Peristaltic Flow of Shear Thinning Fluid via Temperature-Dependent Viscosity and Thermal Conductivity

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Abstract *In this paper Williamson fluid is taken into account to study its peristaltic flow with heat effects. The study is carried out in a wave frame of reference for symmetric channel. Analysis of heat transfer is accomplished by accounting the effects of non-constant thermal conductivity and viscosity and viscous dissipation. Modeling of fundamental equations is followed by the construction of closed form solutions for pressure gradient, stream function and temperature while assuming Reynold's number to be very low and wavelength to be very long. Double perturbation technique is employed, considering Weissenberg number and variable fluid property parameter to be very small. The effects of emerging parameters on pumping, trapping, axial pressure gradient, heat transfer coefficient, pressure rise, velocity profile and temperature are analyzed through the graphical representation. A direct relation is observed between temperature and thermal conductivity whereas the indirect proportionality with viscosity. The heat transfer coefficient is lower for a fluid with variable thermal conductivity and variable viscosity as compared to the fluid with constant thermal conductivity and constant viscosity.*

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Key words: temperature-dependent viscosity, temperature-dependent thermal conductivity, shear thinning fluid, peristalsis

Nomenclature

$\bar{K}$	Dimensional thermal conductivity
$\bar{\mu}$	Dimensional fluid viscosity
K_0	Thermal conductivity at constant temperature
μ_0	Dynamic fluid viscosity at constant temperature
ζ, η	Parametric constants of conductivity & viscosity respectively
$\bar{c}_1$	Half width of channel
$\bar{e}_1$	Wave amplitude
s	Wave speed
(X, Y)	Two dimensional coordinate system
λ	Wavelength
W_z	Weissenberg number
ϵ	Viscosity parameter
α	Thermal conductivity parameter
Re	Reynold number
δ	Wave number
Bk	Brinkman number
Ec	Eckert number
Pr	Prandtl number
θ	Flow rate
$\bar{\tau}$	Stress tensor
$\boldsymbol{A}_1$	First rivlin ericksen tensor
Γ	Fluid parameter
P	Pressure
t	Time

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1 Introduction

Latham[1] presented the pioneering work on peristaltic flows. A strong foundation was laid by him for development of peristalsis theoretically. He examined flow experimentally and analytically, in a two-dimensional channel. Peristalsis plays an important role in many industrial applications like blood pumps in heart and in sanitary fluid transport etc. This mechanism has various biological and biomedical systems, like motion of chyme in the gastrointestinal tract, circulation of blood in the blood vessels, transfer of spermatozoa in the ducts of the male reproductive tracts, transfer of ovum in the female fallopian tube, transportation of urine from kidneys to bladder and circulation of blood in the blood vessels. Shapiro *et al.*[2] proposed the lubrication theory model in which a negligible effect of fluid inertia and wave number is taken into account. Since these works, many researchers have proposed mathematical model with wave trains, of wall generated flow due to difference in phase moving independently of the lower and upper walls. Recently, Rehman *et al.*[3] have explained the peristaltic motion of Jeffrey fluid with effect of wall attributes. Convective boundary and inclined magnetic field effects on the peristaltic mechanism were studied by Noreen and Qasim.[4]

Remarkable progress has been made by several authors during the previous few years in the development of flows for non Newtonian fluids. Heat transfer in peristalsis has also gained attention of researchers for last few decades. The process of transfer of heat may be used to get the details about the attributes of tissues. Currently, Rashidi *et al.*[5−7] have done mentionable studies in investigation of flow of non-Newtonian fluids. Uddin *et al.*[8] also made significant development in investigating the effect of free convection in the flow of real fluids. Bhatti *et al.*[9] described the non-Newtonian fluid flow influenced by non- linear thermal radiation and MHD. Peristalsis has been the main subject of several recent research works. Undesirable tissues, such as cancer can be destroyed by heat. Inspired by above, Noreen and Qasim[10] presented a mathematical study for peristaltic motion of pseudoplastic fluid in a 2-D channel under certain approximations. Mention should be the name of Ramaesh and Devakar[11] for the progressive work in peristaltic flows in vertical channel. The non Newtonian fluid flow that was initiated by peristaltic waves in presence of chemical reaction was described by Noreen and Saleem.[12] Rundora and Makinde[13] synchronized the effects of suction/injection on unsteady non-Newtonian fluid flow in a channel filled with porous medium and convective boundary condition. Noreen[14] studied the induced magnetic field effect in peristaltic flow. A number of researchers are now busy in studying the peristalsis, particularly viscoelastic class of non-Newtonian fluids due to its wide range of applications in industry, engineering and medical science.

Fluid properties such as viscosity, density, thermal conductivity etc. are assumed constant for convenience in many studies. However variable fluid properties have real life applications, which include extrusion processes, fibre and wire coating, food-stuff processing, chemical processing equipment etc. Alvi *et al.*[15] have examined the mixed convective peristaltic flow of Jeffrey nanofluid with variable viscosity, viscous dissipation and Joule heating effects. Latif *et al.*[16] discussed the result of temperature-dependent variable properties on the third order peristaltic flow. Considering viscosity of the fluid as variable, studies[17−18] have also been reported.

Williamson fluid[19] is also a class of non-Newtonian fluids. Williamson fluid model is studied under various aspects in literature. Reddy *et al.*[20] and Malik *et al.*[21] described the Williamson fluid flow over a stretching sheet and stretching cylinder respectively. Few attempts in peristalsis are also available. Nadeem and Akram[22−23] peristaltic flow of Williamson fluid. Nadeem and Akbar[24] presented numerical solutions of Williamson fluid with radially varying MHD. In another article Vajravelu *et al.*[25] presented peristaltic transport of a Williamson fluid with permeable walls. Variable properties effects on peristaltic transport of Williamson fluid are not studied before. The apparent viscosity varies gradually between μ_∞ as the shear rate tends to infinity and μ_0 at zero shear rate. So, we try to fill this gap by studying the effects of variable thermal conductivity as well as variable viscosity on peristaltic transport of Williamson fluid with heat characteristics. The findings of the present study may be applicable in designing the peristaltic-pumps, transport phenomena in chemical engineering and energy systems, channel type solar energy collectors and heat exchangers.

Thermal analysis has been carried out for combined effects of variable conductivity and viscosity on peristaltic flow in the present article. The governing equations are introduced with boundary conditions. Double perturbation technique is employed to solve the system for closed form solution. Section 2 comprises of mathematical development and formulation of our problem. The zeroth and second order systems generated by using Perturbation technique are presented in Sec. 3. Finally the results are discussed in Sec. 4.

2 Problem Development and Formulation

We let that thermal conductivity $\bar{K}$ and viscosity $\bar{\mu}$ of Williamson fluid vary linearly with temperature[16]

$$\bar{K} = K_0[1 + \zeta(\bar{T} - \bar{T}_w)]\,, \tag{1}$$

$$\bar{\mu} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)]\,, \tag{2}$$

where K_0 is the thermal conductivity, μ_0 is fluid dynamic viscosity, T_w is constant temperature and ζ and η are constants.

