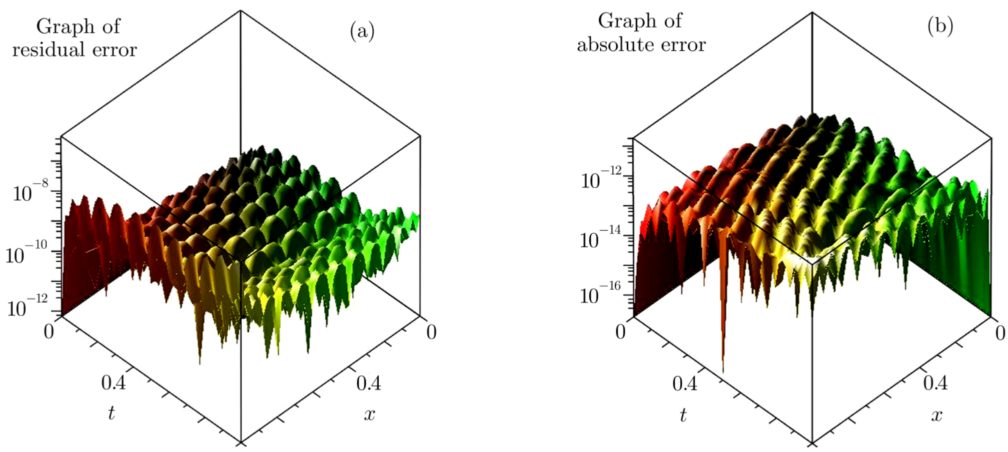

Fig. 2 Graphs of the residual error and the absolute error, for example 1.

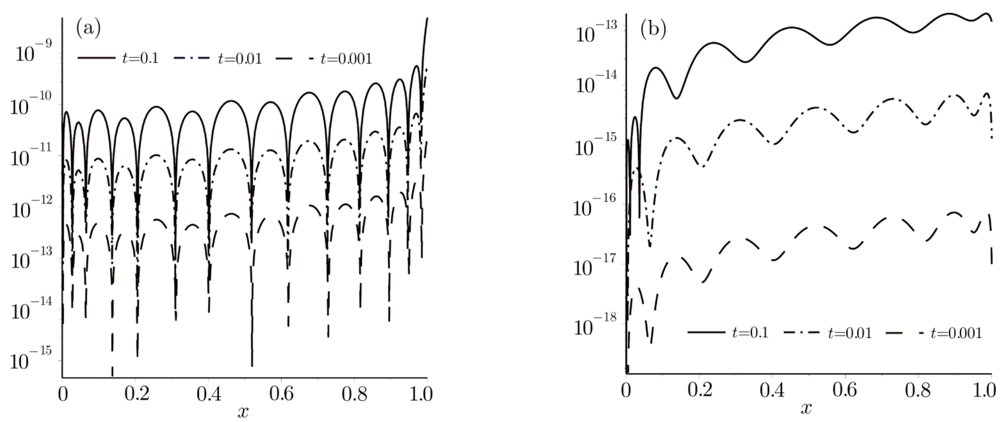

Fig. 3 Graphs of the residual errors and the absolute errors, for example 1 with $t = 0.1$, $t = 0.01$ and $t = 0.001$.

Figure 2 shows the graphs of residual error $\mathrm{Res}_5(x,t)$ of Eq. (20), and the absolute error between the present method and the exact solution (24).

Figure 3 shows the graphs of residual errors $\mathrm{Res}_5(x,t)$ of Eq. (20), and the absolute errors between the present method and the exact solution (24) for $t = 0.1$, $t = 0.01$, and $t = 0.001$.

