 Communications in Theoretical Physics

Commun. Theor. Phys. **72** (2020) 057001 (13pp) https://doi.org/10.1088/1572-9494/ab770b

Topical Review | Editor's Suggestion

Multi-place physics and multi-place nonlocal systems

S Y Lou (楼森岳)

School of Physical Science and Technology, Ningbo University, Ningbo, 315211, China

E-mail: lousenyue@nbu.edu.cn

Received 27 December 2019, revised 1 February 2020
Accepted for publication 14 February 2020
Published 22 April 2020

Abstract

Multi-place nonlocal systems have attracted attention from many scientists. In this paper, we mainly review the recent progresses on two-place nonlocal systems (Alice-Bob systems) and four-place nonlocal models. Multi-place systems can firstly be derived from many physical problems by using a multiple scaling method with a discrete symmetry group including parity, time reversal, charge conjugates, rotations, field reversal and exchange transformations. Multi-place nonlocal systems can also be derived from the symmetry reductions of coupled nonlinear systems via discrete symmetry reductions. On the other hand, to solve multi-place nonlocal systems, one can use the symmetry-antisymmetry separation approach related to a suitable discrete symmetry group, such that the separated systems are coupled local ones. By using the separation method, all the known powerful methods used in local systems can be applied to nonlocal cases. In this review article, we take two-place and four-place nonlocal nonlinear Schrödinger (NLS) systems and Kadomtsev-Petviashvili (KP) equations as simple examples to explain how to derive and solve them. Some types of novel physical and mathematical points related to the nonlocal systems are especially emphasized.

Keywords: multi-place physics, multi-place nonlocal systems, symmetries, integrable systems, parity and time reversal, soliton theory, classical prohibitions

1. Introduction

The first paper on systems with discrete nonlocal places, the nonlinear Schrödinger (NLS) equation

$$\begin{aligned} &\mathrm{i}A_t + A_{xx} \pm A^2B = 0,\\ &B = \hat{f}A = \hat{P}\hat{C}A = A^*(-x, t), \end{aligned} \tag{1}$$

where the operators $\hat{P}$ and $\hat{C}$ are the usual parity and charge conjugation, is proposed by Ablowitz and Musslimani in [1]. In literature, the nonlocal nonlinear Schrödinger equation (1) is also called parity-time reversal ($\hat{P}\hat{T}$) symmetric (more precisely, (1) should be called $\hat{P}\hat{C}$ symmetric because the 'potential' AB is time-dependent and $\hat{P}\hat{C}$ is equivalent to $\hat{P}\hat{T}$ only for the potential in quantum physics is time independent). $\hat{P}$–$\hat{C}$–$\hat{T}$ symmetries play important roles in the quantum physics [2] and many other areas of physics, such as the quantum chromodynamics [3], electric circuits [4], optics [5, 6], Bose–Einstein condensates [7], and so on.

Equation (1) has been studied by many authors via various traditional methods such as the inverse scattering transformation, the Riemann-Chilbert approach, the Hirota's bilinear method, the Darboux transformations and so on [8–20]. For instance, by the Darboux transformation, [21–23] revealed that the defocusing nonlocal NLS (NNLS) equation admits the exponential soliton solutions, rational soliton solutions, and mixed exponential-and-rational soliton solutions over the same nonzero background. Three such types of soliton solutions can display a rich variety of elastic interactions, in which each asymptotic soliton could be of either the dark or antidark type. In addition to these types of soliton solutions, with the stationary solution assumption, [24] the general Jacobi elliptic-function and hyperbolic-function solutions were obtained for equation (1), in which the bounded cases obey either the PT- or anti-PT-symmetric relation. It turns out that the focusing NNLS equation possesses four types of bounded Jacobi elliptic-function solutions, as well as the bright- and dark-soliton solutions, whereas the defocusing NNLS equation has just two types of bounded Jacobi elliptic-function solutions but admits no single-soliton solution.

Notice that the model equation (1) includes two different places $\{x, t\}$ and $\{x' = -x, t' = t\}$, thus, we call all the models including two places $\{x, t\}$ and $\{x', t'\}$ two-place systems or Alice-Bob systems [25]. Two-place systems may be developed to describe various two-place physics, which is the physical theory to explain the correlated/entangled natural phenomena happened at two different spaces and/or times [26].

In addition to the nonlocal NLS system (1), there are many other types of two-place nonlocal models, such as the NLS equations with different non-localities [27], the nonlocal KdV systems [25, 26, 28, 29], nonlocal modified KdV (MKdV) systems [25, 30–35], nonlocal discrete NLS systems [36–38], nonlocal coupled NLS systems [39–45], nonlocal derivative NLS equation [46], nonlocal Davey-Stewartson systems [47–50], generalized nonlocal NLS equation [51], nonlocal nonautonomous KdV equation [52], nonlocal peakon systems [53], nonlocal KP systems, nonlocal sine Gordon systems, nonlocal Toda systems [25, 26], nonlocal Sawada-Kortera equations [54], nonlocal Kaup-Kupershmidt equations [54] and many others [55–65].

Motivated by the AM's work, Yan [18] first introduced two parameters $\{\epsilon_x = \pm 1, \epsilon_t = \pm 1\}$ in the vector NLS equations such that a new unified two-parameter (ϵ_x, ϵ_t) vector NLS equations (simply called the $\mathcal{Q}^{(n)}_{\epsilon_x,\epsilon_t}$ model), containing integrable/non-integrable local/nonlocal vector NLS equations. Particularly, he first established a one-to-one connection between four points $(\epsilon_x, \epsilon_t) = (1, 1), (-1, 1), (1, -1), (-1, -1)$ (or complex numbers $\epsilon_x + i\epsilon_t$) with $\{\mathcal{I}, \mathcal{P}, \mathcal{T}, \mathcal{PT}\}$ symmetries. The two-parameter idea [19] could also be extended to many other nonlinear wave equations including the nonlocal general vector NLS equations [43] and the mixed local and nonlocal NLS equations [44]. Yan et. al gave the bi- and tri-linear forms of the $\mathcal{Q}^{(n)}_{\epsilon_x,\epsilon_t}$ model, solitons, double-periodic solutions [19, 20] and rational solitons of the nonlocal NLS equation. The multi-rational and semi-rational solitons and interactions were also found for the nonlocal two-component NLS equations [45]. It is found that the nonlocal NLS equation with $\mathcal{PT}$-symmetric potentials could support the stable solitons. By using a systematical inverse scattering transformation and solving the corresponding matrix Riemann-Hilbert problems for the focusing and defocusing nonlocal mKdV equations with non-zero boundary conditions at infinity, the solitons and breathers of the nonlocal mKdV are obtained [66].

Motivated by the idea of Ablowitz and Musslimani for nonlocal NLS and Yan's two-parameter idea [18], Ji and Zhu [67] firstly introduced a reverse space-time nonlocal modified KdV (mKdV) equation,

$$q_t(x, t) + 6q(x, t)q(-x, -t)q_x(x, t) + q_{xxx}(x, t) = 0 \quad (2)$$

which can also be found in other references [27, 30, 68]. The soliton solutions of (2) are studied by the Darboux transformation [27, 67] and the inverse scattering transformation [69]. A complex form of (2), which is gauge equivalent to a spin-like model, is studied by Ma, Shen and Zhu [70]. From the gauge equivalence, it is found that a significant difference exists between the nonlocal complex mKdV equation and the classical complex mKdV equation. Through Darboux transformation, a variety of exact solutions for the nonlocal complex mKdV equation including dark soliton, W-type soliton, M-type soliton, and periodic solutions are derived [70]. By using a systematical inverse scattering transformation and solving the corresponding matrix Riemann-Hilbert problems for the focusing and defocusing nonlocal mKdV equations with non-zero boundary conditions at infinity, the solitons and breathers of the nonlocal mKdV are first obtained [66].

Two-place nonlocal discrete models, especially the discrete version of (1)

$$\mathrm{i}q_{n,t} = q_{n+1} + q_{n-1} - 2q_n + q_n q^*_{-n}(q_{n+1} + q_{n-1}), \quad (3)$$

have also attracted the attention of researcher. N-soliton solutions, spatial periodic solutions and singular solutions for this discrete nonlocal NLS equation are given by Ma and Zhu [71]. It is shown that [72], under the gauge transformations, the nonlocal focusing NLS equation and nonlocal defocusing NLS equation are, respectively, gauge equivalent to a Heisenberg-like equation and a modified Heisenberg-like equation. Similarly, the discrete nonlocal NLS equation for the focusing and defocusing case relates to a discrete Heisenberg-like equation and a discrete modified Heisenberg-like equation. This fact significantly impacts the possible physical applications of the nonlocal NLS equation [73].

Two-place nonlocal NLS equations have also been extended to some different coupled versions by many authors [25, 39, 45]. The higher order versions of the NLS systems such as the Sasa-Satsuma equation [74, 75], the Hirota system and the full AKNS hierarchy [76] have also been extended to two-place nonlocal ones.

The Davey-Stewartson (DS) equation is a well-known (2+1)-dimensional NLS equation. Two-place nonlocal versions of DS equation are introduced and studied by many authors including Ablowitz and Musslimani [30], Fokas [49], Zhou [77], Rao-Cheng-He [78], Rao-Zhang-Fokas-He [79], Rao-Cheng-Porsezian-Mihalache-He [80], Yang-Yang [81] and Yang-Chen [82, 83]. In [81], Yang and Yang pointed out that the solutions of nonlocal equations including nonlocal NLS, DS, derivative NLS, mKdV, short pulse equations, nonlinear diffusion equations, nonlocal Sasa-Satsuma equations and many others can be converted to local integrable equations through simple variable transformations. In [82, 83], Yang and Chen have studied the interactions among several rogue waves and the dark and anti-dark rational traveling waves for the nonlocal DS systems.

It is known that both the nonlocal NLS equation (1) and the nonlocal mKdV (2) can be considered as the discrete symmetry reductions of the usual local AKNS hierarchy. In fact, all the nonlocal systems can be considered as discrete symmetry reductions of multi-component local systems [25]. Thus, it is natural that the solutions of the nonlocal systems can also be obtained from the related multi-component local systems via reduction method. In [76, 84], the authors developed a reduction technique of double Wronskians to obtain solutions to the reduced bilinear equations from those of unreduced ones. This technique is simple and valid to both classical and nonlocal integrable systems that allow bilinear forms and double Wronskian solutions. In this method, one can make use of the solutions (most of them are known) of the unreduced bilinear equations, and obtain N-soliton solutions for the whole hierarchy, rather than implementing reductions solution by solution and equation by equation. This method proves general and has applied to many nonlocal systems (e.g. [85–88]), and the reduction idea has been generalized to the Cauchy matrix approach [89].

In natural sciences, more than two events occurred at different places may be correlated or entangled. To describe multi-place problems, it is natural and important to establish some possible multi-place nonlocal models [25, 90].