2.1 *Fluid Model*

Constitutive equation of the Williamson fluid model with non constant viscosity is characterized by

$$\begin{aligned}\bar{\boldsymbol{\tau}} &= \mu_0[1 - \eta(\bar{T} - \bar{T}_w)][(1 - \Gamma\bar{\dot{\gamma}})^{-1}]\boldsymbol{A}_1 \\ &= \mu_0[1 - \eta(\bar{T} - \bar{T}_w)][(1 + \Gamma\bar{\dot{\gamma}})]\boldsymbol{A}_1\,,\end{aligned} \tag{3}$$

with

$$\tau_{\bar{X}\bar{X}} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})\Big(2\frac{\partial \bar{U}}{\partial \bar{X}}\Big), \tag{4}$$

$$\tau_{\bar{X}\bar{Y}} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})\Big(\frac{\partial \bar{U}}{\partial \bar{Y}} + \frac{\partial \bar{V}}{\partial \bar{X}}\Big), \tag{5}$$

$$\tau_{\bar{Y}\bar{Y}} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})\Big(2\frac{\partial \bar{V}}{\partial \bar{Y}}\Big), \tag{6}$$

with

$$\bar{\dot{\gamma}} = \text{trace}\, \boldsymbol{A}_1^2 = 4\Big(\frac{\partial \bar{U}}{\partial \bar{X}}\Big)^2 + 2\Big(\frac{\partial \bar{U}}{\partial \bar{Y}} + \frac{\partial \bar{V}}{\partial \bar{X}}\Big)^2 + 4\Big(\frac{\partial \bar{V}}{\partial \bar{Y}}\Big)^2. \tag{7}$$

The above model reduces to the Newtonian model $\Gamma = 0$.

2.2 *Geometry of Problem*

Let us consider a 2-D channel ($-H < \bar{Y} < \bar{H}$) filled with Williamson fluid, of half width c_1. The walls of the channel are flexible and are subjected to constant temperature T_w. When the sinusoidal waves having small amplitude e_1 with constant speed s propagate on the walls of the channel then the shape of the walls can be defined as

$$Y = \bar{H} = \bar{c}_1 + \bar{e}_1 \cos\Big[\frac{2\pi}{\lambda}(\bar{X} - s\bar{t})\Big]. \tag{8}$$

Here $\bar{X}$ defines direction of wave propagation, $2\bar{c}_1$ defines the channel's width, λ is the wave length and $\bar{t}$ represents the time.

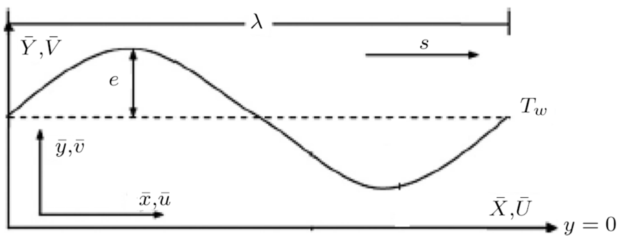

Fig. 1 Flow configuration.

2.3 *Basic Equations*

The governing equations for Williamson fluid flow are:

$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} = 0, \tag{9}$$

$$\rho\Big(\frac{\partial \bar{U}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{U}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{U}}{\partial \bar{Y}}\Big) = \frac{\partial \bar{\tau}_{xx}}{\partial \bar{X}} + \frac{\partial \bar{\tau}_{xy}}{\partial \bar{Y}} - \frac{\partial \bar{P}}{\partial \bar{X}}, \tag{10}$$

$$\rho\Big(\frac{\partial \bar{V}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{V}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{V}}{\partial \bar{Y}}\Big) = \frac{\partial \bar{\tau}_{xy}}{\partial \bar{X}} + \frac{\partial \bar{\tau}_{yy}}{\partial \bar{Y}} - \frac{\partial \bar{P}}{\partial \bar{Y}}, \tag{11}$$

$$\rho c_p\Big(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{U}\frac{\partial \bar{T}}{\partial \bar{X}} + \bar{V}\frac{\partial \bar{T}}{\partial \bar{Y}}\Big) = \frac{\partial}{\partial \bar{X}}\Big(\bar{K}\frac{\partial \bar{T}}{\partial \bar{X}}\Big) + \frac{\partial}{\partial \bar{Y}}\Big(\bar{K}\frac{\partial \bar{T}}{\partial \bar{Y}}\Big)$$
$$+ \bar{\tau}_{xx}\frac{\partial \bar{U}}{\partial \bar{X}} + \bar{\tau}_{yy}\frac{\partial \bar{V}}{\partial \bar{Y}} + \bar{\tau}_{xy}\Big(\frac{\partial \bar{V}}{\partial \bar{X}} + \frac{\partial \bar{U}}{\partial \bar{Y}}\Big). \tag{12}$$

Defining a wave frame $(\bar{x}, \bar{y})$ moving with velocity s with respect to fixed frame $(\bar{X}, \bar{Y})$ by the transformation:

$$\bar{x} = \bar{X} - st, \quad \bar{y} = \bar{Y}, \quad \bar{u} = \bar{U} - s,$$
$$\bar{v} = \bar{V}, \quad \bar{p}(x) = \bar{P}(X, t), \tag{13}$$

yield

$$\frac{\partial(\bar{u} + s)}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0, \tag{14}$$

$$\rho\Big((\bar{u} + s)\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\Big) = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{\bar{x}\bar{x}}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{\bar{x}\bar{y}}}{\partial \bar{y}}, \tag{15}$$

$$\rho\Big((\bar{u} + s)\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\Big) = -\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial \bar{\tau}_{\bar{x}\bar{y}}}{\partial \bar{x}} + \frac{\partial \bar{\tau}_{\bar{y}\bar{y}}}{\partial \bar{y}}, \tag{16}$$

$$\rho c_p\Big((\bar{u} + s)\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\Big) = \frac{\partial}{\partial \bar{x}}\Big(\bar{K}\frac{\partial \bar{T}}{\partial \bar{x}}\Big) + \frac{\partial}{\partial \bar{y}}\Big(\bar{K}\frac{\partial \bar{T}}{\partial \bar{y}}\Big)$$
$$+ \bar{\tau}_{xx}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{\tau}_{yy}\frac{\partial \bar{v}}{\partial \bar{y}} + \bar{\tau}_{xy}\Big(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial \bar{u}}{\partial \bar{y}}\Big), \tag{17}$$

with

$$\bar{\tau}_{xx} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})2\frac{\partial(\bar{u} + s)}{\partial \bar{x}}, \tag{18}$$

$$\bar{\tau}_{xy} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})\Big(\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial(\bar{u} + s)}{\partial \bar{y}}\Big), \tag{19}$$

$$\bar{\tau}_{yy} = \mu_0[1 - \eta(\bar{T} - \bar{T}_w)](1 + \Gamma\bar{\dot{\gamma}})\Big(2\frac{\partial \bar{v}}{\partial \bar{y}}\Big), \tag{20}$$

$$\bar{\dot{\gamma}} = 4\Big(\frac{\partial(\bar{u} + s)}{\partial x}\Big)^2 + 2\Big[\frac{\partial(\bar{u} + s)}{\partial y} + \frac{\partial v}{\partial x}\Big]^2 + 4\Big(\frac{\partial v}{\partial y}\Big)^2. \tag{21}$$

