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Uncertainty Relations for Coherence*

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Abstract *Quantum mechanical uncertainty relations are fundamental consequences of the incompatible nature of noncommuting observables. In terms of the coherence measure based on the Wigner-Yanase skew information, we establish several uncertainty relations for coherence with respect to von Neumann measurements, mutually unbiased bases (MUBs), and general symmetric informationally complete positive operator valued measurements (SIC-POVMs), respectively. Since coherence is intimately connected with quantum uncertainties, the obtained uncertainty relations are of intrinsically quantum nature, in contrast to the conventional uncertainty relations expressed in terms of variance, which are of hybrid nature (mixing both classical and quantum uncertainties). From a dual viewpoint, we also derive some uncertainty relations for coherence of quantum states with respect to a fixed measurement. In particular, it is shown that if the density operators representing the quantum states do not commute, then there is no measurement (reference basis) such that the coherence of these states can be simultaneously small.*

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1 Introduction

The Heisenberg uncertainty principle arising from incompatible (noncommuting) observables asserts a fundamental limit to quantum measurements, and is one of the characteristic consequences of quantum mechanics with deep connections to the Bohr complementarity principle. Uncertainty relations, as manifestations of the Heisenberg uncertainty principle, have been extensively and intensively studied with a wide range of applications in many fields. For example, it is closely related to quantum measurement and signal processing,[1−2] preparation of state,[3] complementarity,[4−5] entanglement detections,[6−9] quantum coherence,[10−13] quantum non-locality.[14−16] There are various quantitative characterizations of uncertainty relations such as entropic uncertainty relations,[17−20] uncertainty relations based on variance and the Wigner-Yanase skew information,[21−25] and so on.

Coherence is intrinsically related to superposition which differentiates quantum mechanics from classical mechanics. In recent years, there are increasing interests in quantitative studies of coherence.[26−40] As a kind of quantum resource, coherence plays an important role in a variety of operational applications in asymmetry,[41−45] metrology,[46] quantum key distributions,[47] thermodynamics,[48−50] quantum computation and communication.[51−53]

Since coherence of a quantum state depends on the choice of measurements (reference bases), it is natural to study the relations of coherence between two or more different measurements, or coherence of different states with respect to a fixed measurement. In fact, there are several investigations on uncertainty relations for quantum coherence,[10−13] as well as the complementarity of coherence in different bases.[54−55] Based on the skew information introduced by Wigner and Yanase,[56] an information-theoretic measure of coherence has been introduced in Refs. [35–37] (see Sec. 2), which has an operational interpretation as quantum uncertainty, in sharp contrast to the conventional notion of variance (which usually involves both classical and quantum uncertainties). The aim of this paper is to employ this coherence measure to characterize uncertainty relations for coherence in an intrinsically quantum fashion.

The paper is organized as follows. In Sec. 2, we establish several uncertainty relations for coherence with respect to arbitrary von Neumann measurements (orthonormal bases), as well as with respect to mutually unbiased bases (MUBs). In Sec. 3, we study coherence with respect to general symmetric informationally complete positive operator valued measurements (SIC-POVMs), and derive the corresponding uncertainty relations. We obtain some uncertainty relations for several quantum states with respect to a common measurement in Sec. 4. Finally, we summarize in Sec. 5.

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2 Uncertainty Relations for Coherence with Respect to von Neumann Measurements

Let ρ be a quantum state (density operator) and $\Pi = \{|i\rangle\langle i|\}$ be a von Neumann measurement (i.e., $\{|i\rangle\}$ constitutes an orthonormal basis for the system Hilbert space), then one can consider coherence of ρ with respect to Π. In this paper, we employ the coherence measure

$$C(\rho,\Pi) = \sum_i I(\rho, |i\rangle\langle i|)\,, \tag{1}$$

introduced in Refs. [35–37] to characterize uncertainty relations. Here

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2$$

is the skew information introduced by Wigner and Yanase in 1963,[56] H is an arbitrary observable (Hermitian operator). It is remarkable that the skew information enjoys many nice properties and has several interpretations as non-commutativity (between ρ and H), quantum Fisher information (of ρ with respect to H), asymmetry (of ρ with respect to H), quantum uncertainty (of H in ρ),[21–24,37–38] etc. More generally, for any POVM $M = \{M_i\}$ (i.e., $M_i \geq 0, \sum_i M_i = \mathbf{1}$), we may define a bona fide measure for coherence of ρ with respect to M as[37]