In section 2, we review two methods, the discrete symmetry reduction method and the consistent correlated bang (CCB) approach, to derive some multi-place nonlocal systems. In section 3, we focus on multi-place nonlocal integrable systems, especially for the two-place and four-place nonlocal NLS equations and KP equations. Section 4 is devoted to investigating special solutions of two special two-place and four-place KP systems. The last section is a summary with some discussions.

2. Generalized aspect to find multi-place nonlocal systems

It is known that most of physically important local integrable nonlinear systems such as the KdV equation, the mKdV equation, the NLS equation and the KP equation are all derived from real physical systems via the multiple scaling approach (MSA). Thus, by means of MSA, we derived the first two-place nonlocal KdV system [26] from a two-vortex model [91] which is a special form of the (2+1)-dimensional rotating fluid model, the Euler equation with rotation effect, or named the nonlinear inviscid dissipative and equivalent barotropic vorticity equation in a β-plane channel [92]. Similarly, two-place nonlocal KdV systems can also be derived from two-layer fluid system [93]. Laterly, some other scientists [28, 51, 52, 94] derived many other nonlocal systems like the nonlocal NLS and generalized NLS equations, the nonlocal mKdV equation and the variable coefficient KdV equation.

It is also known that the nonlocal NLS equation (1) and the nonlocal mKdV (2) can be obtained from the two-component local systems, the second and the third AKNS hierarchy. Thus one can believe that all this types of multi-place nonlocal systems can be derived from local multiple component systems. In this section we focus on two simplest methods. The first method is to find a possible discrete symmetry group with n elements for an m component coupled system such that the discrete symmetry reductions can be found. The second one is to apply the so-called consistent correlated bang (CCB) for a lower component system to get a higher component system so that the first method can be used.

2.1. Multi-place nonlocal systems from multi-component systems

For the m-component system

$$K_i(u_1, u_2,\ldots, u_m) = 0,\ i = 1, 2,\ldots, m, \tag{4}$$

where K_i, $i = 1, 2,\ldots,m$ are functions of u_j, $j = 1, 2,\ldots, m$ and their derivatives with respect to the space and time variables $\mathcal{X} = \{x_1, x_2,\ldots, x_d, t\}$, if we can find an nth-order discrete group

$$\mathcal{G} = \{\hat{g}_0 = I = \text{identity},\ \hat{g}_1,\ldots, \hat{g}_{n-1}\}, \tag{5}$$

then one may find a suitable transformation

$$u_i = U_i(v_1, v_2,\ldots,v_m),\ i = 1, 2,\ldots, m, \tag{6}$$

which transforms the original equation system (4) to a new one

$$\tilde{K}_i(v_1, v_2,\ldots,v_m) = 0,\ i = 1, 2,\ldots,m, \tag{7}$$

thereafter, the $\mathcal{G}$-symmetry reductions can be directly obtained with some v_i, $i = 1, 2,\ldots, m$ related to others by suitable group elements $\hat{g}_j$, $j = 1, 2,\ldots, n$. Usually, the $\mathcal{G}$-symmetry reductions are multi-place nonlocal systems if $\hat{g}_j\mathcal{X} \neq \mathcal{X}$ for some j.

Here is a simple special example. It is clear that the following integrable coupling KP system

$$\begin{cases} (u_t + 6uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0,\\ [v_t + 6(vu)_x + v_{xxx}]_x + \sigma^2 v_{yy} = 0,\\ [w_t + 6(wu)_x + w_{xxx}]_x + \sigma^2 w_{yy} = 0,\\ [z_t + 6(zu)_x + 6(vw)_x + z_{xxx}]_x + \sigma^2 z_{yy} = 0, \end{cases} \tag{8}$$

possesses an eighth-order discrete symmetry group

$$\mathcal{G} = \mathcal{G}_1 \bigcup \hat{C}\mathcal{G}_1,\quad \mathcal{G}_1 \equiv \{1,\ \hat{P}^x\hat{T},\ \hat{P}^y,\ \hat{P}^x\hat{T}\hat{P}^y\}, \tag{9}$$

where the operators $\hat{P}^x$, $\hat{P}^y$, $\hat{T}$ and $\hat{C}$ are the parity for the space variables x and y, time reversal and charge conjugate (complex conjugate in mathematics) defined by

$$\hat{P}^x x = -x,\quad \hat{P}^y y = -y,\quad \hat{T}t = -t,\quad \hat{C}u = u^*, \tag{10}$$

respectively.

Using the symmetry group $\mathcal{G}$, one can directly obtain the following eight discrete symmetry reductions

$$\begin{aligned} &p_{xt} + \{p_{xx} + 6ad(p + p^{\hat{g}})[2d(r - r^{\hat{g}})\\ &\quad + p - p^{\hat{g}}] + 6pu\}_{xx} + 3\sigma^2 p_{yy} = 0,\\ &r_{xt} + \left\{r_{xx} - 3a(p + p^{\hat{g}})[2d(r - r^{\hat{g}}) + p - p^{\hat{g}}]\right.\\ &\quad \left. - \frac{3}{2}u(u - 4r)\right\}_{xx} + 3\sigma^2 r_{yy} = 0,\\ &p^{\hat{g}} \equiv \hat{g}p,\quad \hat{g} \in \mathcal{G}, j = 0, 1,\ldots,7, \end{aligned} \tag{11}$$

where p and r are related to u, v, w and z by the symmetry reduction transformation

$$\begin{aligned} u &= p + p^{\hat{g}} + r + r^{\hat{g}}, \\ v &= \frac{a}{b}(c_0 - c_1 d + c_2 d)(p + p^{\hat{g}}), \\ w &= b(p^{\hat{g}} - p) - 2bd(r - r^{\hat{g}}), \\ z &= c_1 p + c_2 p^{\hat{g}} + c_0(r - r^{\hat{g}}). \end{aligned} \tag{12}$$

For $\hat{g} = \{1, \hat{C}\}$, the reductions (11) are two local integrable coupled KP systems. For $\hat{g} \neq \{1, \hat{C}\}$, the reductions (11) are integrable coupled two-place nonlocal KP systems.

2.2. Multi-place nonlocal systems from single-component systems via CCB

To find multi-place nonlocal systems, we can also use the so-called CCB approach proposed in [95] from lower-component systems, say, single-component systems. There are three basic steps for the CCB approach: (I) banging a single component equation to a multi-component system, (II) making the banged components correlated, and (III) requiring the correlations are consistent.

For simplicity, we just take the KP equation

$$(u_t + 6uu_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \tag{13}$$

as a simple example to show the CCB approach.

(I) **Bang.** To bang the single-component KP equation to an m-component coupled KP system, one can make a transformation $u = F(u_0, u_1, u_2, \ldots, u_{m-1})$, say,

$$u = \sum_{i=0}^{m-1} u_i. \tag{14}$$

Substituting (14) into (13), we have

$$\sum_{i=0}^{m-1}\left[\left(u_{it} + 6u_i \sum_{j=0}^{m-1} u_{jx} + u_{ixxx}\right)_x + \sigma^2 u_{iyy}\right] = 0. \tag{15}$$

It is clear that (15) can be banged to an m component coupled KP system

$$\begin{aligned} &\left(u_{it} + 6u_{ix}\textstyle\sum_{j=0}^{m-1} u_j + u_{ixxx}\right)_x + \sigma^2 u_{iyy} + G_i = 0, \\ &i = 0, 1, 2, \ldots, m-1, \end{aligned} \tag{16}$$

with m arbitrary functionals G_i under only one condition

$$\sum_{i=0}^{m-1} G_i = 0. \tag{17}$$

(II) **Correlation.** To get some nontrivial models, we assume that the banged fields u_i are correlated each other, say, we can write the correlation relations as

$$u_j = \hat{g}_j u_0, \, j = 0, 1, 2, \ldots, m-1. \tag{18}$$

(III) **Consistency.** It is natural that the correlation (18) and the banged system (16) should be consistent. Applying $\hat{g}_k$ on (16) for all $k = 0, 1, 2, \ldots, m-1$, it is straightforward to prove that the set of the correlated operators $\hat{g}$

$$\mathcal{G} = \{\hat{g}_0, \hat{g}_1, \hat{g}_2, \ldots, \hat{g}_{m-1}\} \tag{19}$$

consists of an m order finite group. Furthermore, the condition (17) becomes

$$\sum_{i=0}^{m-1} \hat{g}_i G_0 = 0. \tag{20}$$

It is clear that if take the discrete symmetry group as shown in (9) for $m = 8$, then we get a four-place nonlocal complex KP equation ($u_0 \equiv p$)

$$\left(p_t + 6p_x \sum_{j=0}^{7} p^{\hat{g}_j} + p_{xxx}\right)_x + \sigma^2 p_{yy} + G_0 = 0 \tag{21}$$

$$\begin{aligned} \hat{g}_j \in \mathcal{G} = \{&1, \hat{C}, \hat{P}^x\hat{T}, \hat{P}^y, \hat{C}\hat{P}^x\hat{T}, \\ &\hat{C}\hat{P}^y, \hat{P}^x\hat{T}\hat{P}^y, \hat{C}\hat{P}^x\hat{T}\hat{P}^y\}, \end{aligned} \tag{22}$$

with G_0 being a solution of (20) including $G_0 = 0$ as a special trivial example.

Multi-place nonlocal systems can also be obtained from real physical systems via multiple scaling approximations. The detailed procedures can be found in [25, 26, 28, 51, 52, 93, 94].

3. Two-place and four-place nonlocal integrable systems

In this section, we apply the general theory of the last section to obtain some multi-place nonlocal extensions for several important physical models such as the NLS and KP equations.

3.1. Two-place and four-place nonlocal NLS systems

It is known that one of the most famous NLS equation

$$\mathrm{i}q_t + q_{xx} + 2\sigma|q|^2 q = 0, \; \sigma = \pm 1, \tag{23}$$

is only a simple reduction of the AKNS system

$$\begin{aligned} \mathrm{i}q_t + q_{xx} + 2\sigma q^2 r = 0, \\ -\mathrm{i}r_t + r_{xx} + 2\sigma r^2 q = 0, \end{aligned} \tag{24}$$

by using the reduction relation $r = q^*$ because the AKNS system (24) is invariant under the transformation $\hat{C}\hat{E}_{q,r}$ where $\hat{C}r = r^*$ and $\hat{E}_{q,r}\{r, q\} = \{q, r\}$.

