

General Solutions for Hydromagnetic Free Convection Flow over an Infinite Plate with Newtonian Heating, Mass Diffusion and Chemical Reaction

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Abstract The problem of hydromagnetic free convection flow over a moving infinite vertical plate with Newtonian heating, mass diffusion and chemical reaction in the presence of a heat source is completely solved. Radiative and porous effects are not taken into consideration but they can be immediately included by a simple rescaling of Prandtl number and magnetic parameter. Exact general solutions for the dimensionless velocity and concentration fields and the corresponding Sherwood number and skin friction coefficient are determined under integral form in terms of error function or complementary error function of Gauss. They satisfy all imposed initial and boundary conditions and can generate exact solutions for any problem with technical relevance of this type. As an interesting completion, uncommon in the literature, the differential equations which describe the thermal, concentration and momentum boundary layer, as well as the exact expressions for the thicknesses of thermal, concentration or velocity boundary layers were determined. Numerical results have shown that the thermal boundary layer thickness decreases for increasing values of Prandtl number and the concentration boundary layer thickness is decreasing with Schmidt number. Finally, for illustration, three special cases are considered and the influence of physical parameters on some fundamental motions is graphically underlined and discussed. The required time to reach the flow according with post-transient solution (the steady-state), for cosine/sine oscillating concentrations on the boundary is graphically determined. It is found that, the presence of destructive chemical reaction improves this time for increasing values of chemical reaction parameter.

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1 Introduction

Natural or free convection flows are abundantly met in nature. They are particularly important in oceanic and atmospheric circulation, filtration processes, cooling of nuclear reactors, solar energy collectors and arise in fluids when the temperature changes imply variations of the density leading to buoyancy forces which affect their motion. Details about the applications of free convection flows can be found in the books of Ghoshdastidar^[1] and Nield and Bejan^[2] but one of the oldest and interesting studies regarding the free convection from a heated vertical plate is that of Turnbull^[3] in presence of an electric field. Such flows, which are also affected by the differences in concentration, have been extensively studied due to their multiple applications in engineering and environmental processes. The study of free convection flow in the presence of magnetic field is also important in polymer industry, metallurgy, astrophysics and geophysics and the first authors who took into consideration the effects of magnetic field in their work seem to be Soundalgekar *et al.*^[4]

Hydromagnetic flows combined with heat and mass transfer by free convection have been studied by many

authors due to their diverse applications in science and technology. The mass transfer, that means the transport of a constituent between two regions having different concentrations, is the basis of many biological and chemical processes.^[5] It also appears in the theory of stellar and solar structures. On the other hand, in the last time, hydromagnetic free convection flows involving heat and mass transfer with chemical reaction received a special attention (see for instance the recent works of Reddy *et al.*^[6–7] Rao *et al.*,^[8] Srihari and Chirra Kesava Reddy,^[9] Pattnaik and Biswal,^[10] Seth *et al.*^[11] and therein references). They are important in different areas of sciences and engineering and usually occur in magnetohydrodynamic power generation systems, cooling of nuclear reactors, power and cooling systems as well as in petro-chemical industry.

However, the heat transfer characteristics are strongly dependent on the thermal boundary conditions, and in all above-mentioned papers the free convection flows are driven by a prescribed surface temperature or prescribed surface heat flux. Merkin^[12] was the first author who assumed that the flow is set up by Newtonian heating from the surface. In such flows, which are also called conjugate convective flows and have important applications in many

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engineering devices,^[13] the rate of heat transfer from the plate surface is proportional to the local surface temperature. Effects of Newtonian heating on the free convection flow of a viscous fluid along an infinite vertical or horizontal plate embedded in a porous medium have been studied by Lesnic *et al.*^[14–15] and Pop *et al.*^[16] Other interesting solutions, in the absence of mass transfer, have been also established by Chaudhary and Jain,^[17] Mebine and Adigio,^[18] Narahari and Ishak,^[19] Das *et al.*^[20] and Hussanan *et al.*^[21] The effects of mass transfer on such flows have been studied by Narahari and Nayan,^[22] Narahari *et al.*,^[23] Narahari and Dutta^[24] and Hussanan *et al.*^[25–26] However, none of these works took into consideration, heat source or chemical reaction. Free convection flows with Newtonian heating and mass diffusion in which the plate applies a shear stress to the fluid or slip effects are taken into consideration have been studied by Vieru *et al.*,^[27] Khan *et al.*,^[28] and Fetecau *et al.*^[29] An interesting mathematical study of the free convection with dissipative heating has been developed by Sheremet *et al.*^[30]

The main purpose of this work is to provide a general study of hydromagnetic free convection flow of an incompressible viscous fluid over a moving infinite vertical plate with Newtonian heating, heat source and chemical reaction. Radiative and porous effects are not taken into consideration but, according to Magyari and Pantokratoras^[31] and Fetecau *et al.*,^[32] they can be immediately included by a simple rescaling of Prandtl number and magnetic parameter. Exact analytical solutions are established for the dimensionless velocity and concentration fields and the corresponding Sherwood number and skin friction coefficient. They can generate exact solutions for any flow of this type and, for illustration, three special cases are considered and some known results from the literature are recovered or corrected. The influence of physical parameters on some flows with technical relevance is graphically underlined and discussed. Contributions of mechanical, thermal and concentration components of velocity on the fluid motion are together brought to light for motions due to a highly accelerating plate. The required time to reach the steady-state for cosine or sine oscillations of the concentration on the boundary is also determined.

2 Mathematical Formulation of the Problem

Let us consider the hydromagnetic free convection flow of an electrically conducting, incompressible viscous fluid with Newtonian heating and mass diffusion over a moving infinite non conducting vertical flat plate (Fig. 1). At the initial moment $t = 0$, both the fluid and the plate are at rest with the same temperature T_∞ and the species concentration C_∞ . After time $t = 0^+$ the plate, whose concentration is raised or lowered to the value $C_\infty + C_w g(t)$, is moving in its plane against the gravitational field with an arbitrary velocity $Uf(t)$. Here C_w and U are constants

while the dimensionless functions $f(\cdot)$ and $g(\cdot)$ are piecewise continuous and $f(0) = g(0) = 0$. A transverse magnetic field of uniform strength B , whose magnetic lines of force are fixed relative to the fluid, acts perpendicular to the plate and the magnetic Reynolds number is assumed to be small enough so that the induced magnetic field can be neglected.

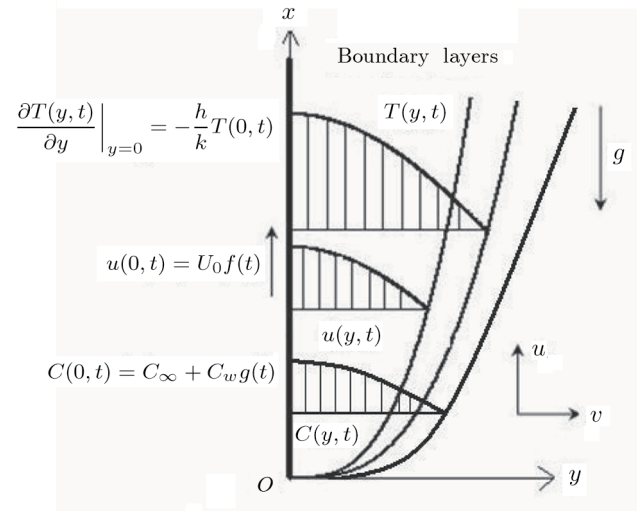


Fig. 1 Schematic diagram of the flow configuration.

Radiative and porous effects are not taken into consideration but, as we already mentioned, they can be immediately included by a simple rescaling of Prandtl number, respectively the magnetic parameter. Our results are obtained in the presence of heat source and chemical reaction, but the viscous dissipation is neglected due to its small size. This assumption can be justified by small velocities usually encountered in free convection flows.^[33] In these conditions, choosing a suitable Cartesian coordinate system and using the usual Boussinesqs approximation, our flow is governed by the following partial differential equations^[10] (the inertia terms are also neglected)

$$\frac{\partial u(y,t)}{\partial t} = \nu \frac{\partial^2 u(y,t)}{\partial y^2} + g\beta_T [T(y,t) - T_\infty] + g\beta_C [C(y,t) - C_\infty] - \frac{\sigma B_0^2}{\rho} u(y,t), \quad y, t > 0, \quad (1)$$

$$\rho C_p \frac{\partial T(y,t)}{\partial t} = k \frac{\partial^2 T(y,t)}{\partial y^2} - Q [T(y,t) - T_\infty], \quad y, t > 0, \quad (2)$$

$$\frac{\partial C(y,t)}{\partial t} = D \frac{\partial^2 C(y,t)}{\partial y^2} - R [C(y,t) - C_\infty], \quad y, t > 0, \quad (3)$$

where u , T , and C are velocity, temperature and species concentration of the fluid, ν is the kinematic viscosity, g is the acceleration due to gravity, β_T is the thermal expansion coefficient, β_C is the volumetric coefficient of concentration expansion, σ is electrical conductivity, ρ is fluid density, c_p is the specific heat at constant pressure, k is the thermal conductivity, Q is the heat generation or

absorption coefficient, D is the chemical molecular diffusivity and R is chemical reaction parameter.

The corresponding initial and boundary conditions are:

$$u(y, 0) = 0, \quad T(y, 0) = T_\infty, \quad C(y, 0) = C_\infty, \quad y \geq 0, \quad (4)$$

$$u(0, t) = Uf(t), \quad \left. \frac{\partial T(y, t)}{\partial y} \right|_{y=0} = -\frac{h}{k}T(0, t),$$

$$C(0, t) = C_\infty + C_w g(t), \quad t > 0, \quad (5)$$

$$u(y, t) \rightarrow 0, \quad T(y, t) \rightarrow T_\infty, \quad C(y, t) \rightarrow C_\infty, \quad \text{as } y \rightarrow \infty, \quad (6)$$

where h is the heat transfer coefficient for Newtonian heating.