$$C(\rho, M) = \sum_i I(\rho, M_i)\,. \tag{2}$$

The above measure reduces to that defined by Eq. (1) when M is a von Neumann measurement. In general, the measurement operators in M may not be mutually orthogonal projections (e.g., coherent states in quantum optics and spin systems), and in this case $C(\rho, M)$ generalizes $C(\rho,\Pi)$ considerably.

Let $\Pi_1 = \{|u_i\rangle\langle u_i|\}$, $\Pi_2 = \{|v_i\rangle\langle v_i|\}$ be two arbitrary von Neumann measurements, then our first result is the following inequality

$$C(\rho,\Pi_1) + C(\rho,\Pi_2) \geq \frac{1}{2}\|[\sqrt{\rho}, U^\dagger\sqrt{\rho}U]\|_F^2\,, \tag{3}$$

which may be regarded as a kind of uncertainty relations for coherence of the quantum state ρ with respect to the measurements Π_1 and Π_2. Here $||A||_F = (\mathrm{tr}A^\dagger A)^{1/2}$ is the Frobenius norm, U is the unitary operator defined by $U|u_i\rangle = |v_i\rangle$, i.e., $U = \sum_i |v_i\rangle\langle u_i|$.

To prove inequality (3), we first recall a mathematical result in Ref. [57], Remark 5.1: If one of the matrices A and B is non-negative, then

$$\|[A, B]\|_F \leq \|A\|_F\|B\|_F\,. \tag{4}$$

Similar to the method in Ref. [58], consider the decompositions of the matrices

$$\sqrt{\rho} = X + D, \qquad U^\dagger\sqrt{\rho}U = Y + D_U\,, \tag{5}$$

where D and D_U are the diagonal parts of ρ and $U^\dagger\rho U$ respectively. It is obvious that X, Y have zero diagonal elements. Then by inequality (4), the triangle inequality of the norm $\|\cdot\|_F$, and the elementary inequality $x^2 + y^2 \geq (x+y)^2/2$ for any real numbers x, y, we have

$$\begin{aligned}
\|[\sqrt{\rho}, U^\dagger\sqrt{\rho}U]\|_F^2 &= \|[X+D, Y+D_U]\|_F^2 \\
&= \|[X+D, Y] + [X+D, D_U]\|_F^2 \\
&\leq (\|[X+D, Y]\|_F + \|[X+D, D_U]\|_F)^2 \\
&= (\|[X+D, Y]\|_F + \|[X, D_U]\|_F)^2 \\
&\leq (\|\sqrt{\rho}\|_F\|Y\|_F + \|X\|_F\|D_U\|_F)^2 \\
&\leq (\|Y\|_F + \|X\|_F)^2 \leq 2\|Y\|_F^2 + 2\|X\|_F^2\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
1/2\|[\sqrt{\rho}, U^\dagger\sqrt{\rho}U]\|_F^2 &\leq \|Y\|_F^2 + \|X\|_F^2 \\
&= \sum_{i\neq j}(|\langle u_i|\sqrt{\rho}|u_j\rangle|^2 + |\langle u_i|U^\dagger\sqrt{\rho}U|u_j\rangle|^2) \\
&= \sum_{i\neq j}|\langle u_i|\sqrt{\rho}|u_j\rangle|^2 + \sum_{i\neq j}|\langle v_i|\sqrt{\rho}|v_j\rangle|^2 \\
&= C(\rho,\Pi_1) + C(\rho,\Pi_2)\,,
\end{aligned}$$

which completes the proof.