Now we define

$$y = \frac{\bar{y}}{c_1}, \quad v = \frac{\bar{v}}{s}, \quad t = \frac{s}{\lambda}\bar{t}, \quad h = \frac{\bar{H}}{c_1}, \quad x = \frac{\bar{x}}{\lambda},$$
$$u = \frac{\bar{u}}{s}, \quad \tau_{xx} = \frac{\bar{c}_1}{\mu_0 s}\bar{\tau}_{xx}, \quad \tau_{xy} = \frac{\bar{c}_1}{\mu_0 s}\bar{\tau}_{xy},$$
$$\tau_{yy} = \frac{\bar{c}_1}{\mu_0 s}\bar{\tau}_{yy}, \quad \mu = \frac{\bar{\mu}}{\mu_0}, \quad \epsilon = \eta T_w,$$
$$\alpha = \zeta T_w, \quad \theta_{\text{temp}} = \frac{\bar{T} - T_w}{T_w}, \quad \dot{\gamma} = \frac{\bar{\dot{\gamma}}\bar{c}_1}{s}, \tag{22}$$
$$\delta = \frac{\bar{c}_1}{\lambda}, \quad Re = \frac{\rho s c_1}{\mu_0}, \quad W_z = \frac{\Gamma s}{c_1},$$
$$P = \frac{\bar{c}_1^2}{s\lambda\mu_0}\bar{P}, \quad e = \frac{\bar{e}_1}{c_1}, \quad Pr = \frac{\mu_0 c_p}{K_0},$$
$$Ec = \frac{c}{T_w}, \quad Bk = \frac{\mu_0 c^2}{T_w K_0}, \quad K = \frac{\bar{K}}{K_0}. \tag{23}$$

After utilizing the dimensionless quantities and then solving the above equations.

$$\delta Re\Big[\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big)u\Big] = -\frac{\partial p}{\partial x} + \delta^2\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}, \tag{24}$$

$$-\delta Re\Big[\Big(u\frac{\partial}{\partial x} - v\frac{\partial}{\partial y}\Big)v\Big] = -\frac{\partial p}{\partial y} + \delta^2\frac{\partial \tau_{xy}}{\partial x} + \delta\frac{\partial \tau_{yy}}{\partial y}, \tag{25}$$

$$\delta Re\Big[(u + 1)\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y}\Big]$$
$$= \delta^2\frac{1}{Pr}\frac{\partial}{\partial x}\Big[(1 + \alpha\theta)\frac{\partial \theta}{\partial x}\Big] + \frac{1}{Pr}\frac{\partial}{\partial y}\Big[(1 + \alpha\theta)\frac{\partial \theta}{\partial y}\Big]$$
$$+ \delta Ec\frac{\tau_{xx}\partial(u + 1)}{\partial x} + \delta Ec\frac{\tau_{yy}\partial v}{\partial y}$$
$$+ Ec\tau_{xy}\Big(\delta^2\frac{\partial v}{\partial x} + \frac{\partial(u + 1)}{\partial y}\Big), \tag{26}$$

$$\tau_{xx} = 2\delta(1-\epsilon\theta)\Big[1+W_z\dot{\gamma}\Big]\frac{\partial u}{\partial x}\,, \tag{27}$$

$$\tau_{xy} = (1-\epsilon\theta)[1+W_z\dot{\gamma}]\Big(\frac{\partial u}{\partial y}+\delta\frac{\partial v}{\partial x}\Big)\,, \tag{28}$$

$$\tau_{yy} = 2(1-\epsilon\theta)[1+W_z\dot{\gamma}]\frac{\partial v}{\partial y}\,, \tag{29}$$

$$\dot{\gamma} = 2\delta^2\Big(\frac{\partial u}{\partial x}\Big)^2+\Big(\frac{\partial u}{\partial y}-\delta^2\frac{\partial v}{\partial x}\Big)^2+2\delta^2\Big(\frac{\partial v}{\partial y}\Big)^2\,. \tag{30}$$

Now introducing stream function $\psi(u = \partial\psi/\partial y,\ v = -\delta\partial\psi/\partial x)$, we arrive at

$$\delta Re\Big[\Big(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Big)\frac{\partial\psi}{\partial y}\Big] = -\frac{\partial p}{\partial x}+\delta^2\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}\,, \tag{31}$$

$$-\delta^1 Re\Big[\Big(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\Big)\frac{\partial\psi}{\partial x}\Big] = -\frac{\partial p}{\partial y}+\delta^2\frac{\partial\tau_{xy}}{\partial x}+\delta\frac{\partial\tau_{yy}}{\partial y}\,, \tag{32}$$

$$\delta Re\Big[\Big(\frac{\partial\psi}{\partial y}+1\Big)\frac{\partial\theta}{\partial x}-\Big(\delta\frac{\partial\psi}{\partial x}\Big)\frac{\partial\theta}{\partial y}\Big] = \delta^2\frac{1}{Pr}\frac{\partial}{\partial x}\Big[(1+\alpha\theta)\frac{\partial\theta}{\partial x}\Big]+\frac{1}{Pr}\frac{\partial}{\partial y}\Big[(1+\alpha\theta)\frac{\partial\theta}{\partial y}\Big]$$
$$+\delta Ec\frac{\tau_{xx}\partial}{\partial x}\Big(\frac{\partial\psi}{\partial y}+1\Big)-\delta^2 Ec\frac{\tau_{yy}\partial}{\partial y}\Big(\frac{\partial\psi}{\partial x}\Big) \tag{33}$$
$$+Ec\ \tau_{xy}\Big(-\delta^1\frac{\partial}{\partial x}\Big(\frac{\partial\psi}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(\frac{\partial\psi}{\partial y}+1\Big)\Big)\,, \tag{34}$$

$$\tau_{xx} = 2\delta(1-\epsilon\theta)\Big[1+W_z\dot{\gamma}\Big]\frac{\partial^2\psi}{\partial x\partial y}\,, \tag{35}$$

$$\tau_{xy} = (1-\epsilon\theta)[1+W_z\dot{\gamma}]\Big(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\Big)\,, \tag{36}$$

$$\tau_{yy} = -2\delta(1-\epsilon\theta)[1+W_z\dot{\gamma}]\frac{\partial^2\psi}{\partial x\partial y}\,, \tag{37}$$

$$\dot{\gamma} = \Big[2\delta^2\Big(\frac{\partial^2\psi}{\partial x\partial y}\Big)^2+\Big(\frac{\partial^2\psi}{\partial y^2}-\delta^2\frac{\partial^2\psi}{\partial x^2}\Big)^2+2\delta^2\Big(\frac{\partial^2\psi}{\partial x\partial y}\Big)^2\Big]^{1/2}\,. \tag{38}$$

Here W_z, Re, Ec, Pr and B_k represent the Weissenberg, Reynolds, Eckert, and Brinkman numbers respectively whereas δ is the wave number. Now applying the approximations of long wave and ignoring the terms of order δ and higher