In fact, the AKNS system (24) possesses a sixteenth-order discrete symmetry group

$$\mathcal{G}_{\mathrm{AKNS}} = \mathcal{S}_1 \bigcup \mathcal{S}_2, \tag{25}$$

$$\mathcal{S}_1 = \hat{E}_{q,r}\{\hat{C}, \hat{F}\hat{C}, \hat{C}\hat{P}, \hat{F}\hat{C}\hat{P}, \hat{T}, \hat{F}\hat{T}, \hat{T}\hat{P}, \hat{F}\hat{T}\hat{P}\}, \tag{26}$$

$$\mathcal{S}_2 = \hat{C}\hat{E}_{q,r}\mathcal{S}_1 = \{1, \hat{F}, \hat{P}, \hat{F}\hat{P}, \hat{C}\hat{T}, \hat{F}\hat{C}\hat{T}, \hat{C}\hat{T}\hat{P}, \hat{F}\hat{C}\hat{T}\hat{P}\}, \tag{27}$$

with four second-order generators, $\{\hat{F}, \hat{P}, \hat{C}\hat{E}_{q,r}, \hat{T}\hat{E}_{q,r}\}$, where $\hat{P}$ is the parity, $\hat{T}$ is the time reversal, $\hat{C}$ is the charge conjugate,

$\hat{F}$ is the field reflection and $\hat{E}_{q,r}$ is the exchange of the fields q and r. $\hat{F}$ and $\hat{E}_{q,r}$ are defined by

$$\hat{F}\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} -q \\ -r \end{pmatrix}, \quad \hat{E}_{q,r}\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} r \\ q \end{pmatrix}. \tag{28}$$

From the definition (28), we know that there are two types of discrete symmetries. The first type of symmetries ($\mathcal{S}_1$) exchanges the fields q and r. However, the second type of symmetries ($\mathcal{S}_2$) does not exchange the field variables, and thus it cannot be used to obtain nontrivial reductions. Consequently, the AKNS system (24) possesses the following eight nontrivial discrete symmetry reductions

$$\begin{aligned} &\mathrm{i}q_t + q_{xx} + 2\sigma q^2 q^{\hat{g}} = 0, \\ &\hat{g} \in \{\hat{C}, \hat{F}\hat{C}, \hat{C}\hat{P}, \hat{F}\hat{C}\hat{P}, \ \hat{T}, \hat{F}\hat{T}, \hat{T}\hat{P}, \hat{F}\hat{T}\hat{P}\}. \end{aligned} \tag{29}$$

Obviously, the reductions (29) include two local reductions for $\hat{g} = \{C, FC\}$ and six two-place nonlocal reductions for $\hat{g} \neq \{\hat{C}, \hat{F}\hat{C}\}$.

To get four-place NLS type nonlocal systems, one has to study the discrete symmetry reductions for some higher component AKNS systems. Here are two special four-component AKNS systems

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(p+q)[2qr + s(q-p)] = 0, \\ \mathrm{i}p_t + p_{xx} + \frac{1}{2}\sigma(p+q)[2ps + r(p-q)] = 0, \\ \mathrm{i}r_t - r_{xx} - \frac{1}{2}\sigma(s+r)[2qr + p(r-s)] = 0, \\ \mathrm{i}s_t - s_{xx} - \frac{1}{2}\sigma(s+r)[2ps + q(s-r)] = 0, \end{cases} \tag{30}$$

and

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2\sigma(qr + ps)q = 0, \\ \mathrm{i}p_t + p_{xx} + 2\sigma(qr + ps)p = 0, \\ \mathrm{i}r_t - r_{xx} - 2\sigma(qr + ps)r = 0, \\ \mathrm{i}s_t - s_{xx} - 2\sigma(qr + ps)s = 0. \end{cases} \tag{31}$$

It is clear that the coupled AKNS systems (30) and (31) will be reduced back to the standard AKNS (23) if $p = q$ and $s = r$.

It is straightforward to find that the coupled AKNS systems (30) and (31) possess an common sixteenth-order discrete symmetry group

$$\begin{aligned} \mathcal{G}_{\mathrm{CAKNS}} &= \mathcal{G}_1 \cup \hat{E}_{pq}^{rs}\mathcal{G}_1 \cup \hat{E}_{qs}^{pr}\mathcal{G}_2 \cup \hat{E}_{qr}^{ps}\mathcal{G}_2, \\ \mathcal{G}_1 &= \{1, \hat{C}\hat{T}, \hat{P}, \hat{P}\hat{C}\hat{T}\}, \quad \mathcal{G}_2 = \hat{C}\mathcal{G}_1, \end{aligned} \tag{32}$$

where the field exchange operators $\hat{E}_{pq}^{rs}$, $\hat{E}_{qs}^{pr}$ and $\hat{E}_{qr}^{ps}$ are defined by

$$\begin{aligned} &\hat{E}_{pq}^{rs}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} q \\ p \\ s \\ r \end{pmatrix}, \quad \hat{E}_{qs}^{pr}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} r \\ s \\ p \\ q \end{pmatrix}, \\ &\hat{E}_{qr}^{ps}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} s \\ r \\ q \\ p \end{pmatrix} = \hat{E}_{pq}^{rs}\hat{E}_{qs}^{pr}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}. \end{aligned} \tag{33}$$

In the discrete symmetry group (32), we have not considered the field reflection operator $\hat{F}$, because the sign change of the fields has been included in the model parameter σ.

Four types of nontrivial and nonequivalent local or nonlocal AKNS systems can be obtained from the reductions of the discrete symmetry group (32).

The first type of reductions can be written from (30) as

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(q^{\hat{f}} + q)[2qr + r^{\hat{f}}(q - q^{\hat{f}})] = 0, \\ \mathrm{i}r_t - r_{xx} - \frac{1}{2}\sigma(r^{\hat{f}} + r)[2qr + q^{\hat{f}}(r - r^{\hat{f}})] = 0, \end{cases} \tag{34}$$

$$\hat{f} \in \mathcal{G}_1 = \{1, \hat{T}\hat{C}, \hat{P}, \hat{P}\hat{T}\hat{C}\}, \quad (p, s) = \hat{f}(q, r). \tag{35}$$

The reduction (34) is local for $\hat{f} = 1$, while the other three reductions of (34) with $\hat{f} \neq 1$ are two-place nonlocal AKNS systems.

The second type of AKNS systems obtained from (31) reads

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2\sigma q(qr + q^{\hat{f}}r^{\hat{f}}) = 0, \\ \mathrm{i}r_t - r_{xx} - 2\sigma r(qr + q^{\hat{f}}r^{\hat{f}}) = 0, \end{cases} \tag{36}$$

$$\hat{f} \in \mathcal{G}_1 = \{1, \hat{T}\hat{C}, \hat{P}, \hat{P}\hat{T}\hat{C}\}, \quad (p, s) = \hat{f}(q, r). \tag{37}$$

As in the first type of reductions (34), the reduction (36) with $\hat{f} = 1$ is the local AKNS while the others are two-place nonlocal AKNS systems.

The third type of discrete symmetry reductions from (30) possesses the forms

$$\begin{cases} \mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(p+q)[2qq^g + p^g(q-p)] = 0, \\ \mathrm{i}p_t + p_{xx} + \frac{1}{2}\sigma(p+q)[2qq^g + p^g(q-p)] = 0, \end{cases} \tag{38}$$

$$\hat{g} \in \mathcal{G}_2 = \{\hat{C}, \hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\}, \quad (r, s) = \hat{g}(q, p). \tag{39}$$

In this case, the local AKNS system is related to $\hat{g} = \hat{C}$, while the two-place nonlocal AKNS reductions are corresponding to $\hat{g} \neq \hat{C}$.

The fourth type of discrete symmetry reductions

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2\sigma q(qq^{\hat{g}} + pp^{\hat{g}}) = 0, \\ \mathrm{i}p_t + p_{xx} + 2\sigma p(qq^{\hat{g}} + pp^{\hat{g}}) = 0, \end{cases} \tag{40}$$

$$\hat{g} \in \mathcal{G}_2 = \{\hat{C}, \hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\}, \quad (r, s) = \hat{g}(q, p), \tag{41}$$

can be obtained from (31). When $\hat{g} = \hat{C}$, the reduction (40) is just the well-known local Manakov system

$$\begin{cases} \mathrm{i}q_t + q_{xx} + 2\sigma q(qq^* + pp^*) = 0, \\ \mathrm{i}p_t + p_{xx} + 2\sigma p(qq^* + pp^*) = 0. \end{cases} \tag{42}$$

When $\hat{g} = \{\hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\}$, the reductions of (40) are two-place nonlocal Manakov models.

The integrability of the coupled AKNS system (30), the nonlocal AKNS systems (34) and (38) can be guaranteed by the following common Lax pair

$$\psi_x = \begin{pmatrix} \lambda & \frac{1}{2}(p+q) & 0 & 0 \\ -\frac{\sigma}{2}(r+s) & -\lambda & 0 & 0 \\ 0 & \frac{1}{2}(q-p) & \lambda & \frac{1}{2}(q+p) \\ \frac{\sigma}{2}(s-r) & 0 & -\frac{\sigma}{2}(s+r) & -\lambda \end{pmatrix}\psi, \tag{43}$$

$$\psi_t = \begin{pmatrix} u & \Diamond(p+q) & 0 & 0 \\ -\sigma\Diamond(s+r) & -u & 0 & 0 \\ -v & \Diamond(q-p) & u & \Diamond(p+q) \\ \Diamond(s-r) & v & -\Diamond(s+r) & -u \end{pmatrix}\psi, \tag{44}$$

where

$$u \equiv \frac{\mathrm{i}}{4}[\sigma(s+r)(p+q) + 8\lambda^2],$$
$$v \equiv \frac{\mathrm{i}}{2}\sigma(sp - qr),\ \Diamond \equiv \frac{\mathrm{i}}{2}(2\lambda + \partial_x).$$

The integrability of the coupled AKNS system (31), the nonlocal AKNS systems (36) and (40) can be ensured by the Lax pair of the two component vector AKNS system,

$$\psi_x = \begin{pmatrix} \lambda_1 - \lambda & q & p \\ -\sigma r & -\lambda & 0 \\ -\sigma s & 0 & -\lambda \end{pmatrix}\psi, \tag{45}$$

$$\psi_t = \mathrm{i}\sigma\begin{pmatrix} \sigma\lambda_1^2 - c\lambda^2 + qr + ps & \sigma(q_x + \lambda_1 q) & \sigma(p_x + \lambda_1 p) \\ r_x - \lambda_1 r & -c\lambda^2 - qr & -pr \\ s_x - \lambda_1 s & -qs & -c\lambda^2 - ps \end{pmatrix}\psi, \tag{46}$$

where c, λ and λ_1 are arbitrary constants.

It is interesting that some known integrable nonlocal NLS (or named ABNLS) systems are just the special reductions of the nonlocal AKNS systems (34), (36) (38) and (40). For instance, taking $p = q = A$, $s = r = B$ in (38), we get the known nonlocal NLS systems (29) and some others such as those in [1, 25, 96] and [76],

$$\mathrm{i}A_t + A_{xx} + 2\sigma A^2 B = 0, \tag{47}$$

$$B = \hat{g}A, \quad \hat{g} \in \{\hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\}. \tag{48}$$

In addition to the known nonlocal NLS reductions (47), one can also obtain some types of ***novel*** local and nonlocal two-place and four-place NLS type systems from the AKNS systems (34), (36), (38) and (40).