From an alternative perspective, we have the following inequality

$$C(\rho,\Pi_1) + C(\rho,\Pi_2) \geq \frac{1}{2}(1 - ||G||_F^2)\,, \tag{6}$$

which is also a kind of uncertainty relations for coherence. Here $G = D_1 T D_2$ with

$$D_1 = \mathrm{diag}(\langle u_1|\rho|u_1\rangle^{1/4}, \langle u_2|\rho|u_2\rangle^{1/4}, \ldots, \langle u_d|\rho|u_d\rangle^{1/4})\,,$$

$$D_2 = \mathrm{diag}(\langle v_1|\rho|v_1\rangle^{1/4}, \langle v_2|\rho|v_2\rangle^{1/4}, \ldots, \langle v_d|\rho|v_d\rangle^{1/4})\,,$$

$$T = (|\langle u_i|v_j\rangle|)\,.$$

The derivation of inequality (6) is as follows.

$$\begin{aligned}
&C(\rho,\Pi_1) + C(\rho,\Pi_2) \\
&\quad= 1 - \mathrm{tr}\sqrt{\rho}\Pi_1(\sqrt{\rho}) + 1 - \mathrm{tr}\sqrt{\rho}\Pi_2(\sqrt{\rho}) \\
&\quad\geq 1 - \mathrm{tr}\sqrt{\rho}\sqrt{\Pi_1(\rho)} + 1 - \mathrm{tr}\sqrt{\rho}\sqrt{\Pi_2(\rho)} \\
&\quad= (D_H^2(\rho,\Pi_1(\rho)) + D_H^2(\rho,\Pi_2(\rho)))/2 \\
&\quad\geq D_H^2(\Pi_1(\rho),\Pi_2(\rho))/4 \\
&\quad= (1 - \mathrm{tr}\sqrt{\Pi_1(\rho)}\sqrt{\Pi_2(\rho)})/2 \\
&\quad= \Big(1 - \sum_{ij}\sqrt{\langle u_i|\rho|u_i\rangle}\sqrt{\langle v_j|\rho|v_j\rangle}|\langle u_i|v_j\rangle|^2\Big)\Big/2 \\
&\quad= \frac{1}{2}(1 - \|G\|_F^2)\,,
\end{aligned}$$

where

$$D_H^2(\rho,\sigma) = ||\sqrt{\rho} - \sqrt{\sigma}||_F^2 = 2(1 - \mathrm{tr}\sqrt{\rho}\sqrt{\sigma})$$

is the square of quantum Hellinger distance between ρ and σ.

The first inequality in the above derivation follows from the Kadison inequality, which states that[59] $\Phi(A)^2 \leq \Phi(A^2)$ for any unital and positive quantum operation Φ and Hermitian operator A, while the second inequality is obtained by the triangle inequality of quantum Hellinger distance.

Combining the above two uncertainty relations, we have

$$C(\rho,\Pi_1)+C(\rho,\Pi_2)\geq\frac{1}{2}\max\{C_1,C_2\}\,,\tag{7}$$

where

$$C_1=\|[\sqrt{\rho},U^{\dagger}\sqrt{\rho}U]\|_F^2\,,\quad C_2=1-\|G\|_F^2\,.$$

Next, we consider coherence with respect to mutually unbiased bases (MUBs). Recall that two orthonormal bases $B_1=\{|b_{1j}\rangle : j=1,2,\ldots,d\}$ and $B_2=\{|b_{2j}\rangle : j=1,2,\ldots,d\}$ of a d-dimensional system Hilbert space are mutually unbiased if[60–61]

$$|\langle b_{1j}|b_{2k}\rangle|^2=1/d\,,\quad \text{for all }\ j,k\,.$$

When the dimension d is a prime power (i.e., $d=p^k$ for a prime number p and a positive integer k), there exists a complete set of $d+1$ MUB $B_\nu=\{|b_{\nu j}\rangle : j=1,2,\ldots,d\}$, $\nu=1,2,\ldots,d+1$.[60–61]

In Ref. [62], we have obtained the following exact uncertainty relation

$$\sum_{\nu=1}^{d+1}C(\rho,B_\nu)=d-(\mathrm{tr}\sqrt{\rho})^2\,,\tag{8}$$

for a complete set of MUBs B_ν, $\nu=1,2,\ldots,d+1$. In particular, for any pure state ρ, we have

$$\sum_{\nu=1}^{d+1}C(\rho,B_\nu)=d-1\,.\tag{9}$$

Here we extend the above exact uncertainty relation to any m MUBs B_ν, $\nu=1,2,\ldots,m$, as follows

$$\sum_{\nu=1}^{m}C(\rho,B_\nu)\geq(m-1)\Big(1-\frac{(\mathrm{tr}\sqrt{\rho})^2}{d}\Big)\,.\tag{10}$$

To establish the above result, noting that[63]

$$\sum_{\nu=1}^{m}P(B_\nu|\rho)\leq\frac{m-1}{d}+\mathrm{tr}\rho^2\,,$$

where $P(B_\nu|\rho)=\sum_j\langle b_{\nu j}|\rho|b_{\nu j}\rangle^2$.