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\tau_{xy}\,,\quad \frac{\partial p}{\partial y} = 0\,, \tag{39}$$

$$\frac{\partial}{\partial y}\Big[(1+\alpha\theta)\frac{\partial\theta}{\partial y}\Big]+Bk\Big\{(1-\epsilon\theta)\Big[\Big(\frac{\partial^2\psi}{\partial y^2}\Big)^2+W_z\Big(\frac{\partial^2\psi}{\partial y^2}\Big)^3\Big]\Big\} = 0\,, \tag{40}$$

$$\tau_{xx} = 0\,,\quad \tau_{yy} = 0\,,$$
$$\tau_{xy} = (1-\epsilon\theta)[1+W_z\dot{\gamma}]\Big(\frac{\partial^2\psi}{\partial y^2}\Big)\,,\quad \dot{\gamma} = \frac{\partial^2\psi}{\partial y^2}\,. \tag{41}$$

Utilizing shear stress from above

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\Big[(1-\epsilon\theta)[1+W_z\dot{\gamma}]\Big(\frac{\partial^2\psi}{\partial y^2}\Big)\Big]\,. \tag{42}$$

Now eliminating pressure, we arrive at

$$\frac{\partial^2}{\partial y^2}\Big[(1-\epsilon\theta)[1+W_z\dot{\gamma}]\Big(\frac{\partial^2\psi}{\partial y^2}\Big)\Big] = 0\,. \tag{43}$$

2.4 *Boundary Conditions*

By aid of stream function ψ, boundary conditions are defined as:

$$\psi = 0\,,\quad \frac{\partial\psi}{\partial y} = 0\,,\quad \frac{\partial\theta_{\rm temp}}{\partial y} = 0\,,\quad \text{for}\quad y = 0\,, \tag{44}$$

$$\psi = F\,,\quad \frac{\partial\psi}{\partial y} = -1\,,\quad \theta_{\rm temp} = 0\,,$$
$$\text{for}\quad y = h(x) = 1+e\cos(2\pi x)\,. \tag{45}$$

2.5 *Volume Flow Rate*

The volume flow rate in the fixed frame is given by

$$\bar{Q} = \int_0^{\bar{h}(\bar{X},\bar{t})}\bar{U}(\bar{X},\bar{Y},\bar{t})\,{\rm d}\bar{Y}\,. \tag{46}$$

In the wave frame, the volume flow rate is defined as

$$q = \int_0^h u(\bar{x},\bar{y})\,{\rm d}\bar{y}\,. \tag{47}$$

The two rates of volume flow are related through

$$Q = q+s\bar{h}(\bar{x})\,. \tag{48}$$

Over a period T, the time mean flow is defined as

$$\bar{Q} = \frac{1}{T}\int_0^T Q\,{\rm d}t\,, \tag{49}$$

$$\bar{Q} = \bar{q}+c_1 s\,. \tag{50}$$

In the wave frame, F and θ the dimensionless time mean flow, are given by

$$F = \frac{q}{c_1 s}\,,\quad \theta = \frac{\bar{Q}}{c_1 s}\,, \tag{51}$$

$$\theta = F+1\,, \tag{52}$$

where

$$F = \int_0^{h(x)}\frac{\partial\psi}{\partial y}\,{\rm d}y = \psi(h(x))-\psi(0)\,. \tag{53}$$

3 Perturbation Solution

The closed form solution of the system of equations that comprises of non linear coupled differential equations is very challenging to find so, by using asymptotic analysis we produce the series solution. We take thermal conductivity parameter α and viscosity parameter ϵ, of the same order of magnitude and asymptotically small, for the purpose of obtaining this. It may also be noticed that thermal conductivity parameter ζ and viscosity parameter η are of same dimension $1/T$. So the heat equation can be written as:

$$\frac{\partial}{\partial y}(1+\epsilon\theta)\frac{\partial\theta}{\partial y}+Bk(1-\epsilon\theta)\Big(\frac{\partial^2\psi}{\partial y^2}\Big)^2+W_z\Big(\frac{\partial^2\psi}{\partial y^2}\Big)^1 = 0\,. \tag{54}$$

For finding the solution we apply the regular perturbation method. We expand ψ, F, and P about fluid parameter W_z and ϵ

$$\psi=(\psi_{00}+W_z\psi_{01}+W_z^2\psi_{02})+\epsilon(\psi_{10}+W_z\psi_{11}+W_z^2\psi_{12})\,,\quad W_z<1\,,\quad \epsilon<1\,, \tag{55}$$

$$F=(F_{00}+W_zF_{01}+W_z^2F_{02})+\epsilon(F_{10}+W_zF_{11}+W_z^2F_{12})\,,\quad W_z<1\,,\quad \epsilon<1\,, \tag{56}$$

$$P=(P_{00}+W_zP_{01}+W_z^2P_{02})+\epsilon(P_{10}+W_zP_{11}+W_z^2P_{12})\,,\quad W_z<1\,,\quad \epsilon<1\,, \tag{57}$$

$$\theta=(\theta_{00}+W_z\theta_{01}+W_z^2\theta_{02})+\epsilon(\theta_{10}+W_z\theta_{11}+W_z^2\theta_{12})\,,\quad W_z<1\,,\quad \epsilon<1\,. \tag{58}$$

Now substituting the above expressions, we obtain the systems given below:

3.1 *Order* (W_z^o,ϵ^0) *System*

$$\frac{\partial^2}{\partial y^2}\Big[\frac{\partial^2\psi_{00}}{\partial y^2}\Big]=0\,, \tag{59}$$

$$\frac{\partial p_0}{\partial x}=\frac{\partial}{\partial y}\Big[\frac{\partial^2\psi_{00}}{\partial y^2}\Big]\,, \tag{60}$$

$$\frac{\partial}{\partial y}\Big[\frac{\partial\theta_{00}}{\partial y}\Big]+Bk\Big[\frac{\partial^2\psi_{00}}{\partial y^2}\Big]^2=0\,, \tag{61}$$

$$\psi_{00}=0\,,\quad \frac{\partial^2\psi_{00}}{\partial y^2}=0\,,\quad \frac{\partial\theta_{00}}{\partial y}=0\,,\quad \text{for}\quad y=0\,, \tag{62}$$

$$\psi_{00}=F_{00}\,,\quad \frac{\partial\psi_{00}}{\partial y}=-1\,,\quad \theta_{00}=0\,,\quad \text{for}\quad y=h(x)=1+e\cos(2\pi x)\,. \tag{63}$$

3.2 *Order* (W_z^1,ϵ^0) *System*

$$\frac{\partial^2}{\partial y^2}\Big[\frac{\partial^2\psi_{01}}{\partial y^2}+\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)^2\Big]=0\,, \tag{64}$$

$$\frac{\partial p_1}{\partial x}=\frac{\partial}{\partial y}\Big[\frac{\partial^2\psi_{01}}{\partial y^2}+\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)^2\Big]\,, \tag{65}$$

$$\frac{\partial}{\partial y}\Big[\frac{\partial\theta_{01}}{\partial y}\Big]+Br\Big[2\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)\Big(\frac{\partial^2\psi_{01}}{\partial y^2}\Big)\Big]+Bk\Big[\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)^3\Big]=0\,, \tag{66}$$

$$\psi_{01}=0\,,\quad \frac{\partial^2\psi_{01}}{\partial y^2}=0\,,\quad \frac{\partial\theta_{01}}{\partial y}=0\,,\quad \text{for}\quad y=0\,, \tag{67}$$

$$\psi_{01}=F_{01}\,,\quad \frac{\partial\psi_{01}}{\partial y}=-1\,,\quad \theta_{01}=0\,,\quad \text{for}\quad y=h(x)=1+e\cos(2\pi x)\,. \tag{68}$$