It is clear that (34) allows a special reduction $r = q^* \equiv A^*$ and then

$$\mathrm{i}A_t + A_{xx} + \frac{1}{2}\sigma(B + A)[2AA^* + B^*(A - B)] = 0, \tag{49}$$

$$B = \hat{f}A, \quad \hat{f} \in \{\hat{P}, \hat{C}\hat{T}, \hat{P}\hat{C}\hat{T}\}. \tag{50}$$

In fact, from the coupled AKNS systems (30) and (31), we can get 32 different types of NLS reductions. Applying the symmetry group $\mathcal{G}$ to (30), we have

$$\mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(q^{\hat{f}} + q)[2qq^{\hat{g}} + q^{\hat{f}\hat{g}}(q - q^{\hat{f}})]$$
$$= 0,\ (p, r, s) = (q^{\hat{f}}, q^{\hat{g}}, q^{\hat{f}\hat{g}}), \tag{51}$$

$$\hat{g} \in \mathcal{G}_1^c = \{\hat{C}, \hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\},$$
$$\hat{f} \in \mathcal{G}_1 = \{1, \hat{P}, \hat{C}\hat{T}, \hat{P}\hat{C}\hat{T}\}. \tag{52}$$

The full $\hat{P}$–$\hat{T}$–$\hat{C}$ symmetry reductions of (31) possess the form

$$\mathrm{i}q_t + q_{xx} + 2\sigma q(qq^{\hat{g}} + q^{\hat{f}}q^{\hat{f}\hat{g}}) = 0, \tag{53}$$

$$(p, r, s) = (\hat{f}q \equiv q^{\hat{f}}, \hat{g}q \equiv q^{\hat{g}}, \hat{f}\hat{g}q \equiv q^{\hat{f}\hat{g}}),$$
$$\hat{g} \in \mathcal{G}_1^c = \{\hat{C}, \hat{T}, \hat{C}\hat{P}, \hat{P}\hat{T}\},$$
$$\hat{f} \in \mathcal{G}_1 = \{1, \hat{C}\hat{T}, \hat{P}, \hat{C}\hat{P}\hat{T}\}. \tag{54}$$

For the sixteen reductions (51), there are one local case ($\hat{f} = 1, \hat{g} = \hat{C}$), nine two-place cases (38) ($\hat{f} = 1, \hat{g} = \{\hat{T}, \hat{P}\hat{C}, \hat{P}\hat{T}\}$), (34) ($\hat{g} = \hat{C}, \hat{f} = \{\hat{P}, \hat{T}\hat{C}, \hat{P}\hat{T}\hat{C}\}$) and the cases related to $\hat{g} = \hat{C}\hat{f}$, $\hat{f} = \{\hat{P}, \hat{T}\hat{C}, \hat{P}\hat{T}\hat{C}\}$,

$$\mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(p + q)[2qr + s(q - p)] = 0, \tag{55}$$

$$(p, r, s) = (\hat{f}q, \hat{f}q^*, q^*), \quad \hat{f} \in \{\hat{P}, \hat{C}\hat{T}, \hat{P}\hat{C}\hat{T}\}. \tag{56}$$

All other six cases, ($\{\hat{f} = \hat{P}, \hat{g} = (\hat{T}, \hat{P}\hat{T})\}$, $\{\hat{f} = \hat{T}\hat{C}, \hat{g} = (\hat{C}\hat{P}, \hat{P}\hat{T})\}$, $\{\hat{f} = \hat{P}\hat{T}\hat{C}, \hat{g} = (\hat{T}, \hat{P}\hat{C})\}$)are four-place nonlocal NLS equations which have not yet appeared in literature. For instance, for $\hat{g} = \hat{C}\hat{P}$ and $\hat{f} = \hat{C}\hat{T}$, the related four-place nonlocal NLS equation (51) becomes

$$\mathrm{i}q_t + q_{xx} + \frac{1}{2}\sigma(q^*(x, -t) + q)[2qq^*(-x, t)$$
$$+ q(-x, -t)(q - q^*(x, -t))] = 0. \tag{57}$$

Systems (55) and (57) are called four-place nonlocal NLS equation because four places (x, t), $(x, -t)$, $(-x, t)$ and $(-x, -t)$ are included.

Similarly, for the sixteen reductions (53), there are one local case, nine two-place nonlocal cases and six four-place nonlocal cases,

$$\mathrm{i}q_t + q_{xx} + 2\sigma q(qq^{\hat{g}} + q^{\hat{f}}q^{\hat{f}\hat{g}})$$
$$=0,\ (p, r, s) = (\hat{f}q, \hat{g}q, \hat{f}\hat{g}q), \tag{58}$$

$$(\hat{g}, f) = (\hat{T}, \hat{P}\{1, \hat{C}\hat{T}\}), (\hat{C}\hat{P}, \hat{C}\hat{T}\{1, \hat{P}\}),$$
$$(\hat{P}\hat{T}, \{\hat{C}\hat{T}, \hat{P}\}). \tag{59}$$

In fact, there are many other coupled (and decoupled) integrable AKNS systems, say, the vector and matrix AKNS systems. Starting from every coupled (and decoupled) AKNS systems, one may obtain some possible multi-place integrable discrete symmetry reductions.

Here, we just list another two sets of integrable local and nonlocal NLS type systems

$$iq_t + q_{xx} + [\alpha(qq^{\hat{g}} + q^{\hat{f}}q^{\hat{f}\hat{g}}) + \beta(q^{\hat{f}}q^{\hat{g}} + qq^{\hat{f}\hat{g}})]q = 0,\quad \hat{f} \in \mathcal{G}_1,\quad \hat{g} \in \mathcal{G}_2, \tag{60}$$

and

$$\begin{aligned} &iq_t + \alpha(q + q^{f})_{xx} + \gamma(q - q^{f})_{xx} \\ &+ [\beta(q + q^{\hat{f}})^2 + \delta(q - q^{\hat{f}})^2]q^{\hat{f}\hat{g}} \\ &+ [\beta(q + q^{\hat{f}})^2 - \delta(q - q^{\hat{f}})^2]q^{\hat{g}} = 0, \\ &\hat{f} \in \mathcal{G}_1,\quad \hat{g} \in \mathcal{G}_2, \end{aligned} \tag{61}$$

with free parameters α, β, γ and δ, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are given by (32).

It is clear that when $\beta = 0$, the models (60) will be degenerated to (53). For convenience, we rewrite (60) as

$$iq_t + q_{xx} + V_{\hat{f},\hat{g}}q = 0, \tag{62}$$

$$V_{\hat{f},\hat{g}} = \alpha(qq^{\hat{g}} + q^{\hat{f}}q^{\hat{f}\hat{g}}) + \beta(q^{\hat{f}}q^{\hat{g}} + qq^{\hat{f}\hat{g}}), \tag{63}$$

where $V_{\hat{f},\hat{g}}$ is clearly $\mathcal{G}_{\hat{f},\hat{g}}$ invariant,

$$\mathcal{G}_{\hat{f},\hat{g}}V_{\hat{f},\hat{g}} = V_{\hat{f},\hat{g}}, \tag{64}$$

$$\mathcal{G}_{\hat{f},\hat{g}} = \{1, \hat{f}, \hat{g}, \hat{f}\hat{g}\}. \tag{65}$$

For concreteness, we list all the independent NLS systems included in (60) (i.e., (62)) below.

$\mathcal{G}_{\hat{C}} \equiv \mathcal{G}_{1,\hat{C}}$ invariant local NLS equation,

$$V_{1,\hat{C}} = 2(\alpha + \beta)qq^*. \tag{66}$$

$\mathcal{G}_{\hat{P}\hat{C}} \equiv \mathcal{G}_{1,\hat{P}\hat{C}}$ invariant two-place nonlocal NLS system,

$$V_{1,\hat{P}\hat{C}} = 2(\alpha + \beta)qq^*(-x, t). \tag{67}$$

$\mathcal{G}_{\hat{P}\hat{T}} \equiv \mathcal{G}_{1,\hat{P}\hat{T}}$ invariant two-place nonlocal NLS system,

$$V_{1,\hat{P}\hat{T}} = 2(\alpha + \beta)qq(-x, -t). \tag{68}$$

$\mathcal{G}_{\hat{T}} \equiv \mathcal{G}_{1,\hat{T}}$ invariant two-place nonlocal NLS system,

$$V_{1,\hat{T}} = 2(\alpha + \beta)qq(x, -t). \tag{69}$$

$\mathcal{G}_{\hat{P},\hat{C}}$ invariant two-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{P},\hat{C}} &= \alpha[qq^* + q(-x, t)q^*(-x, t)] \\ &+ \beta[q(-x, t)q^* + qq^*(-x, t)]. \end{aligned} \tag{70}$$

$\mathcal{G}_{\hat{P},\hat{P}\hat{C}}$ invariant two-place nonlocal NLS system is equivalent to (70) with the exchange of the constants α and β.

$\mathcal{G}_{\hat{P}\hat{T}\hat{C},\hat{C}}$ invariant two-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{P}\hat{T}\hat{C},\hat{C}} &= \alpha[qq^* + q(-x, -t)q^*(-x, -t)] \\ &+ \beta[q^*q^*(-x, -t) + qq(-x, -t)]. \end{aligned} \tag{71}$$

$\mathcal{G}_{\hat{P}\hat{T}\hat{C},\hat{P}\hat{T}}$ invariant two-place nonlocal NLS system is related to (71) by $\alpha \leftrightarrow \beta$.

$\mathcal{G}_{\hat{T}\hat{C},\hat{C}}$ invariant two-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{T}\hat{C},\hat{C}} &= \alpha[qq^* + q(x, -t)q^*(x, -t)] \\ &+ \beta[qq(x, -t) + q^*q^*(x, -t)]. \end{aligned} \tag{72}$$

$\mathcal{G}_{\hat{T}\hat{C},\hat{T}}$ invariant two-place nonlocal NLS system possesses the same form of (72) after using the exchange of α and β.

$\mathcal{G}_{\hat{T}\hat{C},\hat{P}\hat{C}}$ invariant four-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{T}\hat{C},\hat{P}\hat{C}} &= \alpha[qq^*(-x, t) + q(-x, -t)q^*(x, -t)] \\ &+ \beta[qq(-x, -t) + q^*(-x, t)q^*(x, -t)]. \end{aligned} \tag{73}$$

$\mathcal{G}_{\hat{T}\hat{C},\hat{P}\hat{T}}$ invariant four-place nonlocal NLS system is equivalent to (73) because the constants α and β are arbitrary.

$\mathcal{G}_{\hat{P}\hat{T}\hat{C},\hat{P}\hat{C}}$ invariant four-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{P}\hat{T}\hat{C},\hat{P}\hat{C}} &= \alpha[qq^*(-x, t) + q(-x, -t)q^*(x, -t)] \\ &+ \beta[qq(-x, -t) + q^*(-x, t)q^*(x, -t)]. \end{aligned} \tag{74}$$

$\mathcal{G}_{\hat{P}\hat{T}\hat{C},\hat{T}}$ invariant four-place nonlocal NLS system possesses the same form of (74) with $\alpha \leftrightarrow \beta$.