By introducing the dimensionless variables and functions

$$\begin{aligned} y^* &= \frac{h}{k}y, \quad t^* = \nu \left(\frac{h}{k}\right)^2 t, \quad u^* = \frac{u}{U}, \quad T^* = \frac{T - T_\infty}{T_\infty}, \\ C^* &= \frac{C - C_\infty}{C_w}, \quad R^* = \frac{R}{\nu} \left(\frac{k}{h}\right)^2, \quad Q^* = \frac{Q}{\mu c_p} \left(\frac{k}{h}\right)^2, \\ f^*(t^*) &= f\left[\frac{t^*}{\nu} \left(\frac{k}{h}\right)^2\right], \quad g^*(t^*) = g\left[\frac{t^*}{\nu} \left(\frac{k}{h}\right)^2\right], \end{aligned} \quad (7)$$

and dropping out the star notation, we attain to the following dimensionless initial boundary values problem:

$$\frac{\partial u(y, t)}{\partial t} = \frac{\partial^2 u(y, t)}{\partial y^2} + T(y, t) + NC(y, t) - Mu(y, t), \quad y, t > 0, \quad (8)$$

$$\frac{\partial T(y, t)}{\partial t} = \frac{1}{Pr} \frac{\partial^2 T(y, t)}{\partial y^2} - QT(y, t), \quad y, t > 0, \quad (9)$$

$$\frac{\partial C(y, t)}{\partial t} = \frac{1}{Sc} \frac{\partial^2 C(y, t)}{\partial y^2} - RC(y, t), \quad y, t > 0, \quad (10)$$

$$u(y, 0) = 0, \quad T(y, 0) = 0, \quad C(y, 0) = 0, \quad y \geq 0, \quad (11)$$

$$u(0, t) = f(t), \quad \left. \frac{\partial T(y, t)}{\partial y} \right|_{y=0} = -[1 + T(0, t)], \quad C(0, t) = g(t), \quad t \geq 0, \quad (12)$$

$$u(y, t) \rightarrow 0, \quad T(y, t) \rightarrow 0, \quad C(y, t) \rightarrow 0, \quad \text{as } y \rightarrow \infty, \quad (13)$$

where

$$\begin{aligned} Gr &= \frac{g\beta_T}{\nu U} \left(\frac{k}{h}\right)^2 T_\infty, \quad Gm = \frac{g\beta_C}{\nu U} \left(\frac{k}{h}\right)^2 C_w, \quad N = \frac{Gm}{Gr}, \\ M &= \frac{\sigma B^2}{\mu} \left(\frac{k}{h}\right)^2, \quad Pr = \frac{\mu c_p}{k}, \quad Sc = \frac{\nu}{D} \end{aligned}$$

are the thermal Grashof number, the mass Grashof number, the buoyancy ratio parameter, the magnetic parameter, the Prandtl number and the Schmidt number, respectively. Of course, the characteristic velocity U has been taken to be equal with $(g\beta_T/\nu U)(k/h)^2 T_\infty$.

It is worth pointing out that Pr and Sc are transport parameters representing the thermal diffusivity, respectively the mass diffusivity while N gives the relative contribution of the mass transport rate on the flow into consideration.^[24] As β_C can be positive or negative^[34] and β_T is a positive quantity, N can be also positive or negative. If N is positive, the mass and thermal buoyancy

forces act in the same direction. In the contrary case, the two forces are opposite. Of course, $N = 0$ in the absence of mass diffusion.

3 Solution of the Problem

The temperature field $T(y, t)$ corresponding to this problem has been already determined by Vieru *et al.*^[27] in a problem with shear stress on the boundary. Our interest here, is to determine the velocity and concentration fields as well as the corresponding Sherwood number and the skin friction coefficient when $T(y, t)$ is known. For completion, the thermal boundary layer thickness will be also determined. To do that, the Laplace transform technique will be used and the corresponding Laplace transform (Ref. [27], Eq. (19))

$$\bar{T}(y, q) = \frac{1}{q[\sqrt{Pr(q+Q)} - 1]} \exp(-y\sqrt{Pr(q+Q)}), \quad (14)$$

of $T(y, t)$. In the above equation, the Laplace transform is defined by $\bar{T}(y, q) = \int_0^\infty T(y, t) e^{-qt} dt$, q is the transform parameter.

In order to determine the differential equation describing the thickness of the thermal boundary layer,^[35] we integrate Eq. (9) with respect to y from zero to infinity, respectively to δ_{1T} , where δ_{1T} is the thermal layer thickness, introduce the measure of thermal layer

$$\delta_T(t) = \int_0^{\delta_{1T}} T(y, t) dy, \quad (15)$$

and use the boundary conditions (12)₂ and (13)₂. The obtained equation is

$$\frac{d\delta_T(t)}{dt} + Q\delta_T(t) = \frac{1}{Pr}[1 + T(0, t)]. \quad (16)$$

Applying the Laplace transform to Eq. (16) and using Eq. (14) as well as the fact that $\delta_T(0) = 0$, we find that

$$\begin{aligned} \bar{\delta}_T(q) &= \frac{1}{q\sqrt{Pr(q+Q)}} \frac{1}{\sqrt{Pr(q+Q)} - 1} \\ &= \frac{1}{\sqrt{Pr}q} \frac{1}{q} \left(\frac{1}{\sqrt{q+Q} - 1/\sqrt{Pr}} - \frac{1}{\sqrt{q+Q}} \right), \end{aligned} \quad (17)$$

where $\bar{\delta}_T(q)$ is the Laplace transform of $\delta_T(t)$. Direct computations show that the inverse Laplace transform of Eq. (17) is

$$\begin{aligned} \delta_T(t) &= \frac{1}{PrQ-1} \left\{ 1 + \frac{1}{\sqrt{PrQ}} \operatorname{erf}(\sqrt{Qt}) \right. \\ &\quad \left. + \left[\operatorname{erf}\left(\sqrt{\frac{t}{Pr}}\right) - 2 \right] \exp\left(\frac{1-PrQ}{Pr}t\right) \right\}. \end{aligned} \quad (18)$$

3.1 Species Concentration

Applying the Laplace transform to Eq. (10) and using the corresponding initial and boundary conditions, we find that

$$\frac{\partial^2 \bar{C}(y, q)}{\partial y^2} - Sc(q+R)\bar{C}(y, q) = 0, \quad y > 0, \quad (19)$$

where the Laplace transform $\bar{C}(y, q)$ of $C(y, t)$ has to satisfy the conditions

$$\bar{C}(0, q) = G(q), \quad \bar{C}(y, q) \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (20)$$

Here, $G(q)$ is the Laplace transform of $g(t)$ and q is the transform parameter.

The solution of the ordinary differential equation (19) subjected to the boundary conditions (20), is given as

$$\bar{C}(y, q) = G(q) \exp(-y\sqrt{Sc(q+R)}). \quad (21)$$

Applying the inverse Laplace transform to Eq. (21) and using Eq. (A1) from Appendix, the fact that $L^{-1}\{qG(q)\} = g'(t)$ if $g(0) = 0$ and the convolution theorem, we find that

$$C(y, t) = \int_0^t g'(t-s)\Phi(y\sqrt{Sc}, s; R)ds, \quad (22)$$

where the function Φ is defined in Appendix.

The rate of mass transfer from the plate to fluid, in terms of Sherwood number, is given by

$$Sh = -\frac{\partial C(y, t)}{\partial y} \Big|_{y=0}. \quad (23)$$

Introducing the equality (22) into Eq. (23), we find that

$$Sh = \sqrt{RSc} \int_0^t g'(t-s) \operatorname{erf}(\sqrt{Rs})ds + \frac{\sqrt{Sc}}{\sqrt{\pi}} \int_0^t \frac{g'(t-s)}{\sqrt{s}} e^{-Rs} ds. \quad (24)$$

Now, we integrate Eq. (10) across the concentration layer from zero to infinity, respectively to δ_{1c} , where δ_{1c} is the concentration layer thickness, introduce the measure of the concentration boundary layer

$$\delta_c(t) = \int_0^{\delta_{1c}} C(y, t) dy, \quad (25)$$

and use the boundary conditions (12)₃ and (13)₃. It results that

$$\frac{d\delta_c(t)}{dt} + R\delta_c(t) = -\frac{1}{Sc} \frac{\partial \bar{C}(y, q)}{\partial y} \Big|_{y=0}. \quad (26)$$

Applying the Laplace transform to Eq. (26) and using the initial condition $\delta_c(0) = 0$ and Eq. (21) to determine $(\partial \bar{C}(y, q)/\partial y)|_{y=0}$, we find that

$$\bar{\delta}_c(q) = \frac{1}{\sqrt{Sc}} G(q) \frac{1}{\sqrt{q+R}}, \quad (27)$$

whose inverse Laplace transform is

$$\delta_c(t) = \frac{1}{\sqrt{Sc\pi}} \int_0^t \frac{g(t-s)}{\sqrt{s}} e^{-Rs} ds. \quad (28)$$

3.2 Velocity Field

Applying the Laplace transform to Eq. (8) and bearing in mind the corresponding initial and boundary conditions, it results that

$$q\bar{u}(y, q) = \frac{\partial^2 \bar{u}(y, q)}{\partial y^2} + \bar{T}(y, q) + N\bar{C}(y, q) - M\bar{u}(y, q), \quad y > 0, \quad (29)$$

where $\bar{u}(y, q)$ and $F(q)$ are the Laplace transforms of $u(y, t)$ and $f(t)$, and

$$\bar{u}(0, q) = F(q), \quad \text{and} \quad \bar{u}(y, q) \rightarrow 0, \quad \text{as } y \rightarrow \infty. \quad (30)$$

Introducing Eqs. (14) and (21) into Eq. (29), it results that

$$\frac{\partial^2 \bar{u}(y, q)}{\partial y^2} - (q+M)\bar{u}(y, q) = -\frac{1}{q[\sqrt{Pr(q+Q)}-1]} \times e^{-y\sqrt{Pr(q+Q)}} - NG(q)e^{-y\sqrt{Sc(q+R)}}. \quad (31)$$