4.2 *Example 2*

In Eq. (18), it is assumed that:[15]

$$u(x,0) = (2+3x)^{2/3} + 2x + 2\,,$$
$$u_t(x,0) = -4(x+1)((2+3x)^{-1/3} + 1)\,.$$

Baxter *et al.*[14] have proved that the exact solution is as follows:

$$u(x,t) = \frac{(2+3\nu)^{2/3} + 2\nu + 2}{(1+t)^2}, \text{ where } \nu = x(1+t)\,. \tag{26}$$

By applying the technique described in the previous section, we have

$$\begin{aligned}\bar{u}_{r+1}(x,t) = {} & ((2+3x)^{2/3} + 2x + 2) \\ & - 4(x+1)((2+3x)^{-1/3} + 1)t \\ & + t^2 \hat{u}_{r+1}(x,t), \quad r = 0,1,2,\ldots\,,\end{aligned} \tag{27}$$

and the initial guess

$$u_0(x,t) = ((2+3x)^{2/3}+2x+2) - 4(x+1)((2+3x)^{-1/3}+1)t\,.$$

It can be seen that they are satisfied in the initial conditions.

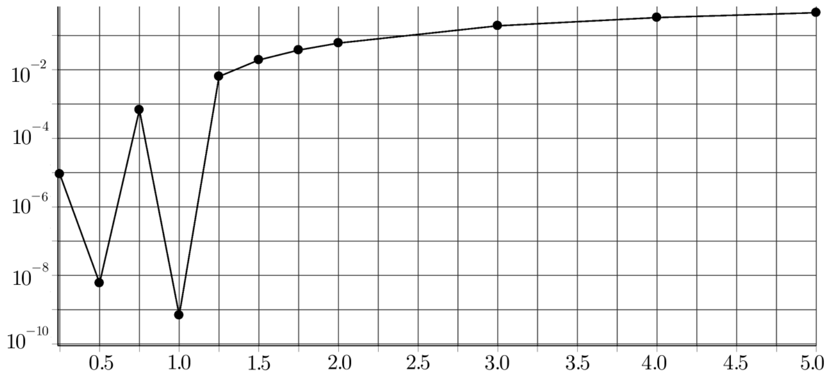

Fig. 4 Graph of the maximum of the absolute errors for various values of α, for example 2.

Table 4 Comparing the obtained results by the present method with the exact solution, for example 2 with $t = 0.1$.

$x/128$	u_{Present}	u_{Exact}	Abs. Err.
1	2.990 248 874 391 214 1	2.990 248 874 396 556 3	5.34×10^{-12}
3	3.041 016 027 824 974 4	3.041 016 027 833 826 7	8.85×10^{-12}
5	3.091 598 936 588 875 6	3.091 598 936 597 216 2	8.34×10^{-12}
7	3.142 003 527 812 484 7	3.142 003 527 821 432 1	8.94×10^{-12}
9	3.192 235 402 784 570 4	3.192 235 402 795 342 7	1.07×10^{-11}
59	4.403 746 615 430 377 1	4.403 746 615 429 786 0	5.91×10^{-13}
61	4.450 784 171 553 887 1	4.450 784 171 553 104 2	7.82×10^{-13}
63	4.497 732 320 607 848 7	4.497 732 320 606 883 6	9.65×10^{-13}
65	4.544 592 754 708 544 8	4.544 592 754 707 423 0	1.12×10^{-12}
67	4.591 367 110 711 451 0	4.591 367 110 710 220 2	1.23×10^{-12}
69	4.638 056 972 753 861 3	4.638 056 972 752 590 5	1.27×10^{-12}
119	5.781 627 663 012 095 6	5.781 627 663 009 410 8	2.68×10^{-12}
121	5.826 547 965 855 359 7	5.826 547 965 852 603 6	2.75×10^{-12}
123	5.871 412 928 437 502 7	5.871 412 928 434 514 9	2.98×10^{-12}
125	5.916 223 285 313 753 2	5.916 223 285 310 265 4	3.48×10^{-12}
127	5.960 979 754 144 010 5	5.960 979 754 140 316 6	3.69×10^{-12}

Figure 4 shows the graph of the maximum of the absolute errors for various values of α. We can see that an appropriate value for the parameter of α is 1.

Tables 4–6 show the obtained results by the present method for various values of x, 225 nodes ($m_1 = m_2 = 14$), 5th iterations, and $t = 0.1$, $t = 0.01$ and $t = 0.001$ respectively, and comparing them with the exact solution. It is seen that the obtained results by the present method are more accurate.