$\mathcal{G}_{\hat{P},\hat{T}}$ invariant four-place nonlocal NLS system,

$$\begin{aligned} V_{\hat{P},\hat{T}} &= \alpha[qq(x, -t) + q(-x, t)q(-x, -t)] \\ &+ \beta[q(-x, t)q(x, -t) + qq(-x, -t)]. \end{aligned} \tag{75}$$

$\mathcal{G}_{\hat{P},\hat{P}\hat{T}}$ invariant four-place nonlocal NLS system can also be written as (75) by using $\alpha \leftrightarrow \beta$.

All sixteen cases of (53) can be obtained from the above cases by setting $\beta = 0$ or $\alpha = 0$.

The first four cases are just known results of the discrete symmetry reductions from the usual AKNS system.

The integrability of (60) (i.e., (62)) is trivial because it is only a special discrete symmetry reduction of the so-called (N +M)-component integrable AKNS system (equations (104), (105) of [97] with $\{\psi_k, \psi_j^*, y\} \to \{q_k, p_j, it\}$)

$$iq_{kt} = -q_{kxx} + \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}q_n p_m q_k,\quad k = 1, 2, \ldots N, \tag{76}$$

$$ip_{jt} = p_{jxx} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}q_n p_m p_j,\quad j = 1, 2, \ldots M, \tag{77}$$

for $M = N = 2$ and special selections of constants a_{nm}. The integrability of (76)–(77) is guaranteed because it is only a symmetry reduction of the KP equation [97, 98].

It is also interesting to mention that using the $\hat{P}$–$\hat{T}$–$\hat{C}$ symmetry group, one can find more discrete symmetry reductions from all the above reduced model equations. For instance, starting from the well-known Manakov systems (42), one can find not only the two-place physically significant nonlocal complex systems listed in [99], but also the following two-place and four-place physically significant nonlocal real nonlinear systems, we omit the details on the similar derivation of these reductions

$$p_t + p_{xx}^{\hat{f}} + 2\sigma p^{\hat{f}}[p^2 + (p^{\hat{f}})^2 + (p^{\hat{g}})^2 + (p^{\hat{f}\hat{g}})^2] = 0,\quad \hat{f} \in \{\hat{T}, \hat{P}\hat{T}\}, \hat{g} \in \{1, \hat{T}, \hat{P}, \hat{P}\hat{T}\}. \tag{78}$$

Especially, if $\hat{g}=1$, two-place models of (78)

$$p_t+p_{xx}(x,-t)+4\sigma p(x,-t)[p^2+p(x,-t)^2]=0, \quad (79)$$

$$p_t+p_{xx}(-x,-t)+4\sigma p(-x,-t)\times[p^2+p(-x,-t)^2]=0, \quad (80)$$

can also be derived from the usual local NLS equation. Only two independent four-place nonlocal systems exist, included in (78),

$$p_t+p_{xx}(x,-t)+2\sigma p(x,-t)[p^2+p(x,-t)^2+p(-x,t)^2+p(-x,-t)^2]=0, \quad (81)$$

and

$$p_t+p_{xx}(-x,-t)+2\sigma p(-x,-t)[p^2+p(x,-t)^2+p(-x,t)^2+p(-x,-t)^2]=0. \quad (82)$$

3.2. Two-place and four-place nonlocal KP systems

To find multi-place nonlocal KP systems, we have to get some multi-component coupled KP equations. To guarantee the integrability, we start from the matrix Lax pairs for matrix KP equations

$$\psi_{xx}+U\psi+\sigma\psi_y=0, \quad (83)$$

$$\psi_t+4\psi_{xxx}+6U\psi_x+3\left(U_x-\int U_y\mathrm{d}x\right)\psi=0, \quad (84)$$

where ψ is an m component vector and U is an $m\times m$ matrix.

The compatibility condition $\psi_{yt}=\psi_{ty}$ of the Lax pair reads

$$(U_t+U_{xxx}+3(U_xU+UU_x)+3\sigma[U,W])_x+3\sigma^2U_{yy}=0, \quad (85)$$

$$[U,W]\equiv UW-WU,\quad W_x=U_y. \quad (86)$$

For the non-Abelian complex matrix KP system (85) with $\sigma=\mathrm{i}=\sqrt{-1}$, its $\hat{P}\hat{T}\hat{C}$ symmetry group is constructed by the generator operators $\hat{P}^x\hat{T}$ and $\hat{C}\hat{P}^y$,

$$\mathcal{G}_n=\{1,\hat{P}^x\hat{T},\hat{C}\hat{P}^y,\hat{C}\hat{P}^y\hat{P}^x\hat{T}\}. \quad (87)$$

For the Abelian matrix KP system, $[U,W]=0$, the $\hat{P}\hat{T}\hat{C}$ symmetry group is the same as given in (9) with three generators $\hat{P}^x\hat{T}$, $\hat{C}$ and $\hat{P}^y$.

Here, we just list some special examples and the related $\hat{P}\hat{T}\hat{C}$ symmetry reductions.

Example 1. The Abelian matrix KP system (85) with

$$U=\begin{pmatrix} u & 0 & 0 & 0\\ w & u & 0 & 0\\ v & 0 & u & 0\\ z & v & w & u\end{pmatrix},$$

$$u=(1+\hat{f})(1+\hat{g})p,\quad v=(1-\hat{f})(1+\hat{g})p,$$
$$w=[(1+\hat{f})(1-\hat{g})p,\ z=(1-\hat{f})(1-\hat{g})p \quad (88)$$

possesses a single component $\hat{P}\hat{T}\hat{C}$ symmetry reduction

$$p_{xt}+\left\{p_{xx}-\frac{3u^2}{4}+6pu+\frac{3}{2}[(p-p^{\hat{f}\hat{g}})^2-(p^{\hat{f}}-p^{\hat{g}})^2]\right\}_{xx}+3\sigma^2p_{yy}=0, \quad (89)$$

$$\hat{f},\ \hat{g},\in\mathcal{G}_n. \quad (90)$$

Example 2. From the Abelian matrix KP system (85) with

$$U=\begin{pmatrix} u & 0 & 0 & 0\\ w & u & 0 & 0\\ v & 0 & u & 0\\ z & 0 & 0 & u\end{pmatrix},$$

$$u=(1+\hat{f})(1+\hat{g})p,\ v=(1-\hat{f})(1+\hat{g})p,$$
$$w=(1+\hat{f})(1-\hat{g})p,\ z=(1-\hat{f})(1-\hat{g})p, \quad (91)$$

we can find a $\hat{P}\hat{T}\hat{C}$ symmetry reduction

$$p_{xt}+\left\{p_{xx}-\frac{3u^2}{4}+6pu\right\}_{xx}+3\sigma^2p_{yy}=0. \quad (92)$$

Example 3. From the non-Abelian matrix KP system (85) with

$$U=\begin{pmatrix} p+2q-r & q-2r+s\\ p-2q+r & s+2r-q\end{pmatrix}, \quad (93)$$

we can find a $\hat{g}\hat{f}$ symmetry reduction

$$p_{xt}+3\sigma^2p_{yy}+3\sigma[(2q_1-r_1)p-(2q-r)\times p_1+s(p_1-2q_1+r_1)-(p-2q+r)s_1]_x+[p_{xx}+3(p-2q+r)s+3p(p+2q-r)]_{xx}=0, \quad (94)$$

$$(p_1,q_1,r_1,s_1)_x=(p,q,r,s)_y, \quad (95)$$

$$q=p^{\hat{f}},\ r=p^{\hat{g}},\ s=p^{\hat{f}\hat{g}},\hat{f}^2=\hat{g}^2=1, \quad (96)$$

where

$$\hat{f},\ \hat{g}\in\mathcal{G}_I=\{1,\hat{P}^x\hat{T},\hat{C}\hat{P}^y,\hat{P}^x\hat{T}\hat{C}\hat{P}^y\} \quad (97)$$

for KPI system ($\sigma=\mathrm{i}=\sqrt{-1}$) and

$$\hat{f},\ \hat{g}\in\mathcal{G}_{II}=\{1,\hat{P}^x\hat{T},\hat{C},\hat{P}^x\hat{T}\hat{C}\} \quad (98)$$

for KPII system ($\sigma=1$).

For the KPI case, the reduction (94) contains one usual local KPI reduction,

$$p_{xt}-3p_{yy}+(p_{xx}+6p^2)_{xx}=0,\hat{f}=\hat{g}, \quad (99)$$

six two-place nonlocal Abel KPI reductions

$$p_{xt}-3p_{yy}+[p_{xx}+6p^2+3(p-p^{\hat{g}})^2]_{xx}=0,$$
$$\hat{f}=1,\ \hat{g}\in\{\hat{P}^x\hat{T},\hat{P}^y\hat{C},\hat{P}^x\hat{T}\hat{P}^y\hat{C}\}, \quad (100)$$

and

$$p_{xt} - 3p_{yy} + [p_{xx} + 6p^{\hat f}(2p - p^{\hat f})]_{xx} = 0,\\ \hat g = 1, \hat f \in \{\hat P^x\hat T, \hat P^y\hat C, \hat P^x\hat T\hat P^y\hat C\}, \tag{101}$$

and six four-place non-Abelian nonlocal systems,

$$\begin{aligned} &p_{xt} - 3p_{yy} + 3\mathrm{i}[(2p_1^{\hat f} - p_1^{\hat g})p - (2p^{\hat f} - p^{\hat g})p_1\\ &+ s(p_1 - 2p_1^{\hat f} + p_1^{\hat g}) - (p - 2p^{\hat f} + p^{\hat g})p_1^{\hat f\hat g}]_x\\ &+ [p_{xx} + 3(p - 2p^{\hat f} + p^{\hat g})p^{\hat f\hat g}\\ &+ 3p(p + 2p^{\hat f} - p^{\hat g})]_{xx} = 0, p_{1x} = p_y,\\ &\{\hat f, \hat g\} \in \{\hat P^x\hat T, \hat P^y\hat C, \hat P^x\hat T\hat P^y\hat C\}, \hat f \neq \hat g. \end{aligned} \tag{102}$$

For the KPII system, we only write down two special Abelian real two-place nonlocal reductions from (94),

$$p_{xt} + 3p_{yy} + [p_{xx} + 6p^2\\ + 3(p - p(-x, y, -t))^2]_{xx} = 0, \tag{103}$$

and

$$p_{xt} + 3p_{yy} + [p_{xx} + 6p(-x, y, -t)\\ \times (2p - p(-x, y, -t))]_{xx} = 0. \tag{104}$$

To end this section, we write down a general vector form of a special local and nonlocal KP system

$$\begin{aligned} &p_{xt} + 3\sigma^2 p_{yy} + [p_{xx} + 6(P|U|P)]_{xx} = 0,\\ &(P|U|P) \equiv \textstyle\sum_{i,j=1}^4 U_{ij}p_ip_j,\ U_{ij} = 0, \forall\, i > j,\\ &p_1 = p, p_2 = p^{\hat f}, p_3 = p^{\hat g}, p_4 = p^{\hat f\hat g},\\ &f, g \in \{1, \hat P^x\hat T, \hat P^y, \hat P^y\hat P^x\hat T\}. \end{aligned} \tag{105}$$

The model equation (105) is a generalization of examples given by (89) and (92).