The solution of the ordinary differential equation (31) with the boundary conditions (30), is given by

$$\bar{u}(y, q) = F(q)e^{-y\sqrt{q+M}} + \frac{1}{1-Pr} \frac{1}{q[\sqrt{Pr(q+Q)}-1]} \times \left[\frac{e^{-y\sqrt{Pr(q+Q)}}}{q-\delta_1} - \frac{e^{-y\sqrt{q+M}}}{q-\delta_1} \right] + \frac{N}{1-Sc} G(q) \left[\frac{e^{-y\sqrt{Sc(q+R)}}}{q-\delta_2} - \frac{e^{-y\sqrt{q+M}}}{q-\delta_2} \right], \quad (32)$$

where

$$\delta_1 = \frac{M-PrQ}{Pr-1}, \quad \delta_2 = \frac{M-ScR}{Sc-1}, \quad Pr \neq 1, \quad Sc \neq 1.$$

Applying the inverse Laplace transform to Eq. (32) and using again the convolution theorem and Eq. (A2) from Appendix, we find the velocity field under the form

$$u(y, t) = u_m(y, t) + u_T(y, t) + u_c(y, t), \quad (33)$$

where

$$u_m(y, t) = \int_0^t f'(t-s)\Phi(y, s; M)ds, \quad (34)$$

$$u_T(y, t) = \frac{1}{1-Pr} \int_0^t h(t-s)[\Phi(y\sqrt{Pr}, s; Q+\delta_1) - \Phi(y, s; M+\delta_1)]e^{\delta_1 s} ds, \quad Pr \neq 1, \quad (35)$$

$$u_c(y, t) = \frac{N}{1-Sc} \int_0^t g(t-s)[\Phi(y\sqrt{Sc}, s; R+\delta_2) - \Phi(y, s; M+\delta_2)]e^{\delta_2 s} ds, \quad Sc \neq 1 \quad (36)$$

are its mechanical, thermal and concentration components and

$$h(t) = \frac{1}{1-PrQ} \left[\left(1 + \operatorname{erf} \sqrt{\frac{t}{Pr}} \right) \exp \left(\frac{1-PrQ}{Pr} t \right) - \sqrt{PrQ} \operatorname{erf} \sqrt{Qt} - 1 \right].$$

General solutions (22) and (33) give the dimensionless velocity and concentration fields corresponding to the studied problem. They satisfy imposed initial and boundary conditions and are valid for all values of the parameters M, Q, R and N . Consequently, the similar solutions corresponding to the same problem in the absence of magnetic effects, heat source, chemical reaction or mass transfer are immediately obtained making $M = 0$, $Q = 0$, $R = 0$, respectively $N = 0$ into these relations. However, the thermal and concentration components of velocity are not valid for $Pr = 1$, respectively $Sc = 1$. In order to determine their expressions in these cases, we substitute $Pr = Sc = 1$ in Eq. (31) and follow the same way as before. Instead of Eq. (32) we get for $\bar{u}(y, q)$ the suitable form

$$\begin{aligned} \bar{u}(y, q) = & F(q) e^{-y\sqrt{q+M}} + \frac{1}{(M-Q)(\sqrt{q+Q}-1)} \\ & \times \left[\frac{e^{-y\sqrt{q+Q}}}{q} - \frac{e^{-y\sqrt{q+M}}}{q} \right] \\ & + \frac{N}{M-R} q G(q) \left[\frac{e^{-y\sqrt{q+R}}}{q} - \frac{e^{-y\sqrt{q+M}}}{q} \right]. \end{aligned} \quad (37)$$

Applying the inverse Laplace transform to Eq. (37), $u_m(y, t)$ remain unchanged while the thermal and concentration components of velocity become (see Eqs. (A1) and (A3)₁)

$$\begin{aligned} u_T(y, t) = & \frac{1}{M-Q} \int_0^t [\Phi(y, t-s; Q) - \Phi(y, t-s; M)] \\ & \times \left[\frac{1}{\sqrt{\pi s}} + e^s \operatorname{erfc}(-\sqrt{s}) \right] e^{-Qs} ds, \end{aligned} \quad (38)$$

$$\begin{aligned} u_C(y, t) = & \frac{N}{M-R} \int_0^t g'(t-s) \\ & \times [\Phi(y, s; R) - \Phi(y, s; M)] ds. \end{aligned} \quad (39)$$

Another physical entity of interest is the skin friction coefficient at the plate^[7,10]

$$\tau = - \frac{\partial u(y, t)}{\partial y} \Big|_{y=0}. \quad (40)$$

Introducing Eq. (33) in Eq. (40), we find the skin friction coefficient

$$\tau = \tau_m + \tau_T + \tau_C, \quad (41)$$

where

$$\tau_m = \sqrt{M} \int_0^t f'(t-s) \operatorname{erf}(\sqrt{Ms}) ds$$

$$+ \frac{1}{\sqrt{\pi}} \int_0^t \frac{f'(t-s)}{\sqrt{s}} e^{-Ms} ds \quad (42)$$

$$\begin{aligned} \tau_T = & \frac{1}{1-Pr} \int_0^t h(t-s) [\sqrt{(Q+\delta_1)Pr} \operatorname{erf}(\sqrt{(Q+\delta_1)s}) \\ & - \sqrt{M+\delta_1} \operatorname{erf}(\sqrt{(M+\delta_1)s})] e^{\delta_1 s} ds \\ & + \frac{1}{(1-Pr)\sqrt{\pi}} \int_0^t \frac{h(t-s)}{\sqrt{s}} \\ & \times (\sqrt{Pr} e^{-Qs} - e^{-Ms}) ds, \end{aligned} \quad (43)$$

$$\begin{aligned} \tau_C = & \frac{N}{1-Sc} \int_0^t g(t-s) [\sqrt{(R+\delta_2)Sc} \operatorname{erf}(\sqrt{(R+\delta_2)s}) \\ & - \sqrt{M+\delta_2} \operatorname{erf}(\sqrt{(M+\delta_2)s})] e^{\delta_2 s} ds \\ & + \frac{N}{(1-Sc)\sqrt{\pi}} \int_0^t \frac{g(t-s)}{\sqrt{s}} \\ & \times (\sqrt{Sc} e^{-Rs} - e^{-Ms}) ds, \end{aligned} \quad (44)$$

are its mechanical, thermal and concentration components.

The differential equation describing the velocity boundary layer thickness, namely

$$\frac{d\delta_v(t)}{dt} + M\delta_v(t) = - \frac{\partial u(y, t)}{\partial y} \Big|_{y=0} + \delta_T(t) + N\delta_C(t) \quad (45)$$

is obtained integrating Eq. (8) across the velocity boundary layer and following the same line as before for temperature and concentration. The solution of this differential equation with the initial condition $\delta_v(0) = 0$ can be also obtained by means of the Laplace transform technique.

Indeed, applying the Laplace transform to Eq. (45) and using Eqs. (17) and (27) for $\bar{\delta}_T(q)$, respectively $\bar{\delta}_C(q)$ and Eq. (32) to determine the Laplace transform of the first term from the right part of Eq. (45), we find that

$$\bar{\delta}_V(q) = F(q) \frac{1}{\sqrt{q+M}} + NG(q) \bar{B}_1(q) + \bar{B}_2(q), \quad (46)$$

where

$$\begin{aligned} \bar{B}_1(q) = & \frac{1}{1-Sc} \left[\frac{1}{\sqrt{Sc}} \frac{1}{(q-\delta_2)\sqrt{q+R}} - \frac{1}{(q-\delta_2)\sqrt{q+M}} \right], \\ \bar{B}_2(q) = & \frac{1}{1-Pr} \frac{1}{q(\sqrt{Pr(q+Q)}-1)} \\ & \times \left[\frac{1}{\sqrt{Pr}} \frac{1}{(q-\delta_1)\sqrt{q+Q}} - \frac{1}{(q-\delta_1)\sqrt{q+M}} \right]. \end{aligned}$$

Applying the inverse Laplace transform to Eq. (46), and using Eqs. (A2) and (A3)₂, it results

$$\delta_v(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(t-s)}{\sqrt{s}} e^{-Ms} ds + \frac{N}{1-Sc} \int_0^t g(t-s) B_1(s) ds + \frac{1}{(1-Pr)(PrQ-1)} B_2(t), \quad (47)$$

where

$$B_1(t) = e^{\delta_2 t} \left[\frac{\operatorname{erf}(\sqrt{(\delta_2+R)t})}{\sqrt{Sc(\delta_2+R)}} - \frac{\operatorname{erf}(\sqrt{(\delta_2+M)t})}{\sqrt{\delta_2+M}} \right],$$

$$B_2(t) = \int_0^t e^{\delta_1(t-s)} \left[\frac{\operatorname{erf}(\sqrt{(\delta_1 + Q)(t-s)})}{\sqrt{Pr(\delta_1 + Q)}} - \frac{\operatorname{erf}(\sqrt{(\delta_1 + M)(t-s)})}{\sqrt{\delta_1 + M}} \right] \\ \times \left[1 + \sqrt{PrQ} \operatorname{erf}(\sqrt{Qs}) - \exp\left(\frac{(1-PrQ)}{Pr}s\right) \operatorname{erf}\left(-\sqrt{\frac{s}{Pr}}\right) \right] ds.$$

4 Special Cases with Engineering Applications

As we previously mentioned, the general expressions that have been here obtained for velocity, concentration, Sherwood number and the skin friction coefficient can generate exact solutions for any hydromagnetic free convection flow of this type. In order to validate their correctness, as well as to get some physical insight of certain fundamental flows with possible engineering applications, three special cases are considered and some results from the existing literature are recovered or corrected.