Replacing ρ with $\sqrt{\rho}/\mathrm{tr}\sqrt{\rho}$ and using the definition of $C(\rho,B_\nu)$, we obtain the desired inequality (10). In particular, for any pure state ρ, we have

$$\sum_{\nu=1}^{m}C(\rho,B_\nu)\geq\frac{(m-1)(d-1)}{d}\,.\tag{11}$$

3 Uncertainty Relations for Coherence with Respect to SIC-POVMs

In this section, we study coherence of a state with respect to a general SIC-POVM, and derive some uncertainty relations for coherence of a state with respect to a family of SIC-POVMs.

Consider a d-dimensional system, let H_d be the set of all $d\times d$ Hermitian operators and T_d be the set of all $d\times d$ traceless Hermitian operators. A set of d^2 nonnegative operators $P=\{P_i : i=1,2,\ldots,d^2\}$ (not necessarily of rank 1) is called a general symmetric informationally complete positive operator valued measurement (SIC-POVMs),[64] if

(i) It is a POVM: $P_i\geq 0$, $\sum_{i=1}^{d^2}P_i=\mathbf{1}$, where $\mathbf{1}$ is the identity matrix.

(ii) It is symmetric: $\mathrm{tr}P_i^2=\mathrm{tr}P_j^2\neq 1/d^3$ for all $i,j=1,2,\ldots,d^2$, and $\mathrm{tr}(P_iP_j)=\mathrm{tr}(P_lP_m)$ for all $i\neq j$ and $l\neq m$.

Gour and Kalev have shown that there is a one-to-one correspondence between SIC-POVMs and orthonormal bases of T_d in the following sense:[64] Let $\{F_i : i=1,2,\ldots,d^2-1\}$ be an orthonormal base of T_d, that is, $\mathrm{tr}F_i=0$ and $\mathrm{tr}F_iF_j=\delta_{ij}$, $i,j=1,2,\ldots,d^2-1$. Put $F=\sum_{i=1}^{d^2-1}F_i$ and

$$t_0=-\frac{1}{d^2}\min\Big\{\frac{1}{\lambda_i} : i=1,2,\ldots,d^2-1\Big\}\,,\tag{12}$$

$$t_1=-\frac{1}{d^2}\max\Big\{\frac{1}{\mu_i} : i=1,2,\ldots,d^2-1\Big\}\,,\tag{13}$$

with λ_i and μ_i the maximum and minimum eigenvalues of $F-d(d+1)F_i$, respectively. For any $0\neq t\in[t_0,t_1]$, take

$$P_i(t)=\frac{1}{d^2}\mathbf{1}+t(F-d(d+1)F_i),\quad i=1,\ldots,d^2-1\,,$$

$$P_{d^2}(t)=\frac{1}{d^2}\mathbf{1}+t(d+1)F\,,$$

then $P(t)=\{P_i(t) : i=1,2,\ldots,d^2\}$ constitutes a general SIC POVM. Conversely, any SIC POVM is of the above form for some orthonormal basis $\{F_i : i=1,2,\ldots,d^2-1\}$ of T_d. Take $F_0=\mathbf{1}$, then it is obvious that $\{F_i : i=0,1,2,\ldots,d^2-1\}$ is an orthonormal basis for the space H_d of all Hermitian operators.

With the above preparation, we can state our result for coherence

$$C(\rho,P(t))=\sum_{i=1}^{d^2}I(\rho,P_i(t))$$

of the state ρ with respect to the SIC-POVM $P(t)=\{P_i(t) : i=1,2,\ldots,d^2\}$ as

$$C(\rho,P(t))=t^2d^2(d+1)^2(d-(\mathrm{tr}\sqrt{\rho})^2)\,,\tag{14}$$

where $t\in[t_0,t_1]$. This may be regarded as an exact uncertainty relation for the family of operators $P_i(t)$ in the state ρ.