4. Exact solutions of multi-place nonlocal KP systems

4.1. Symmetry-antisymmetry separation approach to solve nonlocal systems

For a second-order operator, $\hat g$,

$$\hat g^2 = 1, \tag{106}$$

one can always separate an arbitrary function, A, as a summation of $\hat g$-symmetric and $\hat g$-antisymmetric parts in the following way,

$$A = \frac{1}{2}(A + A^{\hat g}) + \frac{1}{2}(A - A^{\hat g}) \equiv u + v, \tag{107}$$

$$u \equiv \frac{1}{2}(A + A^{\hat g}),\ v \equiv \frac{1}{2}(A - A^{\hat g}). \tag{108}$$

It is clear that u and v defined in (108) are symmetric and antisymmetric, respectively, with respect to $\hat g$, i.e.,

$$\hat g u = u,\ \hat g v = -v. \tag{109}$$

Thus, a two-place nonlocal system

$$F(A, B) = 0, \quad B = A^{\hat g}, g^2 = 1, \tag{110}$$

can be transformed to a coupled local system

$$F_1(u, v) = 0, \quad F_1 = F + \hat g F, \tag{111}$$

$$F_2(u, v) = 0, \quad F_2 = F - \hat g F, \tag{112}$$

by using (107). Therefore, to solve the nonlocal equation (110) is equivalent to solving the local system (111) and (112) with (108).

Similarly, a four-place nonlocal system

$$F(p, q, r, s) = 0, \quad q = p^{\hat f},\\ r = p^{\hat g}, s = p^{\hat f\hat g}, \hat f^2 = \hat g^2 = 1, \tag{113}$$

can be changed to a coupled local system

$$F_1(u, v, w, z) = 0, \quad F_1 = F + \hat g F + \hat f F + \hat f\hat g F, \tag{114}$$

$$F_2(u, v, w, z) = 0, \quad F_2 = F + \hat f F - \hat g F - \hat f\hat g F, \tag{115}$$

$$F_3(u, v, w, z) = 0, \quad F_3 = F + \hat g F - \hat f F - \hat f\hat g F, \tag{116}$$

$$F_4(u, v, w, z) = 0, \quad F_4 = F + \hat f\hat g F - \hat g F - \hat f F, \tag{117}$$

by using the symmetric-antisymmetric separation

$$p = u + v + w + z, \tag{118}$$

such that

$$u = \frac{1}{4}(p + p^{\hat f} + p^{\hat g} + p^{\hat f\hat g}),\\ v = \frac{1}{4}(p + p^{\hat f} - p^{\hat g} - p^{\hat f\hat g}), \tag{119}$$

$$w = \frac{1}{4}(p + p^{\hat g} - p^{\hat f} - p^{\hat f\hat g}),\\ z = \frac{1}{4}(p + p^{\hat f\hat g} - p^{\hat g} - p^{\hat f}). \tag{120}$$

From the definitions (119) and (120), it is not difficult to find that u is group

$$\mathcal{G} = \{1, \hat f, \hat g, \hat f\hat g\}$$

invariant, v is $\hat f$ invariant and $\hat g$ antisymmetric, w is $\hat g$ invariant and $\hat f$ antisymmetric, while z is both $\hat f$ and $\hat g$ antisymmetric. To sum up, we have

$$\begin{aligned} &\hat f u = \hat g u = \hat f\hat g u = u,\\ &\hat f v = -\hat g v = -\hat f\hat g v = v,\\ &\hat g w = -\hat f w = -\hat f\hat g w = w,\\ &\hat f\hat g z = -\hat g z = -\hat f z = z. \end{aligned} \tag{121}$$

Hence, to solve the nonlocal equation (113) is equivalent to solving the local system (114)–(117) with the conditions (121).

4.2. Exact multiple soliton solutions of a two-place nonlocal KP equation

For concreteness, we study the exact solutions of the special two-place nonlocal KP equation

$$A_{xt} + A_{xxxx} + \frac{3}{2}[(A+B)(3A+B)_x]_x + 3\sigma^2 A_{yy} = 0,\ B = A^{\hat{g}},\ \hat{g} \in \{\hat{P}^x\hat{T},\ \hat{P}^y,\ \hat{P}^y\hat{P}^x\hat{T}\}. \tag{122}$$

Using the symmetry-antisymmetry separation procedure,

$$A = u + v,\quad \hat{g}u = u,\quad \hat{g}v = -v. \tag{123}$$

(122) is separated to

$$u_{xt} + (u_{xx} + 6u^2)_{xx} + 3\sigma^2 u_{yy} = 0, \tag{124}$$

$$(v_t + v_{xxx} + 6uv)_x + 3\sigma^2 v_{yy} = 0. \tag{125}$$

The multiple soliton solutions of the KP equation (124) can be simply obtained by using the well-known Hirots's bilinear approach. The bilinear form of (124) can be written as

$$(D_xD_t + D_x^4 + 3\sigma^2 D_y^2)\psi\cdot\psi = 0, \tag{126}$$

by means of the transformation

$$u = (\ln\psi)_{xx}, \tag{127}$$

where the bilinear operators D_x, D_t and D_y are defined by

$$D_x^m D_t^n D_y^p f\cdot g = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n \times (\partial_y - \partial_{y'})^p f(x, y, t)g(x', y', t')|_{x'=x, y'=y, t'=t}.$$

It is interesting that for the equation (125) with (127), we have a special solution

$$v = a(\ln\psi)_x \tag{128}$$

with a being an arbitrary constant.

Though $\{(127), (128)\}$ solves (124) and (125), however, to get the solution of the two-place nonlocal KP equation (122), we have to check the nonlocal conditions (109) for $\hat{g} = \hat{P}^y\hat{P}^x\hat{T}$, $\hat{P}^x\hat{T}$ and $\hat{P}^y$, respectively.

Case 1. $\hat{g} = \hat{P}^y\hat{P}^x\hat{T}$. In this case, the multi-soliton solutions of the two-place KP equation (122) can be written as

$$\begin{aligned} &A = u + v = (\partial_x^2 + a\partial_x)\ln\psi,\\ &\psi = \textstyle\sum_{\{\nu\}} K_{\{\nu\}}\cosh\left(\frac{1}{2}\sum_{j=1}^N \nu_j\eta_j\right),\\ &K_{\{\nu\}} = \textstyle\prod_{i>j}^N \sqrt{3k_i^2k_j^2(k_i - \nu_i\nu_jk_j)^2 - \sigma^2(l_ik_j - l_jk_i)^2},\\ &\eta_j = k_jx + l_jy - (k_j^3 + \sigma^2k_j^{-1}l_j^2)t, \end{aligned} \tag{129}$$

where the summation on $\{\nu\} \equiv \{\nu_1, \nu_2,\ldots, \nu_i,\ldots,\nu_N\}$ should be done for all possible permutations $\nu_i = \{1, -1\}$, $i = 1, 2,\ldots,N$.

Case 2. $\hat{g} = \hat{P}^x\hat{T}$. In this case, the multiple soliton solution of the two-place nonlocal KP equation (122) still possesses the form (129). However, the paired condition has to be satisfied,

$$N = 2n,\ k_{n+i} = \pm k_i,\ l_{n+i} = \mp l_i. \tag{130}$$

The condition (130) implies that the odd numbers of soliton solutions in the form (129) are prohibited for the partially inverse nonlocal system KP system (122) with $\hat{g} = \hat{P}^x\hat{T}$. This kind of prohibition phenomena is first found for the two-place nonlocal Boussinesq equation [93]. Under the condition (130), we have paired traveling wave variables

$$\eta \equiv \{\eta_i = k_ix + l_iy - (k_j^3 + \sigma^2k_j^{-1}l_j^2)t,\ \eta_{n+i} = \pm k_ix \mp l_iy \mp (k_j^3 + \sigma^2k_j^{-1}l_j^2)t,\ i = 1, 2,\ldots,n\} \tag{131}$$

with the property

$$\hat{P}^x\hat{T}\eta = \mp\eta. \tag{132}$$

Thus, the nonlocal condition (109) is naturally satisfied for $\hat{g} = \hat{P}^x\hat{T}$.

For $n = 1$ $(N = 2)$, the solution (129) with (130) becomes

$$A = (\partial_x^2 + a\partial_x)\ln F_2, \tag{133}$$

$$F_2 = 2k_1l_1\sigma\cosh(2l_1y) + 2k_1\sqrt{\sigma^2l_1^2 - 4k_1^4}\cosh \times [2(k_1x - 4k_1^3t - 3k_1^{-1}l_1^2\sigma^2t)]. \tag{134}$$

For $n = 2$ $(N = 4)$, the solution (129) with (130) possesses the form

$$A = (\partial_x^2 + a\partial_x)\ln F_4, \tag{135}$$

$$\begin{aligned} F_4 = {}& K_{\{1,1,1,-1\}}[\cosh(\xi) + \cosh(\hat{g}\xi)]\\ &+ K_{\{1,1,-1,1\}}[\cosh(\eta) + \cosh(\hat{g}\eta)]\\ &+ K_{\{1,1,1,1\}}\cosh(\tau) + K_{\{1,-1,1,-1\}}\cosh(\tau_1)\\ &+ K_{\{1,1,-1,-1\}}\cosh[2(l_1 + l_2)y]\\ &+ K_{\{1,-1,-1,1\}}\cosh[2(l_1 - l_2)y],\\ \xi = {}& 2k_1x + 2l_2y - 2(4k_1^3 + 3k_1^{-1}l_1^2\sigma^2)t,\\ \eta = {}& 2k_2x + 2l_1y - 2(4k_2^3 + 3k_2^{-1}l_2^2\sigma^2)t,\\ \tau = {}& 2(k_1 + k_2)x - 2(4k_1^3 + 3\sigma^2k_1^{-1}l_1^2)\\ &\times t - 2(4k_2^3 + 3\sigma^2k_2^{-1}l_2^2)t. \end{aligned} \tag{136}$$

Case 3. $\hat{g} = \hat{P}^y$. In this case, the multiple soliton solution form (129) is correct only for the conditions (130) and

$$a = 0 \tag{137}$$

being satisfied for the two-place nonlocal KP equation with $\hat{g} = \hat{P}^y$. In other words, for the third kind of two-place nonlocal KP equation (122), we have not yet found $\hat{P}^y$-symmetry breaking multiple soliton solutions.