Case 1 Uniform Motion and Constant Concentration of the Plate

By substituting the functions $f(\cdot)$ and $g(\cdot)$ by $H(\cdot)$ (the Heaviside unit step function) in Eqs. (22), (34), and (36) and bearing in mind the fact that

$$H'(t) = \delta(t), \\ \int_0^t \delta(t-s)f(s)ds = \int_0^t \delta(s)f(t-s)ds = f(t), \quad (48)$$

where $\delta(\cdot)$ is the Dirac delta function, we find the dimensionless fluid concentration

$$C_0(y, t) = \Phi(y\sqrt{Sc}, t; R), \quad (49)$$

and the mechanical and concentration components of ve-

locity (see also Eq. (A4))

$$u_{m0}(y, t) = \Phi(y, t; M), \quad (50)$$

$$u_{c0}(y, t) = \frac{N}{(1-Sc)\delta_2} \\ \times \{ e^{\delta_2 t} [\Phi(y\sqrt{Sc}, t; R + \delta_2) - \Phi(y, t; M + \delta_2)] \\ - \frac{1}{2} [\Phi(y\sqrt{Sc}, t; R) - \Phi(y, t; M)] \},$$

$$Sc \neq 1, \quad (51)$$

corresponding to the hydromagnetic free convection flow over an infinite plate, which is maintained at a constant concentration and is moving in its plane with a constant velocity. The thermal component of velocity remain unchanged while the expressions of $C_0(y, t)$ and $u_{m0}(y, t)$ are identical to those obtained in (Ref. [6], Eqs. (12) and (16)).

The corresponding Sherwood number, namely

$$Sh_0 = \sqrt{ScR} \operatorname{erf}(\sqrt{Rt}) + \frac{\sqrt{Sc}}{\sqrt{\pi t}} e^{-Rt}, \quad (52)$$

is obtained taking $g(t) = H(t)$ in Eq. (24) and using Eq. (48). It corrects the result of Reddy *et al.* (Ref. [6], Eq. (24)), where the authors lost the factor \sqrt{Sc} in the second term. Mechanical and concentration components of the skin friction coefficient (see also Eqs. (48) and (A5))

$$\tau_{m0} = \sqrt{M} \operatorname{erf}(\sqrt{Mt}) + \frac{1}{\sqrt{\pi t}} e^{-Mt}, \quad (53)$$

$$\tau_{c0}(y, t) = \frac{N}{(1-Sc)\delta_2} \{ e^{\delta_2 t} [\sqrt{(R + \delta_2)Sc} \operatorname{erf}(\sqrt{(R + \delta_2)t}) - \sqrt{M + \delta_2} \operatorname{erf}(\sqrt{(M + \delta_2)t})] \\ - [\sqrt{ScR} \operatorname{erf}(\sqrt{Rt}) - \sqrt{M} \operatorname{erf}(\sqrt{Mt})] \}, \quad Sc \neq 1, \quad \delta_2 \neq 0 \quad (54)$$

are obtained substituting $f(t)$ by $H(t)$ in Eqs. (42) and (44). As expected, Eq. (53) is identical to the first term of Eq. (19) from Ref. [6]. In the absence of magnetic effects and chemical reaction, Eqs. (49), (50), (52), and (53) take the simple forms

$$C_0(y, t) = \operatorname{erfc}\left(\frac{y\sqrt{Sc}}{2\sqrt{t}}\right), \quad u_{m0}(y, t) = \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right), \quad Sh_0 = \frac{\sqrt{Sc}}{\sqrt{\pi t}}, \quad \tau_{m0} = \frac{1}{\sqrt{\pi t}}, \quad (55)$$

which are well known in the literature.

By now substituting $f(t)$ and $g(t)$ by $H(t)$ in Eqs. (28) and (47), the expressions of the thickness of the corresponding boundary layers are obtained. The concentration boundary layer thickness, for instance, has the simple form

$$\delta_{C0}(t) = \frac{1}{\sqrt{Sc\pi}} \int_0^t \frac{e^{-Rs}}{\sqrt{s}} ds = \frac{1}{\sqrt{ScR}} \operatorname{erf}(\sqrt{Rt}), \quad (56)$$

which tends to the asymptotic value

$$\delta_{C0}(t) = \frac{1}{\sqrt{ScR}}, \quad (57)$$

when $t \rightarrow \infty$.

Case 2 Accelerated Plate with Ramp-Type Concentration

By now letting $f(t) = g(t)H(t)t^a$ ($a > 0$) into Eqs. (22), (34), and (36), we find solutions

$$C_a(y, t) = a \int_0^t (t-s)^{a-1} \Phi(y\sqrt{Sc}, s; R) ds, \quad (58)$$

$$u_{ma}(y, t) = a \int_0^t (t-s)^{a-1} \Phi(y, s; M) ds, \quad (59)$$

$$u_{Ca}(y, t) = \frac{N}{1 - Sc} \int_0^t (t - s)^a [\Phi(y\sqrt{Sc}, s; R + \delta_2) - \Phi(y, s; M + \delta_2)] e^{\delta_2 s} ds, \quad Sc \neq 1, \quad (60)$$

corresponding to the hydromagnetic free convection flow due to a slowly ($a < 1$), constantly ($a = 1$) or highly ($a >$

1) accelerating plate with ramp-type concentration.^[36] Of course, the corresponding velocity field is

$$u_a(y, t) = u_{ma}(y, t) + u_T(y, t) + u_{Ca}(y, t), \quad (61)$$

where $u_T(y, t)$ is given by Eq. (35).

Making the same substitutions in Eqs. (24), (42), and (44), we find that

$$Sh_a = a\sqrt{RSc} \int_0^t (t - s)^{a-1} \operatorname{erf}(\sqrt{Rs}) ds + \frac{a\sqrt{Sc}}{\sqrt{\pi}} \int_0^t \frac{(t - s)^{a-1}}{\sqrt{s}} e^{-Rs} ds, \quad (62)$$

$$\tau_{ma} = a\sqrt{M} \int_0^t (t - s)^{a-1} \operatorname{erf}(\sqrt{Ms}) ds + \frac{a}{\sqrt{\pi}} \int_0^t \frac{(t - s)^{a-1}}{\sqrt{s}} e^{-Ms} ds, \quad (63)$$

$$\begin{aligned} \tau_{Ca} = & \frac{N}{1 - Sc} \int_0^t (t - s)^a [\sqrt{Sc(R + \delta_2)} \operatorname{erf}(\sqrt{(R + \delta_2)s}) - \sqrt{M + \delta_2} \operatorname{erf}(\sqrt{(M + \delta_2)s})] e^{\delta_2 s} ds \\ & + \frac{N}{(1 - Sc)\sqrt{\pi}} \int_0^t \frac{(t - s)^a}{\sqrt{s}} (\sqrt{Sc} e^{-Rs} - e^{-Ms}) ds, \quad Sc \neq 1. \end{aligned} \quad (64)$$

Of a special interest is the case $a = 1$ corresponding to the free convection flow due to a constantly accelerating plate. By making $a = 1$ in Eqs. (58)–(60), and (62)–(64) and using Eqs. (A4)–(A9), it results that

$$C_1(y, t) = \int_0^t C_0(y, s) ds = t\Phi(y\sqrt{Sc}, t; R) + \frac{y\sqrt{Sc}}{2\sqrt{R}} \Psi(y\sqrt{Sc}, t; R), \quad (65)$$

$$u_{m1}(y, t) = \int_0^t u_{m0}(y, s) ds = t\Phi(y, t; M) + \frac{y}{2\sqrt{M}} \Psi(y, t; M), \quad (66)$$

$$\begin{aligned} u_{C1}(y, t) = \int_0^t u_{C0}(y, s) ds = & \frac{N}{(1 - Sc)\delta_2} \left\{ \frac{e^{\delta_2 t}}{\delta_2} [\Phi(y\sqrt{Sc}, t; R + \delta_2) - \Phi(y, t; M + \delta_2)] \right. \\ & \left. - \frac{1}{2} \left(t + \frac{1}{\delta_2} \right) [\Phi(y\sqrt{Sc}, t; R) - \Phi(y, t; M)] - \frac{y}{4} \left[\sqrt{Sc} \frac{\Psi(y\sqrt{Sc}, t; R)}{\sqrt{R}} - \frac{\Psi(y, t; M)}{\sqrt{M}} \right] \right\}, \end{aligned} \quad (67)$$

$$Sh_1 = \int_0^t Sh_0(s) ds = \sqrt{RSc} \left(t + \frac{1}{2R} \right) \operatorname{erf}(\sqrt{Rt}) + \frac{\sqrt{Sc}t}{\sqrt{\pi}} e^{-Rt}, \quad (68)$$

$$\tau_{m1} = \int_0^t \tau_{m0}(s) ds = \sqrt{M} \left(t + \frac{1}{2M} \right) \operatorname{erf}(\sqrt{Mt}) + \frac{\sqrt{t}}{\pi} e^{-Mt}, \quad (69)$$

$$\begin{aligned} \tau_{C1} = \int_0^t \tau_{C0}(s) ds = & \frac{N}{(1 - Sc)\delta_2} \left\{ \frac{e^{\delta_2 t}}{\delta_2} [\sqrt{(R + \delta_2)Sc} \operatorname{erf}(\sqrt{(R + \delta_2)t}) - \sqrt{M + \delta_2} \operatorname{erf}(\sqrt{(M + \delta_2)t})] \right. \\ & - \left(t + \frac{1}{\delta_2} \right) [\sqrt{RSc} \operatorname{erf}(\sqrt{Rt}) - \sqrt{M} \operatorname{erf}(\sqrt{Mt})] - \frac{1}{2} \left[\sqrt{Sc} \frac{\operatorname{erf}(\sqrt{Rt})}{\sqrt{R}} - \frac{\operatorname{erf}(\sqrt{Mt})}{\sqrt{M}} \right] \\ & \left. - \sqrt{\frac{t}{\pi}} (\sqrt{Sc} e^{-Rt} - e^{-Mt}) \right\}, \quad Sc \neq 1, \quad \delta_2 \neq 0. \end{aligned} \quad (70)$$