To prove Eq. (14), noting that from Ref. [24], we know that

$$\sum_{i=0}^{d^2-1}I(\rho,F_i)=d-(\mathrm{tr}\sqrt{\rho})^2\,,$$

from which we have

$$\begin{aligned}C(\rho,P(t))&=\sum_{i=1}^{d^2}I(\rho,P_i(t))\\&=-\frac{1}{2}\sum_{i=1}^{d^2}\mathrm{tr}[\sqrt{\rho},P_i(t)]^2\\&=-\frac{t^2}{2}\Big(\sum_{i=1}^{d^2-1}\mathrm{tr}[\sqrt{\rho},F-d(d+1)F_i]^2\\&\quad+\mathrm{tr}[\sqrt{\rho},(d+1)F]^2\Big)\end{aligned}$$

$$= t^2 d^2 (d+1)^2 \sum_{i=1}^{d^2-1} (\mathrm{tr}\rho F_i^2 - \mathrm{tr}\sqrt{\rho} F_i \sqrt{\rho} F_i)$$
$$= t^2 d^2 (d+1)^2 \sum_{i=1}^{d^2-1} I(\rho, F_i)$$
$$= t^2 d^2 (d+1)^2 \sum_{i=0}^{d^2-1} I(\rho, F_i)$$
$$= t^2 d^2 (d+1)^2 (d - (\mathrm{tr}\sqrt{\rho})^2) .$$

From the above result, we readily obtain that for arbitrary m SIC-POVMs $P^{(\nu)}(t_\nu), \nu = 1, 2, \ldots, m$, where t_ν is the corresponding t constants in their representations in terms of orthonormal bases of T_d, we have

$$\sum_{\nu=1}^{m} C(\rho, P^{(\nu)}(t_\nu)) = \Big(\sum_{\nu=1}^{m} t_\nu^2\Big)(d^2+d)^2 (d - (\mathrm{tr}\sqrt{\rho})^2 .$$

In particular, for any pure state ρ, we have

$$\sum_{\nu=1}^{m} C(\rho, P^{(\nu)}(t_\nu)) = \Big(\sum_{\nu=1}^{m} t_\nu^2\Big)(d^3-d)^2 . \tag{15}$$

These are uncertainty relations for coherence with respect to several SIC-POVMs.

4 Uncertainty Relations for Coherence of Quantum States with Respect to a Common Measurement

Coherence is a relative concept involving both quantum states and measurements. In the previous sections, we have discussed uncertainty relations with respect to different measurements by fixing a quantum state. From a dual viewpoint, we study uncertainty relations of different quantum states by fixing a measurement in this section, and obtain the following result

$$C(\rho, \Pi) + C(\sigma, \Pi) \geq \frac{1}{2} \| [\sqrt{\rho}, \sqrt{\sigma}] \|_F^2 , \tag{16}$$

which may be regarded as an uncertainty relation for coherence of two quantum states with respect to a common measurement (reference basis). Here ρ and σ are arbitrary quantum states, and $\Pi = \{|i\rangle\langle i|\}$ is any von Neumann measurement.

Equation (16) can be derived by decomposing $\sqrt{\rho}$ and $\sqrt{\sigma}$ as Eq. (5). This shows that, if two states are not commutative, then there is no measurement (reference basis) such that the coherence of these two states can be simultaneously small.