4.3. Exact multiple soliton solutions of a four-place nonlocal KP equation

In this subsection, we study the possible multiple soliton solutions for the four-place nonlocal KP equation (105). By using the symmetric-antisymmetric separation relations (118)–(120), the four-place nonlocal KP equation (105) can be equivalent to

$$u_{xt} + (u_{xx} + c_+u^2 + c_-z^2 + e_+v^2 + e_-w^2)_{xx} + 3\sigma^2u_{yy} = 0, \tag{138}$$

$$v_{xt} + (v_{xx} + d_+uv + d_-wz)_{xx} + 3\sigma^2v_{yy} = 0, \tag{139}$$

$$w_{xt} + (w_{xx} + b_+ uw + b_- vz)_{xx} + 3\sigma^2 w_{yy} = 0, \tag{140}$$

$$z_{xt} + (z_{xx} + a_+ wv + a_- uz)_{xx} + 3\sigma^2 z_{yy} = 0 \tag{141}$$

with the symmetric-antisymmetric conditions (121) and the constant relations

$$\begin{aligned} c_\pm &= U_{11} \pm U_{12} \pm U_{13} + U_{14} + U_{22} \\ &\quad + U_{23} \pm U_{24} + U_{33} \pm U_{34} + U_{44}, \\ e_\pm &= U_{11} \mp U_{12} \pm U_{13} - U_{14} + U_{22} \\ &\quad - U_{23} \pm U_{24} + U_{33} \mp U_{34} + U_{44}, \\ d_\pm &= 2(U_{11} \pm U_{13} - U_{22} \mp U_{24} + U_{33} - U_{44}), \\ b_\pm &= 2(U_{11} \pm U_{12} + U_{22} - U_{33} \mp U_{34} - U_{44}), \\ a_\pm &= 2(U_{11} \mp U_{14} - U_{22} \pm U_{23} - U_{33} + U_{44}). \end{aligned}$$

The system of equations (138)–(141) is not integrable for arbitrary constants $\{a_\pm, b_\pm, c_\pm, d_\pm, e_\pm\}$. For some special fixed parameters, for instance,

$$\begin{aligned} c_+ &= 3,\ c_- = e_+ = e_- = d_- = b_- \\ &= 0,\ d_+ = b_+ = a_+ = a_- = 6, \end{aligned} \tag{142}$$

the four-place nonlocal equation (105) becomes

$$\begin{aligned} &p_{xt} + 3\sigma^2 p_{yy} + \left[p_{xx} - 3u^2 + 6pu \right. \\ &\left. + \frac{3}{8}(p - p^{\hat{f}\hat{g}})^2 - \frac{3}{8}(p^{\hat{f}} - p^{\hat{g}})^2 \right]_{xx} = 0, \\ &u \equiv \frac{1}{4}(p + p^{\hat{f}} + p^{\hat{g}} + p^{\hat{f}\hat{g}}), \quad \hat{f} \equiv \hat{P}^x\hat{T},\ \hat{g} \equiv \hat{P}^y, \end{aligned} \tag{143}$$

while the related symmetric-antisymmetric system (138)–(141) becomes an integrable coupling system

$$u_{xt} + (u_{xx} + 3u^2)_{xx} + 3\sigma^2 u_{yy} = 0, \tag{144}$$

$$v_{xt} + (v_{xx} + 6uv)_{xx} + 3\sigma^2 v_{yy} = 0, \tag{145}$$

$$w_{xt} + (w_{xx} + 6uw)_{xx} + 3\sigma^2 w_{yy} = 0, \tag{146}$$

$$z_{xt} + (z_{xx} + 6wv + 6uz)_{xx} + 3\sigma^2 z_{yy} = 0. \tag{147}$$

Because (145) and (146) is just the symmetry equations of (144) and the system of equations (146) and (147) is also a symmetry system of (144) and (145), it is not difficult to find some special solutions of (144)–(147) and then the solutions of the four-place nonlocal KP equation (143). A special multiple soliton solutions of (143) can be written as

$$p = 2(1 + \beta_1\partial_y + \beta_2\partial_x + \beta_1\beta_2\partial_x\partial_y)(\ln\psi)_{xx}, \tag{148}$$

where ψ is given in (129) with the paired condition (130) satisfying the symmetric-antisymmetric conditions (121). $\psi = F_2$ with (134) and $\psi = F_4$ with (136) are two simplest two-soliton and four-soliton examples. The condition (130) implies that the odd number of soliton solutions in the form (148) are prohibited for the four-place nonlocal KP equation (143). This kind of classical prohibition property is firstly found for the Boussinesq system [93] and it may be universal for partially reversal nonlocal systems. A partially reversal nonlocal system is defined as a multi-place system possessing at least one of the places is not fully space-time reversal.

5. Summary and discussions

In summary, to describe multi-events happened at different places and times, multi-place nonlocal integrable (and non-integrable) nonlinear models have been systematically derived by means of the discrete symmetry reductions of the coupled local systems. Especially, various two-place and four-place nonlocal integrable models are obtained.

Starting from every multi-component AKNS system, one may derive some local and nonlocal multi-place AKNS, NLS and Manakov systems. For instance, from the two-component AKNS system (31), one can obtain the usual local AKNS system (34) with $\hat{f} = 1$, local NLS equation (34) with $\{\hat{f} = 1, r = q^*\}$, local Manakov system (40) with $\hat{g} = \hat{C}$, three types of two-place nonlocal AKNS systems (34) with $\hat{f} \neq 1$, three types of two-place nonlocal Manakov models (40) with $\hat{g} \neq \hat{C}$, nine types of two-place nonlocal NLS equations (62) with (67)–(72) and $\{\alpha, \beta\} = \{\alpha, 0\}$ or $\{\alpha, \beta\} = \{0, \alpha\}$, and six types of four-place nonlocal NLS systems (62) with (73)–(74) and $\{\alpha, \beta\} = \{\alpha, 0\}$ or $\{\alpha, \beta\} = \{0, \alpha\}$.

In fact, starting from every coupled nonlinear systems, one may also find some types of multi-place nonlocal systems via discrete symmetry reductions. In addition to the NLS equation, the (2+1)-dimensional KP equation is another important physically applicable model. To find some types of multi-place extensions of the KP equation, the matrix KP equations are best candidates. In this paper, some types of multi-place nonlocal KP equations are obtained from the $\hat{P}\hat{T}\hat{C}$ symmetry reductions from some special Abelian and non-Abelian matrix KP equations.

Because many nonlocal nonlinear systems can be derived from the $\hat{P}\hat{T}\hat{C}$ symmetry reductions, the nonlocal systems may be solved via $\hat{P}$–$\hat{T}$–$\hat{C}$ symmetric-antisymmetric separation approach (SASA). Using SASA, the two-place nonlocal KP equation (122) and four-place nonlocal KP system (143) are explicitly solved for special types of multiple soliton solutions. Similar to the two-place nonlocal Boussinesq equation with partially space-time reversal nonlocalities [93], the odd numbers of soliton solutions are prohibited for the four-place nonlocal KP equation (143).

Because all the known multi-place nonlocal systems (and their solutions) mentioned in all the references of this paper can be considered as discrete symmetry reductions of multiple component coupled local ones, it is natural to ask the following important question:

What is physically and/or mathematically new for these kinds of nonlocal systems?

To answer this question, by summarizing all the known results, we list some significant novel points to end this paper.

(i). *Multi-place correlations*. As pointed out in the title and the introduction section, the multi-place systems describes the correlations and or entanglements among multiple events happened at different space-times [25, 26]. Some of two-place nonlocal systems can be used to approximately solve the real physical problems [26, 28, 51, 52, 93, 94].

(ii). *Classical prohibitions.* It is well known in the quantum case, there are some kinds of quantum prohibitions which have not yet found in classical physics. For partially reversal multi-place systems, one can find that there are classical prohibitions as mentioned in the last section for the four-place nonlocal KP equations. For the usual local Boussinesq system, there are N solitons for arbitrary positive integer numbers and the soliton interactions may be both pursuit interactions and head on interactions with arbitrary wave numbers and velocities. However, for the two-place nonlocal Boussinesq equation, the odd numbers of solitons are prohibited, the pursuit interactions are not allowed and only the head-on interactions with the same velocities are permitted [93, 100, 101].

(iii). *Transitions caused by nonlocality.* In nonlinear optics, it is well known that there are two different types of materials, normal and abnormal dispersion materials. The bright solitons can only be found in abnormal dispersion material while the dark solitons can only be found in normal dispersion materials. However, for the multi-place nonlocal systems, one can find that there are possible transformations such that the bright solitons can be changed to dark solitons under the soliton interactions [27, 32]. Similar transitions can also be caused by other types of nonlocalities [102, 103].

(iv). *Structure modifications.* Rogue waves/instantons and lumps (more generally rational solutions) are recent important topics [104–107]. It is known that the usual lowest order rogue waves (lumps) possess four-leaf structure. However, because of the introduce of the multi-place nonlocalities in the model, the structure of the rogue waves and lumps may be changed from four leaves to five leaves and six leaves as parameter or time changes [93, 100, 101].

(v). *Nonlinear excitations with special reversal symmetries of initial and/or boundary conditions.* In [99], a nonlocal NLS equation is obtained from a special reduction of the Manakov system which governs wave propagation in a wide variety of physical systems. Thus, this kind of nonlocal systems possess clear physical meanings. In fact, the general coupled local NLS system (76) and (77) is obtained from the well-known physically significant KP equation [97], its reductions (62) with (66)–(75) have clear physical meanings. If the initial conditions and/or boundary conditions possess some types of full or partial reversal symmetries then the related of nonlinear excitations of the original local physical models satisfy their related reversal symmetric nonlocal reductions.

(vi). *Weaken the Hirota's integrable sense.* Usually, if a model possesses n-soliton solutions with arbitrary n, one may call it Hirota integrable. Almost all the integrable systems in other senses, say, Lax integrable, are also Hirota integrable. However, there are some special Hirota integrable examples are not integrable under other senses. After introducing the multi-place nonlocalities, one can find that there are infinitely many nonintegrable systems possess n-solitons [26].

(vii). *Integrable systems without n-solitons for arbitrary n.* Because of the classical prohibition property for the multi-place nonlocal systems [93, 100, 101], we known that an integrable (in any sense) nonlocal system may not possess n-solitons for arbitrary n.

(viii) *Mixing of linear and nonlinear waves.* It is known that the linear superposition theorem is not valid for nonlinear systems. It is also interesting that some types of linear waves and nonlinear waves may be linearly mixed if some kinds of nonlocalities are introduced [93].

(ix) *Existences of many first and second order integrable systems.* It is known that in local case, there are very few integrable systems for the first- and second-order nonlinear partial differential equations. However, if the multiple place non-localities are considered, one can find various first- and second-order integrable systems.

(x) *New methods to solve nonlinear systems.* In the study of nonlinear multi-place nonlocal systems, in addition to the well-known traditional powerful approaches, some new types of methods to solve nonlinear systems have been established. For instance, the symmetric-antisymmetric separation approach with respect to the discrete symmetry operators can be successfully applied to solve nonlocal systems. The full reversal invariance method [25, 26, 108, 109] can be used to find invariant solutions for numerous types of multi-place nonlocal systems.

Acknowledgments

The author is grateful to thank Professors W L Yang, Z N Zhu, Z Y Yan, D J Zhang, X Y Tang, T Xu, Q P Liu, X B Hu, Y Q Li, J S He and Y Chen for their helpful discussions. The work was sponsored by the National Natural Science Foundations of China (No. 11 975 131, 11 435 005) and K C Wong Magna Fund in Ningbo University.