It is worth pointing out the fact that $u_{m1}(y, t)$ from Eq. (66) is identical to the result of Reddy *et al.* (Ref. [6] Eq. (17) and Seth *et al.* (Ref. [37], Eq. (2.11)) while the expression of τ_{m1} from Eq. (69) corrects the similar result of Ref. [6]. The corresponding expressions of the associated boundary layers thickness can be immediately obtained putting $f(t) = g(t) = H(t)t^a$ in Eqs. (28) and (47). For $a = 1$ Eq. (28) reduces to

$$\delta_{C1}(t) = \int_0^t \delta_{C0}(s) ds = \frac{1}{\sqrt{ScR}} \left(t - \frac{1}{2R} \right) \operatorname{erf}(\sqrt{Rt}) + \frac{\sqrt{t}}{R\sqrt{Sc\pi}} e^{-Rt}. \quad (71)$$

Case 3 Oscillating Plate with Oscillatory Concentration

Let us now assume that the plate, with oscillatory concentration on the boundary, is oscillating in its plane with the same frequency ω as well as the concentration. The dimensionless solutions corresponding to the free convection flow due to cosine or sine oscillations of the concentration on the boundary, namely

$$C_c(y, t) = \Phi(y\sqrt{Sc}, t; R) - \omega \int_0^t \sin[\omega(t - s)] \Phi(y\sqrt{Sc}, s; R) ds, \quad (72)$$

$$u_{mc}(y, t) = \Phi(y, t; M) - \omega \int_0^t \sin[\omega(t-s)] \Phi(y, s; M) ds, \quad (73)$$

$$u_{Cc}(y, t) = \frac{N}{1-Sc} \int_0^t \cos[\omega(t-s)] [\Phi(y\sqrt{Sc}, s; R+\delta_2) - \Phi(y, s; M+\delta_2)] e^{\delta_2 s} ds, \quad Sc \neq 1, \quad (74)$$

$$Sh_c = \sqrt{ScR} \operatorname{erf}(\sqrt{Rt}) + \frac{\sqrt{Sc} e^{-Rt}}{\sqrt{\pi t}} - \omega \sqrt{RSc} \int_0^t \sin[\omega(t-s)] \operatorname{erf}(\sqrt{Rs}) ds - \frac{\omega \sqrt{Sc}}{\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{\sqrt{s}} e^{-Rs} ds, \quad (75)$$

$$\tau_{mc} = \sqrt{M} \operatorname{erf}(\sqrt{Mt}) + \frac{e^{-Mt}}{\sqrt{\pi t}} - \omega \sqrt{M} \int_0^t \sin[\omega(t-s)] \operatorname{erf}(\sqrt{Ms}) ds - \frac{\omega}{\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{\sqrt{s}} e^{-Ms} ds, \quad (76)$$

$$\begin{aligned} \tau_{Cc} = & \frac{N}{1-Sc} \int_0^t \cos[\omega(t-s)] [\sqrt{R+\delta_2} \operatorname{erf}(\sqrt{(R+\delta_2)s}) - \sqrt{M+\delta_2} \operatorname{erf}(\sqrt{(M+\delta_2)s})] e^{\delta_2 s} ds \\ & + \frac{N}{(1-Sc)\sqrt{\pi}} \int_0^t \frac{\cos[\omega(t-s)]}{\sqrt{s}} (\sqrt{Sc} e^{-Rs} - e^{-Ms}) ds, \end{aligned} \quad (77)$$

$$C_s(y, t) = \omega \int_0^t \cos[\omega(t-s)] \Phi(y\sqrt{Sc}, s; R) ds, \quad (78)$$

$$u_{ms}(y, t) = \omega \int_0^t \cos[\omega(t-s)] \Phi(y, s; M) ds, \quad (79)$$

$$u_{Cs}(y, t) = \frac{N}{1-Sc} \int_0^t \sin[\omega(t-s)] [\Phi(y\sqrt{Sc}, s; R+\delta_2) - \Phi(y, s; M+\delta_2)] e^{\delta_2 s} ds, \quad Sc \neq 1, \quad (80)$$

$$Sh_s = \omega \sqrt{RSc} \int_0^t \cos[\omega(t-s)] \operatorname{erf}(\sqrt{Rs}) ds + \frac{\omega \sqrt{Sc}}{\sqrt{\pi}} \int_0^t \frac{\cos[\omega(t-s)]}{\sqrt{s}} e^{-Rs} ds, \quad (81)$$

$$\tau_{ms} = \omega \sqrt{M} \int_0^t \cos[\omega(t-s)] \operatorname{erf}(\sqrt{Rs}) ds + \frac{\omega}{\sqrt{\pi}} \int_0^t \frac{\cos[\omega(t-s)]}{\sqrt{s}} e^{-Ms} ds, \quad (82)$$

$$\begin{aligned} \tau_{Cs} = & \frac{N}{1-Sc} \int_0^t \sin[\omega(t-s)] [\sqrt{R+\delta_2} \operatorname{erf}(\sqrt{(R+\delta_2)s}) - \sqrt{M+\delta_2} \operatorname{erf}(\sqrt{(M+\delta_2)s})] e^{\delta_2 s} ds \\ & + \frac{N}{(1-Sc)\sqrt{\pi}} \int_0^t \frac{\sin[\omega(t-s)]}{\sqrt{s}} (\sqrt{Sc} e^{-Rs} - e^{-Ms}) ds \end{aligned} \quad (83)$$

are obtained substituting $f(t)$ and $g(t)$ by $H(t) \cos(\omega t)$ or $H(t) \sin(\omega t)$ in Eqs. (22), (24), (34), (36), (42), and (44), respectively. As expected, the solutions (72)–(77) reduce to those given by Eqs. (49)–(54) if the frequency ω of oscillations tends to zero (for $u_{Cc}(y, t)$, $u_{C_0}(y, t)$ and τ_{Cc} , τ_{C_0} see also the general solutions (36) and (44)). Furthermore, all solutions corresponding to this subsection can be written as a sum of steady-state (permanent) and transient solutions. The steady-state solutions corresponding to $C_c(y, t)$ and $C_s(y, t)$, for instance, can be given by the equalities (see also Eq. (A10))

$$\begin{aligned} C_{cp}(y, t) = & \frac{y\sqrt{Sc}}{2\sqrt{\pi}} \int_0^\infty \frac{\cos[\omega(t-s)]}{s\sqrt{s}} \\ & \times \exp\left(-\frac{y^2 Sc}{4s} - Rs\right) ds, \end{aligned} \quad (84)$$

$$\begin{aligned} C_{sp}(y, t) = & \frac{y\sqrt{Sc}}{2\sqrt{\pi}} \int_0^\infty \frac{\sin[\omega(t-s)]}{s\sqrt{s}} \\ & \times \exp\left(-\frac{y^2 Sc}{4s} - Rs\right) ds. \end{aligned} \quad (85)$$

Moreover, lengthy but straightforward computation show that these solutions can be written in the simple forms

(see Eqs. (A11) and (A12))

$$\begin{aligned} C_{cp}(y, t) &= e^{-py} \cos(\omega t - qy), \\ C_{sp}(y, t) &= e^{-py} \sin(\omega t - qy), \end{aligned} \quad (86)$$

where

$$p = \sqrt{\frac{\sqrt{R^2 + \omega^2} + R}{2} Sc}, \quad q = \sqrt{\frac{\sqrt{R^2 + \omega^2} - R}{2} Sc}.$$

5 Numerical Results and Discussions

In order to gain some physical insight of results that have been here obtained and to avoid repetition, the effects of buoyancy ratio parameter (N), heat generation or absorption coefficient (Q), Schmidt number (Sc) and chemical reaction parameter (R) on dimensionless concentration and velocity fields are graphically underlined in Figs. 2–7 for fluid motions induced by a highly accelerating plate ($f(t) = H(t)t^{3/2}$) with ramp-type concentration ($g(t) = H(t)t^a$). Variations of Sherwood number (Sh) with respect to Sc and R are presented in Fig. 8 while the diagrams of the skin friction coefficient τ against t are given in Fig. 9. Finally, for completion, the contributions of mechanical, thermal and concentration components of velocity on the fluid motion are brought to light

by Fig. 10 and the required time to reach the steady-state of mass transfer is graphically obtained in Figs. 11 and 12 for flows with cosine or sine oscillations of concentration at the plate. Time variation of thermal or concentration boundary layer thickness is presented in Figs. 13 and 14 for different values of physical parameters when the species concentration is constant on the boundary.

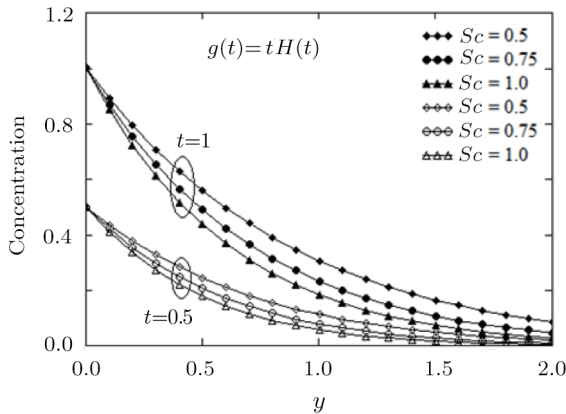


Fig. 2 Profiles of the dimensionless concentration $C_1(y, t)$ against y for $R = 1.5$, $t = 0.5$, and 1 with different values of Sc .