Figure 5 shows the graphs of residual error $\mathrm{Res}_5(x,t)$ of Eq. (20), and the absolute error between the present method and the exact solution (26).

Figure 6 shows the graphs of residual errors $\mathrm{Res}_5(x,t)$ of Eq. (20), and the absolute errors between the present method and the exact solution (26) for $t=0.1$, $t=0.01$ and $t=0.001$.

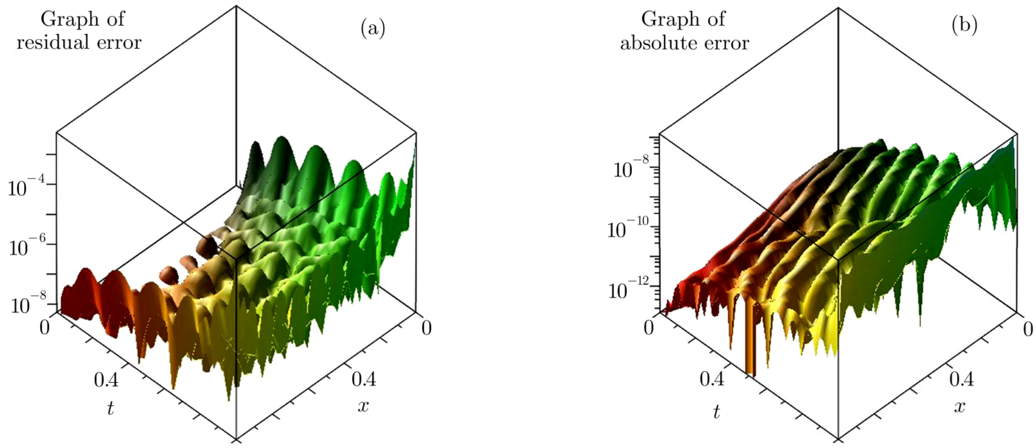

Fig. 5 Graphs of the residual error and the absolute error, for example 2.

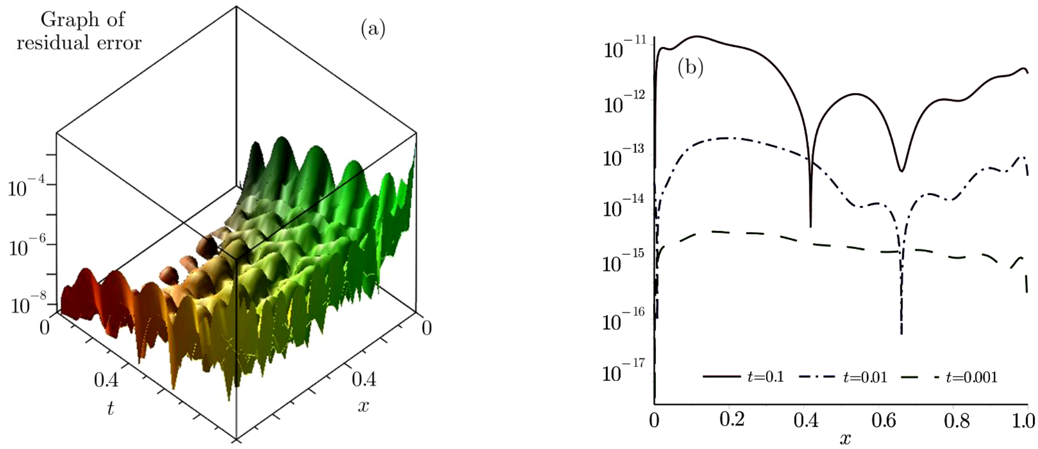

Fig. 6 Graphs of the residual errors and the absolute errors, for example 2 with $t=0.1$, $t=0.01$, and $t=0.001$.

Table 5 Comparing the obtained results by the present method with the exact solution, for example 2 with $t=0.01$.