We further reveal a link between uncertainty relations for coherence and quantumness of ensembles. Consider a quantum ensemble $\mathcal{E} = \{(p_i, \rho_i) : i = 1, 2, \ldots, n\}$ with

$$Q(\mathcal{E}) = - \sum_{i,j=1}^{n} \sqrt{p_i p_j} \mathrm{tr}[\sqrt{\rho_i}, \sqrt{\rho_j}]^2 ,$$

as a measure of quantumness in Ref. [65]. Here ρ_i are density operators and $p_i \geq 0, \sum_{i=1}^{n} p_i = 1$. From the above discussion, we get that

$$Q(\mathcal{E}) \leq 2 \sum_{i,j=1}^{n} \sqrt{p_i p_j} (C(\rho_i, \Pi) + C(\rho_j, \Pi))$$
$$= 4 \Big(\sum_{i=1}^{n} \sqrt{p_i}\Big) \Big(\sum_{i=1}^{n} \sqrt{p_i} C(\rho_i, \Pi)\Big)$$
$$\leq 4\sqrt{n} \sum_{i=1}^{n} \sqrt{p_i} C(\rho_i, \Pi) .$$

Consequently, the weighted average of coherence

$$\sum_{i=1}^{n} \sqrt{p_i} C(\rho_i, \Pi)$$

of the quantum ensemble satisfies

$$\sum_{i=1}^{n} \sqrt{p_i} C(\rho_i, \Pi) \geq \frac{Q(\mathcal{E})}{4\sqrt{n}} .$$

This relation sets a lower bound to coherence in terms of quantumness of the quantum ensemble.

5 Summary

By employing the coherence measure based on skew information, we have derived some uncertainty relations with respect to von Neumann measurements as well as with respect to SIC POVMs. In the special cases of a family of von Neumann measurements, we have considered uncertainty relations for coherence with respect to any MUBs (not necessary two MUBs), and have proved that the lower bound is a positive constant for pure states.

From a dual perspective, we have obtained some tradeoff relations for coherence of different quantum states and quantum ensembles with respect to a common measurement. These results imply that if two density operators are not commutative, then there is no reference basis such that their coherence are simultaneously small.

We emphasize that the coherence measure based on the skew information has a natural interpretation as quantum uncertainty,[36–37] consequently the uncertainty relations for coherence obtained here can be regarded as genuinely quantum uncertainty relations.

References

[1] E. Candes, J. Romberg, and T. Tao, IEEE Trans. Inf. Theory **52** (2006) 489.

[2] Y. Huang, Phys. Rev. A **86** (2012) 024101.

[3] P. Busch, T. Heinonen, and P. Lahti, Phys. Rep. **452** (2007) 155.

[4] M. J. W. Hall, Phys. Rev. A **69** (2004) 052113.

[5] A. Trifonov, G. Björk, and J. Söderholm, Phys. Rev. Lett. **86** (2001) 4423.

[6] H. F. Hofmann and S. Takeuchi, Phys. Rev. A **68** (2003) 032103.

[7] O. Gühne, Phys. Rev. Lett. **92** (2004) 117903.

[8] C. Qian, J. L. Li, and C. F. Qiao, Quantum Inf. Process **17** (2018) 84.

[9] Z. W. Wang, Y. F. Huang, X. F. Ren, *et al.*, Europhys. Lett. **78** (2007) 40002.

[10] U. Singh, A. K. Pati, and M. N. Bera, Mathematics **4** (2016) 47.

[11] F. Liu, F. Li, J. Chen, and W. Xing, Quantum Inf. Process. **15** (2016) 3459.

[12] X. Yuan, G. Bai, T. Peng, and X. Ma, Phys. Rev. A **96** (2017) 032313.

[13] A. E. Rastegin, arXive:1702.01409v3(2017).

[14] J. Oppenheim and S. Wehner, Science **330** (2010) 1072.

[15] M. Banik, M. R. Gazi, S. Ghosh, and G. Kar, Phys. Rev. A **87** (2013) 052125.

[16] G. Kar, S. Ghosh, S. Choudhary, and M. Banik, Mathematics **4** (2016) 52.

[17] I. Bialynicki-Birula, Phys. Rev. A **74** (2006) 052101.

[18] J. Kaniewski, M. Tomamichel, and S. Wehner, Phys. Rev. A **90** (2014) 012332.

[19] K. Baek, T. Farrow, and W. Son, Phys. Rev. A **89** (2014) 032108.

[20] J. Zhang, Y. Zhang, and C. Yu, Quantum Inf. Process. **14** (2015) 2239.

[21] S. Luo, Phys. Rev. Lett. **91** (2003) 180403.

[22] S. Luo, Phys. Rev. A **72** (2005) 042110.