3.3 *Order* (W_z^0,ϵ^1) *System*

$$\frac{\partial^2}{\partial y^2}\Big[\frac{\partial^2\psi_{10}}{\partial y^2}-\theta_{00}\frac{\partial^2\psi_{00}}{\partial y^2}\Big]=0\,, \tag{69}$$

$$\frac{\partial p_2}{\partial x}=\frac{\partial}{\partial y}\Big[\frac{\partial^2\psi_{10}}{\partial y^2}-\theta_{00}\frac{\partial^2\psi_{00}}{\partial y^2}\Big]\,, \tag{70}$$

$$\frac{\partial}{\partial y}\Big[\frac{\partial\theta_{10}}{\partial y}+\theta_{00}\frac{\partial\theta_{00}}{\partial y}\Big]+Bk\Big[2\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)\Big(\frac{\partial^2\psi_{10}}{\partial y^2}\Big)-\theta_{00}\Big(\frac{\partial^2\psi_{00}}{\partial y^2}\Big)^2\Big]=0\,, \tag{71}$$

$$\psi_{10}=0\,,\quad \frac{\partial^2\psi_{10}}{\partial y^2}=0\,,\quad \frac{\partial\theta_{10}}{\partial y}=0\,,\quad \text{for}\quad y=0\,, \tag{72}$$

$$\psi_{10}=F_{10}\,,\quad \frac{\partial\psi_{10}}{\partial y}=-1\,,\quad \theta_{10}=0\,,\quad \text{for}\quad y=h(x)=1+e\cos(2\pi x)\,. \tag{73}$$

3.4 *Solution for System of Order* (W_z^0,ϵ^0)

$$\psi_{00}=\frac{1}{2h^1}(1F_{00}+h)y-(F_{00}+h)y^1\,, \tag{74}$$

$$\frac{\mathrm{d}p_0}{\mathrm{d}x}=-\frac{1}{h^1}[1(F_{00}+h)]\,, \tag{75}$$

$$\theta_{00}=\frac{1}{4h^6}[1Bk(F_{00}+h)^2(h^4-y^4)]\,. \tag{76}$$

3.5 *Solution for System of Order* (W_z^1,ϵ^0)

$$\psi_{01}=\frac{1}{8h^6}[-1h^1(F_{00}^2+2F_{00}h+(1-4F_{01})h^2)y+h(9F_{00}^2+18F_{00}h+(9-4F_{01})h^2)y^1-6(F_{00}+h)^2y^4]\,, \tag{77}$$

$$\frac{\mathrm{d}p_1}{\mathrm{d}x}=\frac{1}{4h^6}\Big[1(h^2((9-4F_{01})h-24y)+1F_{00}^2(1h-8y)+6F_{00}h(1h-8y))+2\Big(-\frac{1}{h^1}(1F_{00}+h)y\Big)\Big(-\frac{1}{h^1}(1F_{00}+h)\Big)\Big]\,, \tag{78}$$

$$\theta_{01}=\frac{1}{40h^9}[1Bk(-27F_{00}^1h^5-81F_{00}^2h^6-81F_{00}h^7+20F_{00}F_{01}h^7-27h^8+20F_{01}h^8+45F_{00}^1hy^4+115F_{00}^2h^2y^4+115F_{00}h^1y^4-20F_{00}F_{01}h^1y^4+45h^4y^4-20F_{01}h^4y^4-18F_{00}^1y^5-54F_{00}^2hy^5-54F_{00}h^2y^5-18h^1y^5)]\,.$$

3.6 *Solution for System of Order* (W_z^0,ϵ^1)

$$\psi_{10}=\frac{1}{56h^9}(6BkF_{00}h^6y+18BkF_{00}h^7y+18BkF_{00}h^8y+84F_{10}h^8y+6Bkh^9y-9BkF_{00}^1h^4y^1-27BkF_{00}^2h^5y^1-27BkF_{00}h^6y^1-28F_{10}h^6y^1-9Bkh^7y^1+1BkF_{00}^1y^7+9BkF_{00}^2hy^7+9BkF_{00}h^2y^7+1Bkh^1y^7)\,, \tag{79}$$

$$\frac{\mathrm{d}p_2}{\mathrm{d}x} = \frac{1}{7h^5}[1(-72h^2 + 1Br(F_0 + h)^1)]\,, \tag{80}$$

$$\begin{aligned}\theta_{10} = \frac{1}{56h^{12}}[&19Br^2F_0^4h^8 + 16Br^2F_0h^9 + 54Br^2F_0^2h^{10} - 28BrF_0F_2h^{10} + 16Br^2F_0h^{11}\\ &- 28BrF^2h^{11} + 9Br^2h^{12} - 12Br^2F_0^4h^4y^4 - 48Br^2F_0^1h^5y^4 - 72Br^2F_0^2h^6y^4\\ &+ 28BrF_0F_2h^6y^4 - 48Br^2F_0h^7y^4 + 28BrF_2h^7y^4 - 12Br^2h^8y^4 + 1Br^2F_0^4y^8\\ &+ 12Br^2F_0^1hy^8 + 18Br^2F_0^2h^2y^8 + 12Br^2F_0h^1y^8 + 1Br^2h^4y^8]\,.\end{aligned} \tag{81}$$

Using solution of above systems and

$$F_{00} = F - W_zF_{01} - \epsilon F_{10}\,, \tag{82}$$

net results could be stated as:

$$\begin{aligned}\psi = &-\frac{1}{2h^1}(h^2(1F + h)y - (F + h)y^1)\\ &+ W_z\Big[-\frac{1}{8h^6}(1F + h)^2(h - y)^2y(h + 2y)\Big]\\ &+ \epsilon\Big[\frac{1}{56h^9}(1Bk(F + h)^1(2h^6y - 1h^4y^1 + y^7))\Big]\,,\end{aligned} \tag{83}$$

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}x} = &-\frac{1}{h^1}(F + h) + W_z\Big[\frac{27}{4h^5}(F + h)^2\Big]\\ &+ \epsilon\Big[\frac{1}{7h^5}(9Bk(F + h)^1)\Big]\,,\end{aligned} \tag{84}$$

$$\begin{aligned}\theta = &\frac{1Bk}{4h^6}(F + h)^2(h^4 - y^4) + W_z\\ &-\frac{1}{40h^9}(27Bk(F_0 + h)^1(1h^5 - 5hy^4 + 2y^5))\\ &\times \epsilon\Big[-\frac{9Bk^2}{56h^{12}}((F + h)^4(1h^8 - 4h^4y^4 + y^8))\Big]\,.\end{aligned} \tag{85}$$

The expression for pressure rise and heat transfer coefficient is

$$\Delta P_\lambda = \int_0^1 \frac{\mathrm{d}P}{\mathrm{d}x}\mathrm{d}x\,, \tag{86}$$

$$Z_T = \frac{\partial\theta}{\partial y}\frac{\partial h}{\partial x}\Big|_{y=h}\,. \tag{87}$$