$x/128$	u_{Present}	u_{Exact}	Abs. Err.
1	3.544 440 063 528 169 2	3.544 440 063 528 170 0	7.95×10^{-16}
3	3.599 747 692 922 393 1	3.599 747 692 922 413 9	2.08×10^{-14}
5	3.654 870 331 118 512 9	3.654 870 331 118 5478	3.49×10^{-14}
7	3.709 813 471 108 776 7	3.709 813 471 108 835 7	5.89×10^{-14}
9	3.764 582 326 651 650 5	3.764 582 326 651 739 4	8.89×10^{-14}
59	5.088 424 229 176 983 5	5.088 424 229 177 026 8	4.33×10^{-14}
61	5.139 900 464 055 950 8	5.139 900 464 055 983 7	3.28×10^{-14}
63	5.191 282 811 005 050 9	5.191 282 811 005 075 0	2.41×10^{-14}
65	5.242 572 963 839 631 3	5.242 572 963 839 648 9	1.76×10^{-14}
67	5.293 772 563 611 455 6	5.293 772 563 611 469 0	1.34×10^{-14}
69	5.344 883 200 925 859 7	5.344 883 200 925 871 1	1.14×10^{-14}
119	6.597 568 382 060 775 1	6.597 568 382 060 725 6	4.94×10^{-14}
121	6.646 799 617 613 847 5	6.646 799 617 613 796 7	5.07×10^{-14}
123	6.695 971 607 860 443 6	6.695 971 607 860 378 4	6.51×10^{-14}
125	6.745 085 113 030 094 6	6.745 085 113 030 005 6	8.89×10^{-14}
127	6.794 140 876 443 237 1	6.794 140 876 443 155 0	8.21×10^{-14}

Table 6 Comparing the obtained results by the present method with the exact solution, for example 2 with $t = 0.001$.

$x/128$	$u_{\rm Present}$	$u_{\rm Exact}$	Abs. Err.
1	3.608 211 473 488 318 8	3.608 211 473 488 317 8	1.02×10^{-15}
3	3.664 018 059 021 949 7	3.664 018 059 021 947 6	2.04×10^{-15}
5	3.719 639 578 393 518 6	3.719 639 578 393 516 3	2.29×10^{-15}
7	3.775 081 480 722 098 0	3.775 081 480 722 095 6	2.41×10^{-15}
9	3.830 348 940 397 089 8	3.830 348 940 397 087 2	2.64×10^{-15}
59	5.166 543 573 333 404 5	5.166 543 573 333 402 3	2.19×10^{-15}
61	5.218 508 251 180 510 0	5.218 508 251 180 507 8	2.17×10^{-15}
63	5.270 378 571 394 656 9	5.270 378 571 394 654 7	2.14×10^{-15}
65	5.322 156 227 560 871 1	5.322 156 227 560 869 0	2.09×10^{-15}
67	5.373 842 860 773 953 1	5.373 842 860 773 951 0	2.04×10^{-15}
69	5.425 440 061 932 150 9	5.425 440 061 932 148 9	1.97×10^{-15}
119	6.690 140 788 072 044 1	6.690 140 788 072 043 2	8.97×10^{-16}
121	6.739 846 942 800 212 9	6.739 846 942 800 212 1	7.95×10^{-16}
123	6.789 493 435 192 629 4	6.789 493 435 192 628 4	9.61×10^{-16}
125	6.839 081 028 041 753 5	6.839 081 028 041 752 2	1.28×10^{-15}
127	6.888 610 467 235 188 6	6.888 610 467 235 187 7	9.22×10^{-16}

5 Conclusion

The fundamental goal of the paper has been to construct an approximation to the solution of nonlinear Hunter–Saxton equation. To achieve this goal, a hybrid numerical method based on the quasilinearization method and the bivariate generalized fractional order of the Chebyshev functions (B-GFCF) collocation method is applied. The obtained results of the present method are more accurate than the results that calculated by other methods for fewer collocation points and are in a good agreement with the exact solutions. So it can be concluded that the present method is very convenient for solving other nonlinear partial differential equations.

Acknowledgments

The authors are very grateful to reviewers and editor for carefully reading the paper and for their comments and suggestions which have improved the paper.

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