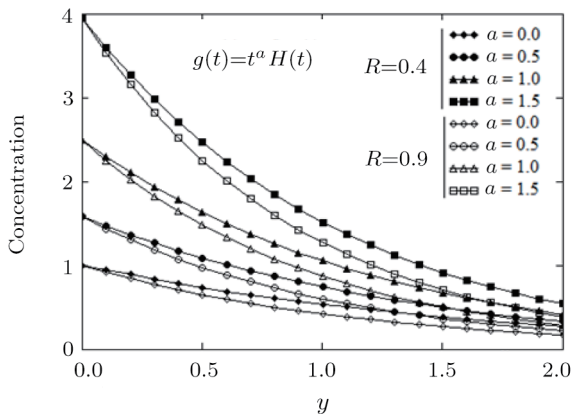


Fig. 3 Profiles of the dimensionless concentration $C_a(y, t)$ against y for $t = 2.5$, $Sc = 0.8$, $R = 0.4$, and 0.9 with different values of a .

Profiles of the concentration $C_a(y, t)$ against y are presented in Figs. 2 and 3 for different values of Sc , R , a and the time t . The species concentration, as expected, is an increasing function with respect to a and t but it decreases for increasing values of Sc and R . As it is known,^[5] a diminution in the Schmidt number Sc means an increase in mass diffusivity which enhances the species concentration in fluid. Consequently, an increase of Sc or R lowers the concentration level of the fluid. In all cases, the concentration profiles smoothly descend from maximum values on the wall to the zero value for large values of y .

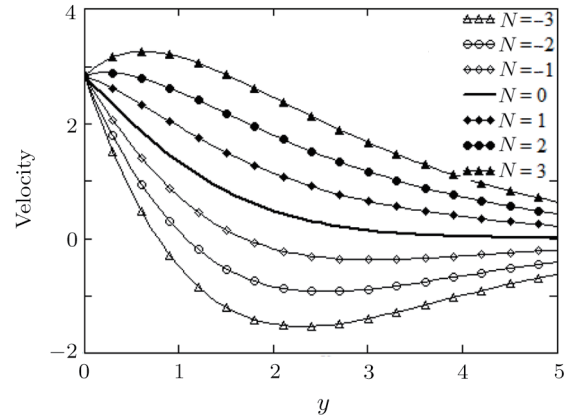


Fig. 4 Profiles of the dimensionless velocity $u_{3/2}(y, t)$ against y at time $t = 2$, $M = 0.2$, $Pr = 1.5$, $Q = 0.7$, $Sc = 0.2$, $R = 0.3$, and different values of N .

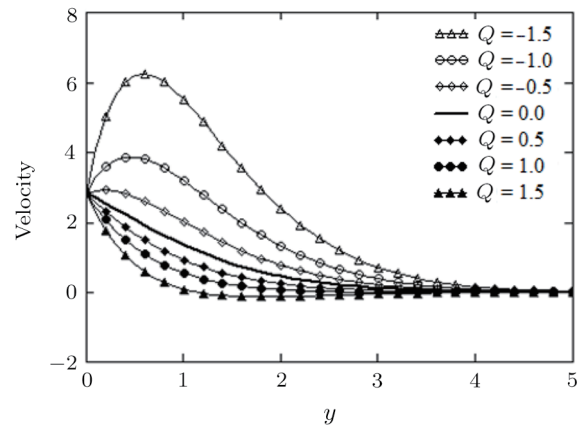


Fig. 5 Profiles of the dimensionless velocity $u_{3/2}(y, t)$ against y at time $t = 2$, for $M = 1$, $N = 2$, $Pr = 1.5$, $Sc = 0.5$, $R = 0.7$, and different values of Q .

Numerical values of the fluid velocity $u_a(y, t)$, given by Eq. (61), are graphically displayed in Figs. 4–7 for $a = 3/2$ the plate concentration $C(0, t) = tH(t)$ and different value of physical parameters. Velocity profiles against y are presented in Fig. 4 for aiding ($N > 0$) and opposing ($N < 0$) flows at the time $t = 2$. In the first case, when thermal and mass buoyancy forces act in the same direction, the fluid velocity increases for increasing values of N as a result of the growth of concentration. $N = 0$ implies the mass Grashof number $Gm = 0$ and the mass diffusion phenomenon is absent. If $N < 0$, the mass buoyancy forces are negative and the fluid velocity is significantly diminished. However, it increases for increasing values of N . For positive values of N greater than a critical N_c value (about 1.9), in the plate vicinity, the fluid velocity increases from the common value on the wall up to a maximum value and then decreases to the stream value for large values of y .

Effects of the heat generation or absorption coefficient Q on the fluid motion are displayed in Fig. 5. The presence of heat generation ($Q < 0$) generates thermal energy, which increases the fluid temperature. As a result,

the fluid velocity increases due to the increasing thermal buoyancy force. An opposite effect appears in the case of heat absorption. More exactly, due to the heat absorption ($Q > 0$), the fluid temperature diminishes and the thermal buoyancy force decreases. This implies a reduction of fluid velocity with increasing values of Q . However, in the case of heat generation, for each Q less than a critical value Q_c (about -0.5), the fluid velocity increases from the common value on the plate up to a maximum value and then smoothly decreases to the zero value for increasing values of y .

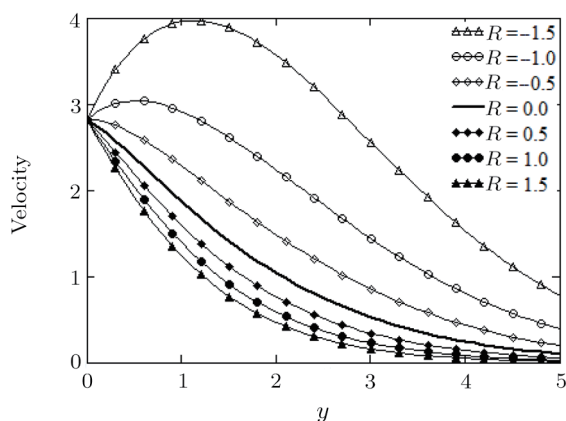


Fig. 6 Profiles of the dimensionless velocity $u_{3/2}(y, t)$ against y at time $t = 2$, for $M = 1$, $N = 2$, $Pr = 1.5$, $Q = 0.5$, $Sc = 0.5$, and different values of R .

Figure 6 displays the influence of chemical parameter R on the fluid velocity. The presence of destructive chemical reaction ($R > 0$), as it results from Fig. 3, diminishes the species concentration and implicitly reduces the mass buoyancy force. As a result, the fluid velocity decreases for increasing values of R . Of course, an opposite trend appears in the case of non-destructive chemical reactions when $R < 0$. Variations of the fluid velocity are also presented in Fig. 7 for two values of Pr and different values of Sc while the other parameters are fixed. From this figure, it clearly results that the velocity is a decreasing

function both with Pr and Sc . Consequently, the viscous forces predominate thermal diffusion or mass diffusion effects for increasing values of Prandtl, respectively Schmidt number.

The variation of Sherwood number Sh in time is graphically presented in Fig. 8 at different values of Sc and R for flows with ramp-type surface concentration. It is found that the rate of mass transfer at the plate is an almost linearly increasing function of t . It also increases for increasing values of Sc and R . Consequently, the destructive chemical reaction enhances the rate of mass transfer at the plate. An opposite effect produces the increase of the chemical molecular diffusivity D .

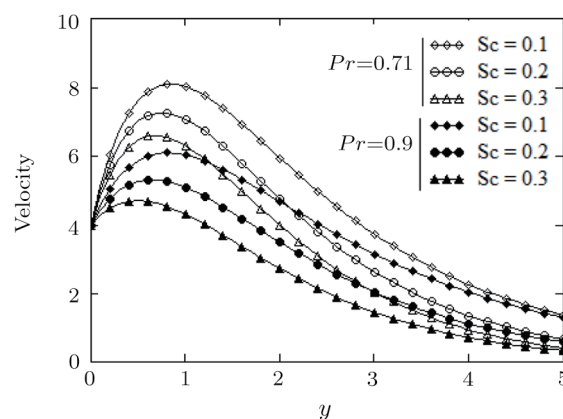


Fig. 7 Profiles of the dimensionless velocity $u_{3/2}(y, t)$ against y at time $t = 2.5$, for $M = 1$, $N = 2$, $Q = 0.5$, $R = 0.5$, $Pr = 0.71$, and 0.9 with different values of Sc .

Figure 9 shows the skin friction coefficient variation against t under the influence of Q and N , respectively R and Pr . The skin friction coefficient is an increasing function with respect to t , Q , N , Pr and decreases for increasing positive values of R . It increases almost linearly in t for $N = 3$ with $Q = 0.6$ and 0.7 or $Pr = 3.5$ with $R = 0.5, 1.0$ and 1.5 . From physical point of view, it means that a destructive chemical reaction diminishes the viscous drag at the plate while the heat absorption enhances it.

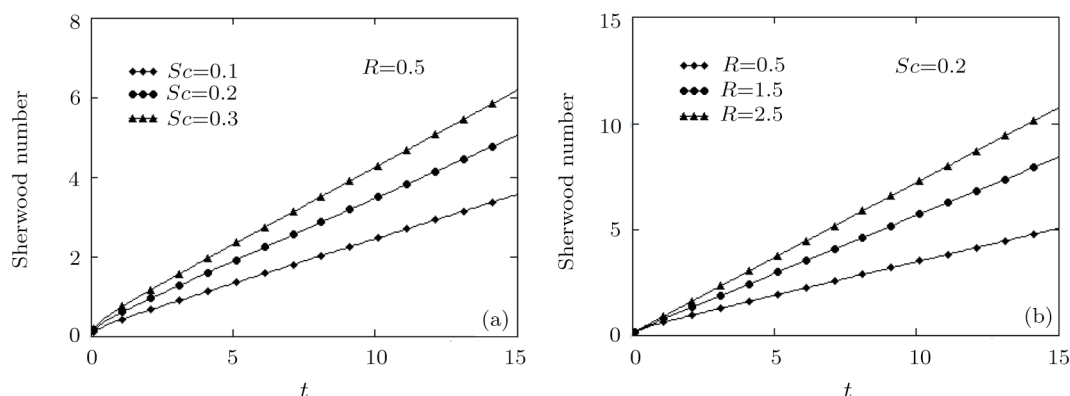


Fig. 8 Variation of Sherwood number Sh , given by Eq. (68), with respect to Sc and R .