[23] S. Luo, Theor. Math. Phys. **143** (2005) 681.

[24] S. Luo, Phys. Rev. A **73** (2006) 022324.

[25] D. Li, X. Li, F. Wang, *et al.*, Phys. Rev. A **79** (2009) 052106.

[26] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113** (2014) 140401.

[27] D. Girolami, Phys. Rev. Lett. **113** (2014) 170401.

[28] L. H. Shao, Z. Xi, H. Fan, and Y. Li, Phys. Rev. A **91** (2015) 042120.

[29] A. Streltsov, U. Singh, and H. S. Dhar, Phys. Rev. Lett. **115** (2015) 020403.

[30] X. Yuan, H. Zhou, Z. Cao, and X. Ma, Phys. Rev. A **92** (2015) 022124.

[31] C. Napoli, T. R. Bromley, M. Cianciaruso, *et al.*, Phys. Rev. Lett. **116** (2016) 150502.

[32] A. Winter and D. Yang, Phys. Rev. Lett. **116** (2016) 120404.

[33] K. Bu, U. Singh, S. M. Fei, *et al.*, Phys. Rev. Lett. **119** (2017) 150405.

[34] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. **89** (2017) 041003.

[35] C. Yu, Phys. Rev. A **95** (2017) 042337.

[36] S. Luo and Y. Sun, Phys. Rev. A **96** (2017) 022136.

[37] S. Luo and Y. Sun, Phys. Rev. A **96** (2017) 022130.

[38] Y. Sun, Y. Mao, and S. Luo, Europhys. Lett. **118** (2017) 60007.

[39] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Phys. Rev. A **92** (2015) 012118.

[40] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, Phys. Rev. Lett. **116** (2016) 160406.

[41] G. Gour and R. W. Spekkens, New J. Phys. **10** (2008) 033023.

[42] I. Marvian and R. W. Spekkens, Nat. Commun. **5** (2014) 3821.

[43] I. Marvian and R. W. Spekkens, Phys. Rev. A **94** (2016) 052324.

[44] M. Piani, M. Cianciaruso, T. R. Bromley, *et al.*, Phys. Rev. A **93** (2016) 042107.

[45] H. Kwon, C. Y. Park, K. C. Tan, *et al.*, Phys. Rev. A **97** (2018) 012326.

[46] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photon. **5** (2011) 222.

[47] A. K. Ekert, Phys. Rev. Lett. **67** (1991) 661.

[48] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5** (2015) 021001.

[49] M. Lostaglio, D. Jennings, and T. Rudolph, Nat. Commun. **6** (2015) 6383.

[50] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **115** (2015) 210403.

[51] H. L. Shi, S. Y. Liu, X. H. Wang, *et al.*, Phys. Rev. A **95** (2017) 032307.

[52] Y. L. Su, S. Y. Liu, X. H. Wang, *et al.*, Sci. Rep. **8** (2018) 11081.

[53] A. E. Rastegin, Quantum Inf. Process **17** (2018) 179.

[54] F. Pan, L. Qiu, and Z. Liu, Sci. Rep. **7** (2017) 43919.

[55] Y. Peng, Y. R. Zhang, and Z. Y. Fan, arXiv:1608. 07950(2016).

[56] E. P. Wigner and M. M. Yanase, Proc. Nat. Acad. Sci. USA **49** (1963) 910.

[57] A. Böttcher and D. Wenzel, Linear Algebra Appl. **403** (2005) 216.

[58] K. Glashoff and M. M. Bronstein, arXiv:1305.2135 (2013).

[59] R. Kadison, Ann. Math. **56** (1952) 494.

[60] W. K. Wooters, Found. Phys. **16** (1986) 391.

[61] W. K. Wooters and B. D. Fields, Ann. Phys. **191** (1989) 363.

[62] S. Luo and Y. Sun, Phys. Lett. A **383** (2019) 2869.

[63] S. Wu, S. Yu, and K. Mölmer, Phys. Rev. A **79** (2009) 022104.

[64] G. Gour and A. Kalev, J. Phys. A **47** (2014) 335302.

[65] N. Li, S. Luo, and Y. Mao, Phys. Rev. A **96** (2017) 022132.