ORCID iDs

S Y Lou (楼森岳) https://orcid.org/0000-0002-9208-3450

References

[1] Ablowitz M J and Musslimani Z H 2013 *Phys. Rev. Lett.* **110** 064105
[2] Bender C M 2007 *Rep. Prog. Phys.* **70** 947
[3] Markum H, Pullirsch R and Wettig T 1999 *Phys. Rev. Lett.* **83** 484
[4] Lin Z, Schindler J, Ellis F M and Kottos T 2012 *Phys. Rev.* A **85** 050101
[5] Ruter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 *Nat. Phys.* **6** 192
[6] Musslimani Z H, Makris K G, El-Ganainy R and Christodoulides D N 2008 *Phys. Rev. Lett.* **100** 030402
[7] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463
[8] Sinha D and Ghosh K P 2015 *Phys. Rev.* E **91** 042908
[9] Valchev T 2015 *Phys. Lett.* A **379** 1877
[10] Huang X and Ling L M 2016 *Euro. Phys. J. Plus* **131** 148
[11] Gerdjikov V S and Saxena A 2017 *J. Math. Phys.* **58** 013502
[12] Ablowitz M J, Luo X D and Musslimani Z H 2018 *J. Math. Phys.* **59** 011501

[13] Rybalko Y and Shepelsky D 2018 *J. Math. Phys.* **60** 031504
[14] Rusin R, Kusdiantara R and Susanto H 2019 *Phys. Lett.* A **383** 2039
[15] Feng W and Zhao S L 2019 *Rep. Math. Phys.* **84** 75
[16] Yang B and Chen Y 2018 *Nonl. Dyn.* **94** 489
[17] Yang B and Chen Y 2018 *Chaos* **28** 053104
[18] Yan Z 2015 *Appl. Math. Lett.* **47** 61
[19] Wen X, Yan Z and Yang Y 2016 *Chaos* **26** 063123
[20] Zhang G, Yan Z and Chen Y 2017 *Appl. Math. Lett.* **69** 113
[21] Li M and Xu T 2015 *Phys. Rev.* E **91** 033202
[22] Li M, Xu T and Meng D X 2016 *J. Phys. Soc. Jpn.* **85** 124001
[23] Xu T, Lan S, Li M, Li L L and Zhang G W 2019 *Physica* D **390** 47
[24] Xu T, Chen Y, Li M and Meng D X 2019 *Chaos* **29** 123124
[25] Lou S Y 2018 *J. Math. Phys.* **59** 083507
[26] Lou S Y and Huang F 2017 *Sci. Rep.* **7** 869
[27] Li C C, Lou S Y and Jia M 2018 *Nonl. Dyn.* **93** 1799
[28] Tang X Y, Liang Z F and Hao X Z 2018 *Nonl. Sci. Numer. Simul.* **60** 62
[29] Jia M and Lou S Y 2016 *Phys. Lett.* A **382** 1157
[30] Ablowitz M J and Musslimani Z H 2016 *Nonlinearity* **29** 915
[31] Ji J L and Zhu Z N 2017 *J. Math. Anal. Appl.* **453** 973
[32] Li C C, Xia Q and Lou S Y 2018 *Commun. Theor. Phys.* **70** 7
[33] Luo X D 2019 *Chaos* **29** 073118
[34] Li L, Duan C N and Yu F J 2019 *Phys. Lett.* A **383** 1578
[35] Ma Z Y, Fei J X and Chen J C 2018 *Commun. Theor. Phys.* **70** 31
[36] Ablowitz M J and Musslimani Z H 2014 *Phys. Rev.* E **90** 032912
[37] Hu Y H and Chen J C 2018 *Chin. Phys. Lett.* **35** 110201
[38] Ji J L, Xu Z W and Zhu Z N 2019 *Chaos* **29** 103129
[39] Song C Q, Xiao D M and Zhu Z N 2017 *Commun. Nonl. Sci. Numer. Simulat* **45** 13
[40] Zhou H J, Li C Z and Lin Y L 2018 *Adv. Math. Phys.* **2018** 9216286
[41] Grahovski G G, Mustafa J I and Susanto H 2018 *Theor. Math. Phys.* **197** 1430
[42] Fan R and Yu F J 2018 *Commun. Theor. Phys.* **70** 651
[43] Yan Z 2016 *Appl. Math. Lett.* **62** 101
[44] Yan Z 2018 *Appl. Math. Lett.* **79** 123
[45] Zhang G and Yan Z 2017 *EPL (Europhysics Letters)* **118** 60004
[46] Wu W and He J 2016 *Rom. Rep. Phys.* **68** 79
[47] Dimakos M and Fokas A S 2013 *J. Math. Phys.* **54** 081504
[48] Fokas A S 2006 *Phys. Rev. Lett.* **96** 190201
[49] Fokas A S 2016 *Nonlinearity* **29** 319
[50] Ioannou-Sougleridis I, Fyantzeskakis J D and Horikis P T 2020 *Stud. Appl. Math.* **144** 3
[51] Tang X Y and Liang Z F 2018 *Nonl. Dyn.* **92** 815
[52] Tang X Y, Liu S J, Liang Z F and Wang J Y 2018 *Nonl. Dyn.* **94** 693
[53] Lou S Y and Qiao Z J 2017 *Chin. Phys. Lett.* **34** 100201
[54] Zhao Q L, Jia M and Lou S Y 2019 *Commun. Theor. Phys.* **71** 1149
[55] Cen J, Correa F and Fring A 2019 *J. Math. Phys.* **60** 081508
[56] Zhu X M and Zuo D F 2019 *Appl. Math. Lett.* **91** 181
[57] Grahovski G G, Mohammed A J and Susanto H 2018 *Theor. Math. Phys.* **197** 1412
[58] Ablowitz M J, Feng B F and Luo X D 2018 *Stud. Appl. Math.* **141** 267
[59] Brunelli J C 2018 *Braz. J. Phys.* **48** 421
[60] Demontis F, Ortenzi G and van der Mee C 2018 *J. Geom. Phys.* **127** 84
[61] Gurses M 2017 *Phys. Lett.* A **381** 1791
[62] Gerdjikov V S, Grahovski G G and Ivanov R I 2016 *Theor. Math. Phys.* **188** 1305
[63] Liu W *et al* 2016 *Commun. Theor. Phys.* **65** 671
[64] Ji J L, Huang Z L and Zhu Z N 2017 *Anal Math. Sci. Appl.* **2** 409
[65] Ji J L, Yang J and Zhu Z N 2020 *Commun. Nonlinear Sci. Numer. Simul.* **82** 105028
[66] Zhang G and Yan Z 2020 *Physica* D **402** 132170
[67] Ji J L and Zhu Z N 2017 *Commun. Nonl. Sci. Numer. Simul.* **42** 699
[68] Wen Z and Yan Z 2017 *Chaos* **27** 053105
[69] Ji J L and Zhu Z N 2017 *J. Math. Anal. Appl.* **453** 973
[70] Ma L Y, Shen S F and Zhu Z N 2017 *J. Math. Phys.* **58** 103501
[71] Ma L Y and Zhu Z N 2016 *Appl. Math. Lett.* **59** 115
[72] Ma L Y and Zhu Z N 2016 *J. Math. Phys.* **57** 083507
[73] Feng B F, Luo X D, Ablowitz M J and Musslimani Z H 2018 *Nonlinearity* **31** 5385
[74] Song C Q, Xiao D M and Zhu Z N 2017 *J. Phys. Soc. Jpn.* **86** 054001
[75] Ma L Y, Zhao H Q and Gu H 2018 *Nonlinear Dyn.* **91** 1909
[76] Chen K, Deng X, Lou S Y and Zhang D J 2018 *Stud. Appl. Math.* **141** 113
[77] Zhou Z X 2018 *Stud. Appl. Math.* **141** 186
[78] Rao J, Cheng Y and He J S 2017 *Stud. Appl. Math.* **139** 568
[79] Rao J, Zhang Y, Fokas A S and He J 2018 *Nonlinearity* **31** 4090
[80] Rao J, Cheng Y, Porsezian K, Mihalache S and He J 2020 *Physica* D **401** 132180
[81] Yang B and Yang J K 2018 *Stud. Appl. Math.* **140** 178
[82] Yang B and Chen Y 2019 *Commun. Nonl. Sci. Numer. Simu.* **69** 287
[83] Yang B and Chen Y 2018 *Appl. Math. Lett.* **82** 43
[84] Chen K and Zhang D J 2018 *Appl. Math. Lett.* **75** 82
[85] Deng X, Lou S Y and Zhang D J 2018 *Appl. Math. Comput.* **332** 477
[86] Chen K, Liu S M and Zhang D J 2019 *Appl. Math. Lett.* **88** 230
[87] Shi Y, Shen S F and Zhao S L 2019 *Nonlinear Dyn.* **95** 1257
[88] Liu S M, Wu H and Zhang D J 2019 New dynamics of the classical and nonlocal Gross–Pitaevskii equation with a parabolic potential (arXiv:2003.01865 [nlin.SI])
[89] Feng W and Zhao S L 2019 *Rep. Math. Phys.* **84** 75
[90] Lou S Y 2019 Multi-place nonlocal systems (arXiv:1901.02828 [nlin.SI])
[91] Jia M, Gao Y, Huang F, Lou S Y, Sun J L and Tang X Y 2012 *Nonl. Anal. Real Word Appl.* **13** 2079
[92] Pedlosky J 1979 *Geophysical Fluid Dynamics* (Berlin: Springer)
[93] Lou S Y 2019 *Stud. Appl. Math.* **143** 123
[94] Ablowitz M J and Musslimani Z H 2019 *J. Phys. A: Math. Theor.* **52** 15LT02
[95] Lou S Y 2017 *Chin. Phys. Lett.* **34** 060201
[96] Ablowitz M J and Musslimani Z H 2016 *Stud. Appl. Math.* **139** 7
[97] Lou S Y and Hu X B 1997 *J. Math. Phys.* **38** 6401
[98] Lou S Y, Chen C L and Tang X Y 2002 *J. Math. Phys.* **43** 4078
[99] Yang J K 2018 *Phys. Rev.* E **98** 042202
[100] Li H and Lou S Y 2019 *Chin. Phys. Lett.* **36** 050501
[101] Hu W B and Lou S Y 2019 *Commun. Theor. Phys.* **71** 629
[102] Liang G *et al* 2019 *Phys. Rev.* A **99** 063808
[103] Hu Y H and Lou S Y 2015 *Commun. Theor. Phys.* **64** 665
[104] Dudley J M, Genty G, Mussot A, Chabchoub A and Dias F 2019 *Nat. Rev. Phys.* **1** 675
[105] Onorato M, Residori S, Bortolozzo U, Montinad A and Arecchi F T 2013 *Phys. Rep.* **528** 47
[106] Tang X Y, Lou S Y and Zhang Y 2002 *Phys. Rev. E.* **66** 046601
[107] Du X and Lou S Y 2019 *Commun. Theor. Phys.* **71** 633
[108] Lou S Y 2020 *Acta. Phys. Sin.* **69** 010503 (in Chinese)
[109] Yan Z W and Lou S Y 2020 *Appl. Math. Lett.* **104** 106271