4 Discussion

Influence of variable fluid properties on peristaltic flow of Williamson fluid has been discussed. The salient features of several physical parameters like velocity, pressure rise per wavelength, pressure gradient, heat transfer coefficient, temperature and streamlines have been described graphically. The reduced version of present study for fluid parameter W_z and Brinkman number Bk are in agreement with studies.[22–23]

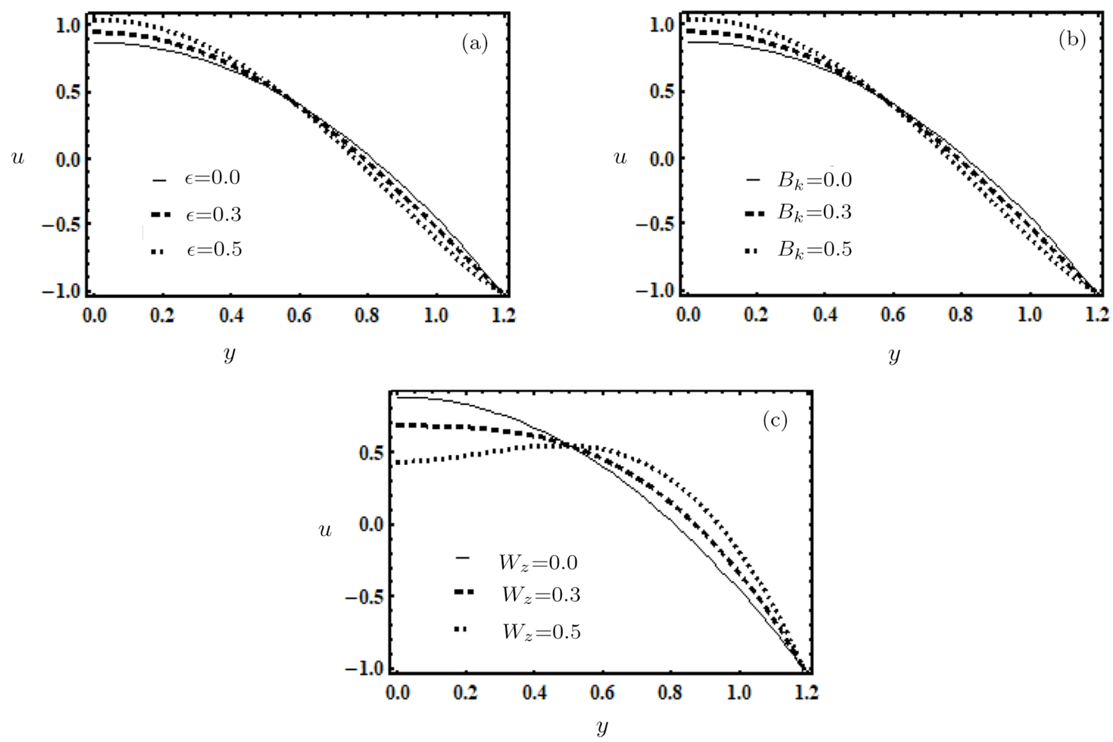

Fig. 2 (a) Influence of ϵ on u for $W_z = 0.01$, $e = 0.6$, $\theta = 1.1$, $x = 0.2$, and $Bk = 0.9$. (b) Influence of Bk on u for $W_z = 0.01$, $e = 0.6$, $\theta = -1.5$, $\epsilon = 0.1$, and $x = 0.2$. (c) Influence of W_z on u for $Bk = 2$, $e = 0.6$, $\theta = -1.5$, $\epsilon = 0.1$, and $x = 0.2$.

Figure 2 depicts the behavior of ϵ, Bk and W_z on velocity. At the center of the channel and near the channel walls, the behavior of the velocity is opposite. Figure 2(a) shows that at the center of the channel, the velocity increases as ϵ increases whereas the velocity decreases near the channel wall as ϵ increases. Brinkman number is the ratio between heat transported by molecular conduction and production of heat by viscous dissipation. Figure 2(b) shows that velocity increases at the center of the channel as Bk increases while velocity decreases at the center of the channel as Bk increases. Figure 2(c) represents decrease in velocity at the center of the channel as W_z increases.

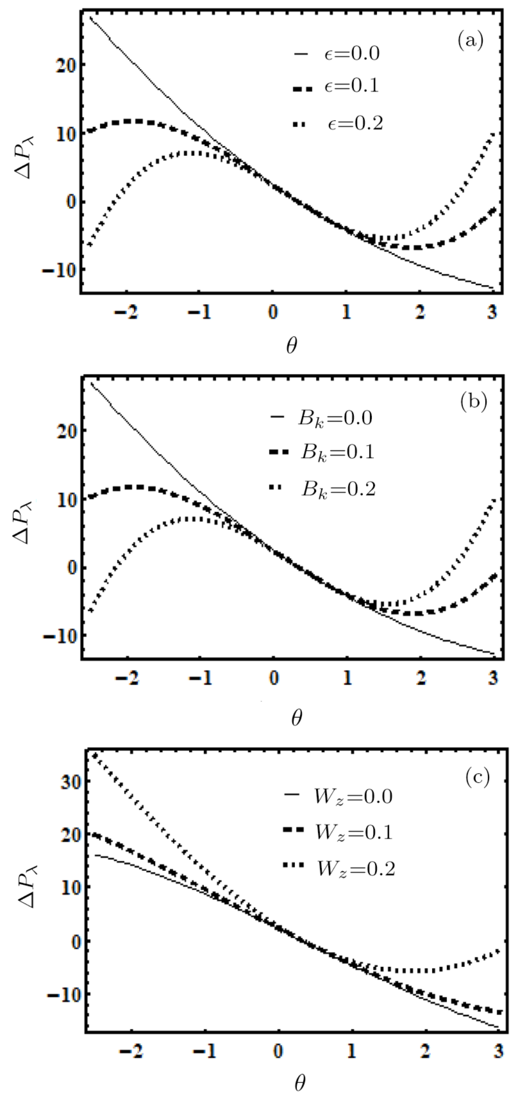

Fig. 3 (a) Influence of ϵ on ΔP_λ for $Bk = 0.8$, $e = 0.5$, and $W_z = 0.02$. (b) Influence of Bk on ΔP_λ for $\epsilon = 0.8$, $e = 0.5$, and $W_z = 0.02$. (c) Influence of W_z on ΔP_λ for $Bk = 0.8$, $e = 0.5$, and $\epsilon = 0.02$.