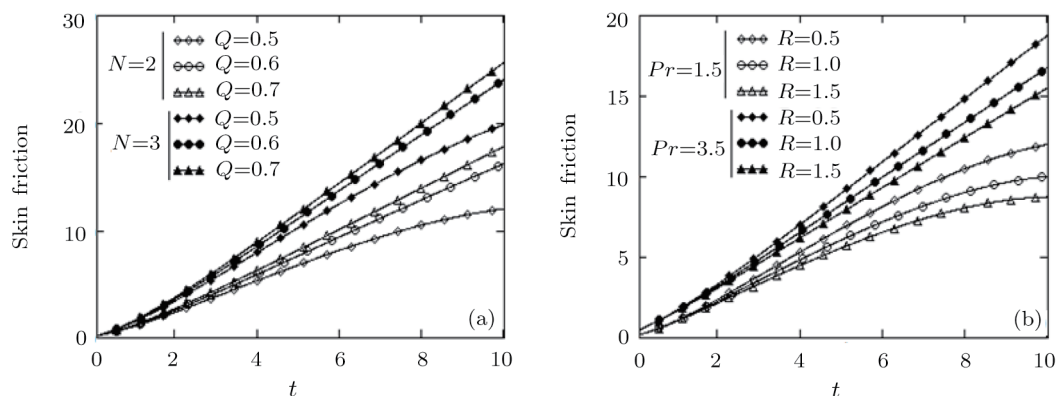


Fig. 9 Variation of skin friction τ given by Eq. (41) against t for $M=0.4$, $Sc=0.5$, $f(t) = t^{3/2}$, $g(t) = t$ and different values of Q , N , R , and Pr .

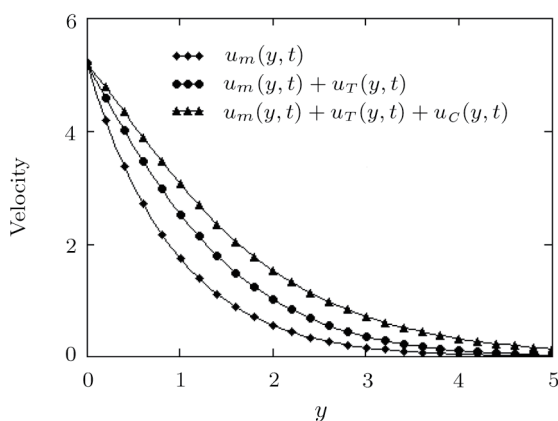


Fig. 10 Profiles of the dimensionless velocities $u_{m3/2}(y, t)$, $u_{m3/2}(y, t) + u_T(y, t)$, and $u_{m3/2}(y, t) + u_T(y, t) + u_C(y, t)$ against y for $Pr = 1.5$, $Q = 0.5$, $Sc = 0.5$, $R = 0.7$, $M = 0.6$, $N = 2$, and $t = 3$.

In order to evaluate the importance of thermal or mass diffusion effects on free convection flows of viscous fluids,

the contributions of mechanical, thermal and concentration components of velocity $u_{3/2}(y, t)$ on the fluid motion are together brought to light in Fig. 10. As it clearly results from this figure, each component significantly influences the fluid velocity and cannot be neglected.

In Figs. 11 and 12, the required time to reach the steady-state for mass diffusion is graphically determined for flows with cosine or sine oscillations of concentration at the plate for two values of chemical reaction parameter R . This is the time after which the diagrams of starting solutions (72) or (78) are almost identical to those of steady-state solutions (86)₁, respectively (86)₂. At small values of t , the difference between the corresponding solutions is significant but it quickly disappears and the required time to reach the steady-state is higher for sine in comparison to cosine oscillations of concentration at the wall. This is obvious, because at time $t = 0$ the concentration level at the plate is zero for sine oscillations. Furthermore, as it clearly results from these figures, the presence of destructive chemical reaction improves this time for increasing values of $R > 0$.

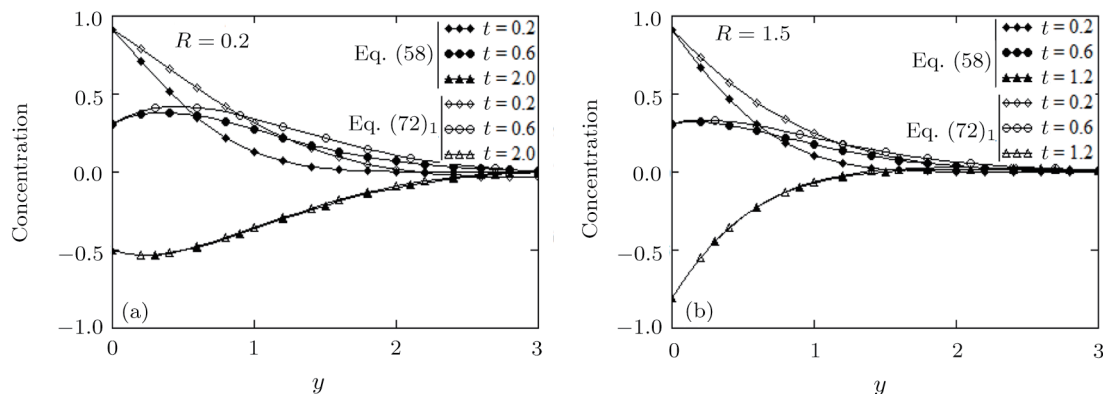


Fig. 11 Required time to reach the steady-state of mass transfer for cosine oscillations of concentration at the plate at $Sc = 0.9$ and $\omega = 2\pi/3$.

Figures 13 and 14 bring to light the time variation of the thickness of thermal or concentration boundary layers with respect to Pr and Q , respectively Sc and R and $g(t) = H(t)$. In all cases, the boundary layer thickness significantly

increases up to a critical value of t (around $t = 10$) and then rapidly tends to the asymptotic value. The thermal boundary layer thickness is a decreasing function with respect to Pr or Q and it rather reaches the asymptotic value for greater values of these parameters. A similar behavior appears from Fig. 14 for the concentration boundary layer thickness with regard to R and Sc .

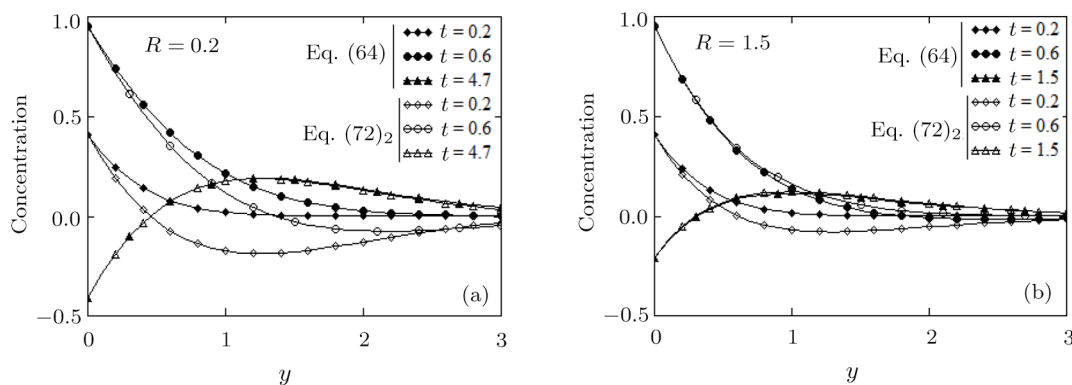


Fig. 12 Required time to reach the steady-state of mass transfer for sine oscillations of concentration at the plate at $Sc = 0.9$ and $\omega = 2\pi/3$.

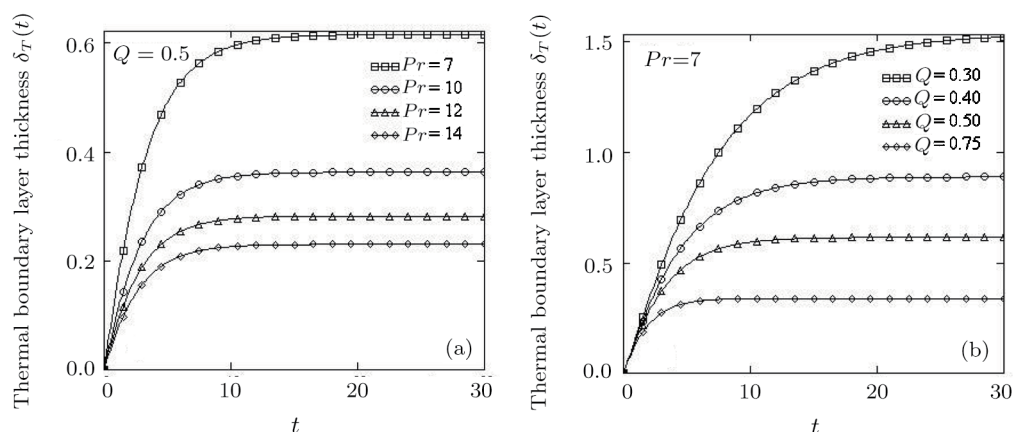


Fig. 13 Time variation of thermal boundary layer thickness for different values of Prandtl number Pr and heat generation/absorption parameter Q .