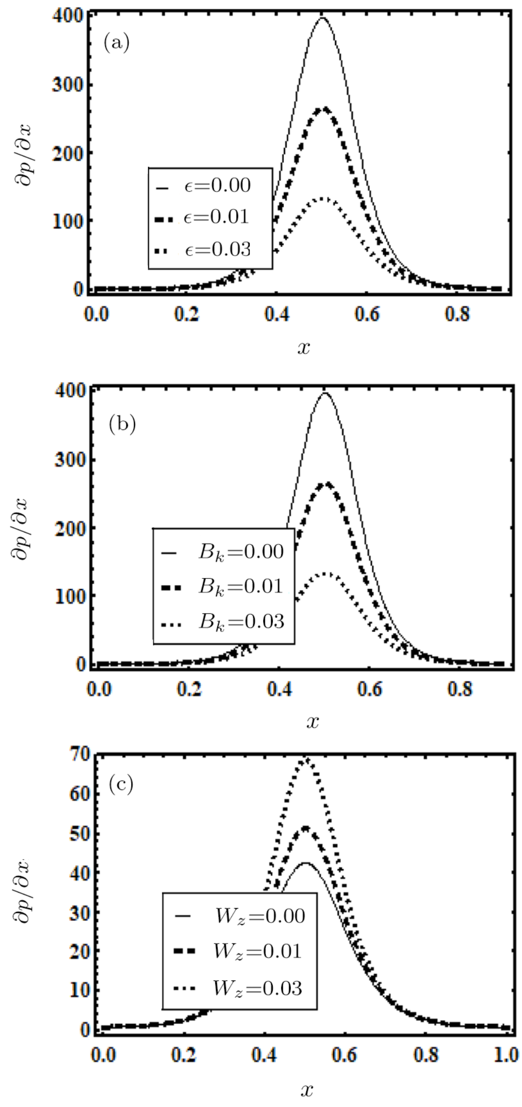

Fig. 4 (a) Influence of ϵ on $\mathrm{d}p/\mathrm{d}x$ for $Bk = 0.8$, $e = 0.5$, and $W_z = 0.02$. (b) Influence of Bk on $\mathrm{d}p/\mathrm{d}x$ for $W_z = 0.8$, $e = 0.5$, and $\epsilon = 0.02$. (c) Influence of W_z on $\mathrm{d}p/\mathrm{d}x$ for $Bk = 0.8$, $e = 0.5$, and $\epsilon = 0.02$.

Figure 3 shows the behavior of ϵ, Bk and W_z on pressure rise. The pumping against pressure rise is the most significant aspect of peristalsis. The retrograde pumping region is where $\Delta P_\lambda > 0$ and $\theta < 0$. The fluid flow in this region is due to pressure gradient. The region where $\Delta P_\lambda > 0$ and $\theta > 0$ is known as peristaltic pumping region. The fluid that is moved in forward direction and the peristalsis of walls in this region overcomes the resistance of pressure gradient. The free pumping zone is where $\Delta P_\lambda = 0$ and the volume flow rate θ is known as free pumping flux. In the region where $\Delta P_\lambda < 0$ and $\theta > 0$ is the copumping region. It is observed that increase in ϵ means increase in the thermal conductivity/variable viscosity. Figure 3(a) shows that the pressure decreases as ϵ increases in the retrograde region while in the copumping region it behaves oppositely. The effect on ΔP_λ for Bk is the same as that of ϵ in Fig. 3(b). It is noticed that ΔP_λ in Fig. 3(c), increases as W_z increases in the retrograde region whereas its behavior is opposite in copumping region. It is noticed that in the peristaltic pumping region, ΔP_λ shows no deviation under all type of variations.

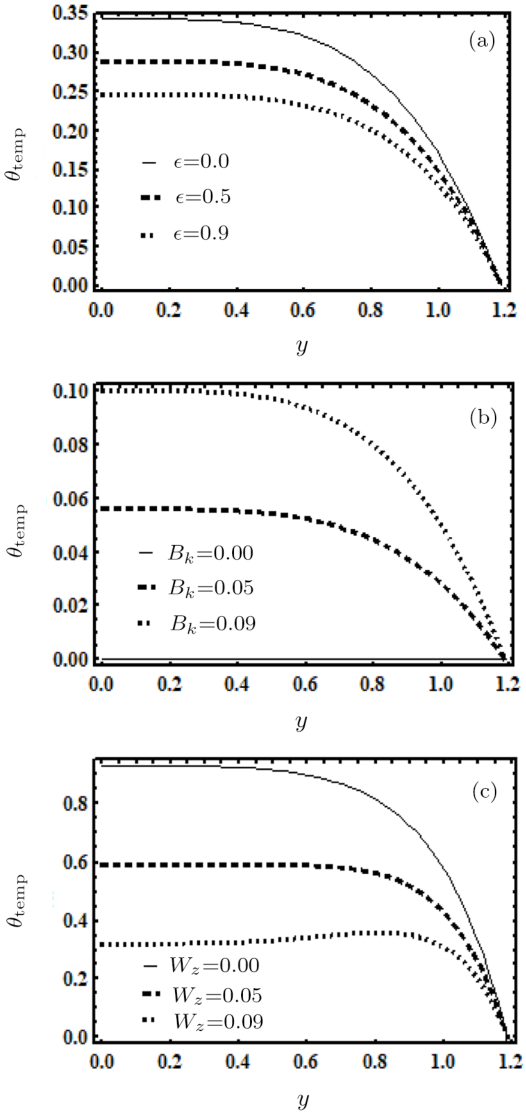

Fig. 5 (a) Influence of ϵ on $\theta_{\rm temp}$ for $Bk = 0.1$, $e = 0.6$, $x = 0.2$, and $W_z = 0.01$. (b) Influence of Bk on $\theta_{\rm temp}$ for $W_z = 0.01$, $e = 0.6$, $\theta = 1.1$, $x = 0.2$, and $\alpha = 0.1$. (c) Influence of W_z on $\theta_{\rm temp}$ for $Bk = 2$, $e = 0.5$, $x = 0.2$, $\theta = 1.1$, and $\epsilon = 0.1$.

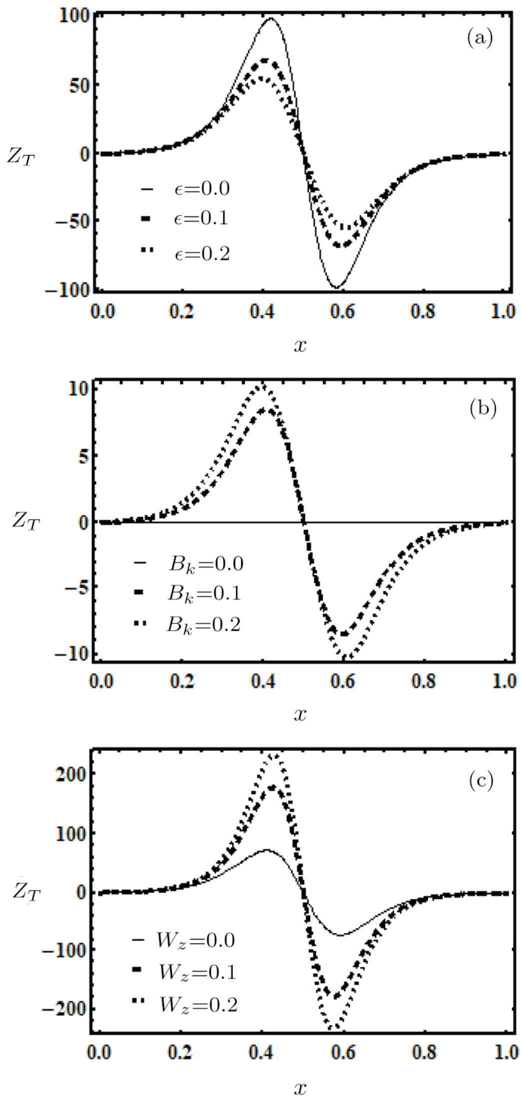

Fig. 6 (a) Influence of ϵ on Z_T for $Bk = 0.8$, $e = 0.5$, $x = 0.2$, $\theta = -1.5$, and $W_z = 0.02$. (b) Influence of Bk on Z_T for $\epsilon = 0.8$, $e = 0.5$, $x = 0.2$, $\theta = -1.5$, and $W_z = 0.02$. (c) Influence of W_z on Z_T for $Bk = 0.8$, $e = 0.5$, $x = 0.2$, $\theta = -1.5$, and $\epsilon = 0.02$.