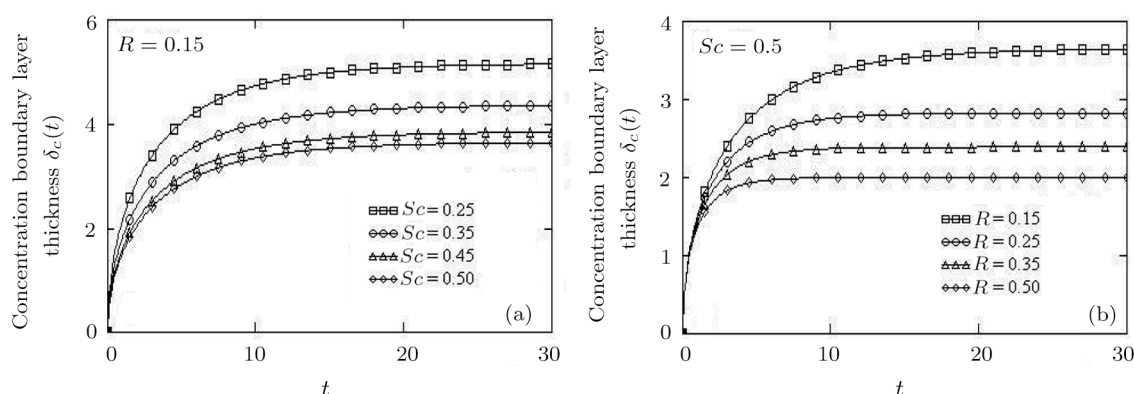


Fig. 14 Time variation of thermal boundary layer concentration for different values of Schmidt number Sc and chemical reaction parameter R .

6 Conclusions

Hydromagnetic free convection flow of an electrically conducting, incompressible viscous fluid over a moving in-

finite vertical plate with Newtonian heating, heat source, mass diffusion and chemical reaction is completely solved. Exact analytic solutions are established for velocity, con-

centration, Sherwood number and skin friction coefficient when the plate is moving in its plane with an arbitrary velocity and the concentration at the wall is a time-dependent function. They satisfy all imposed initial and boundary conditions and can generate exact solutions for any free convection flow of this type. For illustration, as well as to get some physical insight of the obtained results, three special cases with technical relevance are considered and some results from the existing literature are recovered or corrected. Radiative and porous effects are not taken into consideration but they can be immediately included by a simple rescaling of Prandtl number and magnetic parameter.^[31–32]

The solutions corresponding to the motion due to a plate with uniform velocity (Stokes first problem) and constant concentration at the wall, as well as those induced by a constantly accelerating plate with ramp-type concentration at the wall,^[36] are presented in simple forms in terms of exponential function and error function or complementary error function of Gauss. In addition, the solutions of the second problem can be written as simple integrals of the similar solutions corresponding to the first problem of Stokes. The solutions corresponding to motions due to an oscillating plate (Stokes second problem) with oscillatory concentration at the wall can be written as sum of steady-state (permanent) and transient solutions. These solutions, which are independent of the initial conditions but satisfy the boundary conditions and governing equations, are important for those who want to eliminate the transients from their experiments. Moreover, as it was to be expected, all solutions corresponding to cosine oscillations of the plate and of the concentration at the wall reduce to the similar solutions of Stokes first problem when the oscillation frequency ω tends to zero.

Finally, in order to bring to light some physical penetration of results that have been obtained, the diagrams of dimensionless concentration and velocity fields, Sherwood number and skin friction coefficient are presented in different situations for typical values of pertinent parameters. However, in order to avoid repetition, their profiles have been here presented and discussed only for variations of physical parameters N , Q , Sc and R with ramp-type concentration at the wall. Contributions of mechanical, thermal and concentration components of velocity on the

fluid motion are together underlined for motions due to highly accelerating plate. The required time to reach the steady-state of mass diffusion for cosine or sine oscillations of the concentration at the plate has been graphically determined and the main results that have been here obtained are:

(i) The problem in consideration has been completely solved. Obtained results can generate exact solutions for any free convection flow of this type.

(ii) Species concentration is increasing function in time and ramp-type parameter a .

(iii) The increase of mass diffusivity brings up the concentration level of the fluid while the presence of destructive chemical reaction diminishes it.

(iv) For aiding flows ($N > 0$), velocity of the fluid is increasing function with respect to N . An opposite trend appears in the case of opposing flows when ($N < 0$).

(v) Heat absorption ($Q > 0$) causes a reduction of velocity for increasing value of Q . This is due to the fact the fluid temperature diminishes and the thermal buoyancy force decreases. A reverse trend appears in the presence of heat generation $Q < 0$.

(vi) Destructive chemical reaction ($R > 0$) reduces the mass buoyancy force and the fluid velocity decreases for increasing values of R through the boundary layer region. The non-destructive chemical reaction ($R < 0$) enhances the fluid velocity.

(vii) Destructive chemical reaction enhances the rate of mass transfer at the plate.

(viii) The presence of heat absorption enhances the viscous drag at the plate while the destructive chemical reaction diminishes it.

(ix) Mechanical, thermal or concentration effects significantly influence the fluid motion and they cannot be neglected.

(x) Required time to reach the steady-state for the mass transfer is higher for sine in comparison to cosine oscillations of concentration of the plate and it is improved in the presence of destructive chemical reaction.

(xi) Thermal or concentration boundary layer thickness significantly increases up to a critical value of t and then it rather reaches the asymptotic value for greater values of Q or Pr , respectively R or Sc . It is a decreasing function with respect to each of the respective parameters.

Appendix

$$L^{-1} \left\{ \frac{\exp(-y\sqrt{q+a})}{q-b} \right\} = e^{bt} \Phi(y, t; a+b),$$

$$\text{where } \Phi(y, t; a) = \frac{1}{2} \left\{ e^{y\sqrt{a}} \operatorname{erfc} \left(\frac{y}{2\sqrt{t}} + \sqrt{at} \right) + e^{-y\sqrt{a}} \operatorname{erfc} \left(\frac{y}{2\sqrt{t}} - \sqrt{at} \right) \right\}, \quad (\text{A1})$$

$$L^{-1} \left\{ \frac{1}{q[\sqrt{a(q+b)}-1]} \right\} = \frac{1}{1-ab} \left\{ \left[1 + \operatorname{erf} \left(\sqrt{\frac{t}{a}} \right) \right] \exp \left(\frac{1-ab}{a} t \right) - \sqrt{ab} \operatorname{erf}(\sqrt{bt}) - 1 \right\}, \quad (\text{A2})$$

$$L^{-1}\left\{\frac{1}{\sqrt{q+a}}\right\} = \frac{1}{\sqrt{\pi t}} - a e^{a^2 t} \operatorname{erfc}(a\sqrt{t}); L^{-1}\left\{\frac{1}{(q-a)\sqrt{q+b}}\right\} = \frac{e^{at}}{\sqrt{a+b}} \operatorname{erf}(\sqrt{(a+b)t}), \quad (\text{A3})$$

$$\int_0^t \Phi(y, s; a+b) e^{bs} ds = \frac{1}{2b} [2 e^{bt} \Phi(y, t; a+b) - \Phi(y, t; a)], \quad (\text{A4})$$

$$\int_0^t e^{bs} \operatorname{erf}(\sqrt{(a+b)s}) ds = \frac{e^{bt}}{b} \operatorname{erf}(\sqrt{(a+b)t}) - \frac{\sqrt{a+b}}{b\sqrt{a}} \operatorname{erf}(\sqrt{at}), \quad (\text{A5})$$

$$\int_0^t \operatorname{erf}(\sqrt{as}) ds = \left(t - \frac{1}{2a}\right) \operatorname{erf}(\sqrt{at}) + \frac{\sqrt{t}}{\sqrt{\pi a}} e^{-at}, \quad (\text{A6})$$

$$\int_0^t s e^{bs} \operatorname{erf}(\sqrt{(a+b)s}) ds = \frac{bt-1}{b^2} e^{bt} \operatorname{erf}(\sqrt{(a+b)t}) + \frac{(2a-b)\sqrt{a+b}}{2ab^2\sqrt{a}} \operatorname{erf}(\sqrt{at}) + \frac{\sqrt{(a+b)t}}{ab\sqrt{\pi}} e^{-at}, \quad (\text{A7})$$

$$\int_0^t \Phi(y, s; a) ds = t\Phi(y, t; a) + \frac{y}{2\sqrt{a}} \Psi(y, t; a),$$

where

$$\Psi(y, t; a) = \frac{1}{2} \left\{ e^{y\sqrt{a}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} + \sqrt{at}\right) + e^{-y\sqrt{a}} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}} - \sqrt{at}\right) \right\}, \quad (\text{A8})$$

$$\int_0^t s e^{bs} \Phi(y, s; a+b) = \frac{bt-1}{b^2} e^{bt} \Phi(y, t; a+b) + \frac{1}{2b^2} \Phi(y, t; a) + \frac{y}{4b\sqrt{a}} \Psi(y, t; a), \quad (\text{A9})$$

$$\int_0^t \frac{1}{s\sqrt{s}} \exp\left(-\frac{a^2}{4s} - bs\right) ds = \frac{\sqrt{\pi}}{a} \left\{ e^{a\sqrt{b}} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} + \sqrt{bt}\right) + e^{-a\sqrt{b}} \operatorname{erfc}\left(\frac{a}{2\sqrt{t}} - \sqrt{bt}\right) \right\}, \quad a \neq 0, \quad (\text{A10})$$

$$\int_0^\infty \exp\left(-m^2 s^2 - \frac{n^2}{s^2}\right) \cos\left(a^2 s^2 + \frac{b^2}{s^2}\right) ds = \frac{\sqrt{\pi}}{2\sqrt[4]{m^4 + a^4}} e^{-2c \cos(\alpha+\beta)} \cos[\alpha + 2c \sin(\alpha + \beta)], \quad (\text{A11})$$

$$\int_0^\infty \exp\left(-m^2 s^2 - \frac{n^2}{s^2}\right) \sin\left(a^2 s^2 + \frac{b^2}{s^2}\right) ds = \frac{\sqrt{\pi}}{2\sqrt[4]{m^4 + a^4}} e^{-2c \cos(\alpha+\beta)} \sin[\alpha + 2c \sin(\alpha + \beta)], \quad (\text{A12})$$

where

$$\alpha = \frac{1}{2} \arctg\left(\frac{a^2}{m^2}\right), \quad \beta = \frac{1}{2} \arctg\left(\frac{b^2}{n^2}\right), \quad c = \sqrt[4]{(m^4 + a^4)(n^4 + b^4)}.$$

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