Figure 4 illustrates the behavior of ϵ, Bk and W_z on pressure gradient. It is observed that at the wider part of the channel when $x = 0$, the pressure gradient is very small. This can be justified physically because without the assistance of huge pressure gradient, the fluid can pass easily. Whereas in the narrow part of the channel huge pressure gradient is required for maintaining the same flux of fluid to pass through it. Figures 4(a) and 4(b) show that the pressure gradient decreases as ϵ and Bk increase. In Fig. 4(c) pressure gradient increases as W_z increases.

Figure 5 depicts the behavior of ϵ, Bk and W_z on temperature. Here $\theta_{\rm temp}$ is plotted against y. Figure 5(a) shows that when $\theta_{\rm temp}$ decreases ϵ increases. While Figs. 5(b) and 5(c) show increase of $\theta_{\rm temp}$ as Bk and W_z increase.

Figure 6 depicts the variation of heat transfer coefficient Z_T for several values of parameters at $y = h(x)$. Figure 6(a) shows that as ϵ increases, the value of Z_T decreases. Whereas in Figs. 6(b) and 6(c) Z_T increases by increasing the values of Bk and W_z.

An important process in the transport of the fluid is trapping. Under some conditions, a bolus which is trapped is enclosed by the splitting of streamlines and it is carried out along the wave in the wave frame. The following process is known as trapping. Figure 7 represents the phenomenon of trapping by sketching streamlines. The bolus which is trapped, an increasing behavior is found, as the size of the bolus increases by increasing the parameters ϵ, Bk, and W_z respectively.

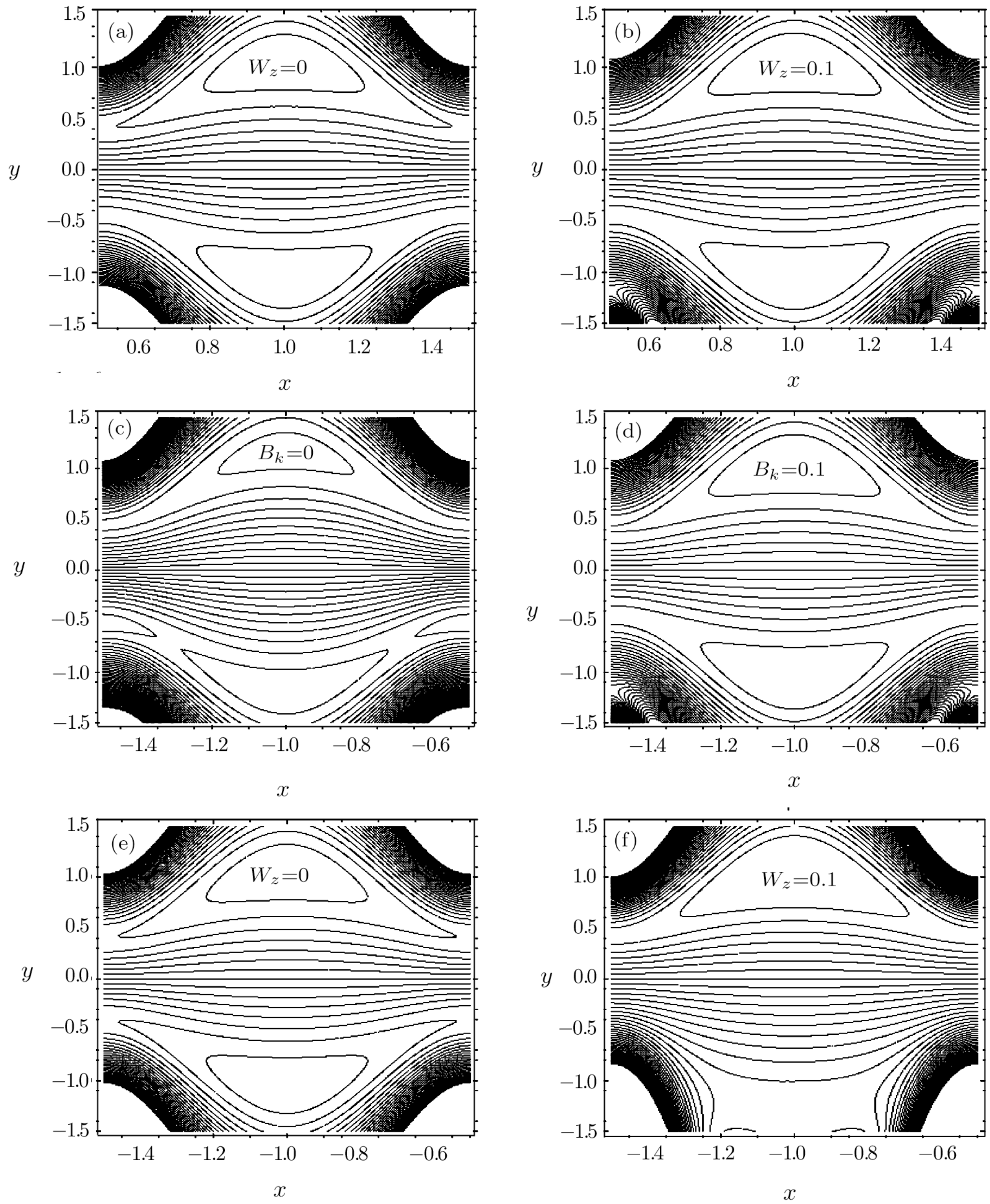

Fig. 7 (a)–(b) Influence of streamlines for values of for $\epsilon = 0$, 0.1, $Bk = 0.2$, $e = 0.5$, $W_z = 0.01$, and $\theta = 1.5$. (c)–(d) Influence of streamlines for values of for $Bk = 0$, 0.1, $\epsilon = 0.2$, $e = 0.5$, $W_z = 0.01$, and $\theta = 1.5$. (e)–(f) Influence of streamlines for values of for $W_z = 0$, 0.1, $Bk = 0.2$, $e = 0.5$, $\epsilon = 0.01$, and $\theta = 1.5$.

5 Conclusion

In the following paper we have examined the influence of variable fluid properties on peristaltic flow of Williamson fluid. By the help of perturbation method series solutions are found. The observations are concluded as follows:

(i) It is observed that the behavior of ϵ on pressure gradient and pressure rise per wavelength, is opposite.

(ii) It is seen that W_z and ϵ, in the narrow part of the channel cause better variation as compared to the wider part of the channel.

(iii) As W_z and ϵ increase, the pressure gradient decreases.

(iv) It is seen that when temperature increases thermal conductivity also increases whereas the temperature has negative relation with viscosity.

(v) The heat transfer coefficient is lower for a fluid with variable thermal conductivity and variable viscosity as compared to the fluid with constant thermal conductivity and constant viscosity.

(vi) When the values of ϵ and W_z increase, the bolus which is trapped, its size increases.

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