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Soliton molecules and some novel hybrid solutions for the (2+1)-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation

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Abstract

Soliton molecules have become one of the hot topics in recent years. In this article, we investigate soliton molecules and some novel hybrid solutions for the (2+1)-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt (gKDKK) equation by using the velocity resonance, module resonance, and long wave limits methods. By selecting some specific parameters, we can obtain soliton molecules and asymmetric soliton molecules of the gKDKK equation. And the interactions among N-soliton molecules are elastic. Furthermore, some novel hybrid solutions of the gKDKK equation can be obtained, which are composed of lumps, breathers, soliton molecules and asymmetric soliton molecules. Finally, the images of soliton molecules and some novel hybrid solutions are given, and their dynamic behavior is analyzed.

Keywords: the (2+1)-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt equation, soliton molecules, hybrid solutions, velocity resonance, long-wave limit

(Some figures may appear in colour only in the online journal)

1. Introduction

Soliton molecules are soliton bound states, which have been observed in some fields [1–6]. In 2012, the numerical prediction of a soliton molecule in a Bose–Einstein condensate is made. In 2017, Herink *et al* [7] obtained the evolution of femtosecond soliton molecules in the cavity of a few-cycle mode-locked laser through an emerging time-stretching technique. In 2013, Rohrmann *et al* [8] studied the existence state and stability of bound states two and three solitons in dispersion managed fibers. Lou *et al* [9] theoretically obtained (1+1)-dimensional soliton molecules and asymmetric solitons of fluid systems by using velocity resonance in 2019. Zhang *et al* [10] obtained soliton molecules, asymmetric solitons and hybrid solutions for (2+1)-dimensional fifth-order KdV equation in 2019. After that, Yan and Lou [11] obtained the kink molecules, half periodic kink molecules and breathing soliton molecules of (1+1)-dimensional Sharma–Tasso–Olver–Burgers equation in 2020. Yang *et al* [12] derived some novel soliton molecules, breather waves and lump waves of (2+1)-dimensional B-type Kadomtsev–Petviashvili equation by applying velocity resonance, module resonance and long wave limit method. Based on Darboux transformation, Zhang *et al* [13] derived a molecule consisting of two identical soliton waves and molecules containing a plurality of solitons for modified KdV equation by applying velocity resonance. Dong *et al* [14] obtained soliton molecules, asymmetric solitons and mixed solutions of the (2+1)-dimensional bidirectional Sawada–Kotera (SK) equation in 2020. So the soliton molecular problem has become one of the most advanced problems in research.

In the field of nonlinear science, nonlinear local waves and interaction solutions are hot topics. In recent years, many scholars have obtained different types of new excited states

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by soliton resonances, such as lumps [15, 16], breathers [17, 18] (caused by the module resonance of wave numbers, say, $|k_l| = k_2$, i.e. $k_2 = \pm k_1^*$), rogue wave [19] and so on. The researchers also studied local wave interactions such as lump-kink [20], cross-kink [21] and so on. In this work, we will investigate soliton molecules and some novel hybrid solutions for the (2+1)-dimensional generalized Konopelchenko–Dubrovsky–Kaup–Kupershmidt (gKDKK) equation by using velocity resonance, module resonance, and long wave limits methods. The (2+1)-dimensional gKDKK equation [22] is

$$\begin{aligned} &u_t + h_1 u_{xxx} + h_2 u u_x + h_3 u_{xxxxx} + h_4 v_y \\ &\quad + h_5 u_{xxy} + h_6(u_x v + u v_x) \\ &\quad + h_7(u_x uxx + u u_{xxx}) + h_8 u^2 u_x = 0, \\ &u_y = v_x, \end{aligned} \tag{1}$$

where u and v are function of the variables x, y and t, and the coefficients h_i $(i = 1, 2, \ldots, 8)$ are the real parameters. When selecting different parameters for h_i, we can obtain the (2+1)-dimensional SK equation [23], the Caudrey–Dodd–Gibbon–Sawada–Kotera equation [24], the Bogoyavlensky–Konoplechenko equation [25] and the isospectral BKP equation [26]. Obviously, equation (1) can describe many physical phenomena. In 2016, Feng *et al* [22] derived the bilinear form of equation (1), and obtained N-soliton solution, the periodic wave solution and the asymptotic behavior of the gKDKK equation. Liu *et al* [27] investigated the lumpoff waves, lump waves, and rogue waves of the gKDKK equation based on the Hirota's bilinear method. Soliton solutions, lump waves, rogue waves of the gKDKK equation have been studied, but soliton molecules and asymmetric solitons for the gKDKK equation have not been explored. So we will introduce soliton molecules, asymmetric solitons and some novel hybrid solutions for the gKDKK equation in this paper.

In section 1, we will introduce N-soliton solutions form about the (2+1)-dimensional gKDKK equation. In section 2, based on the velocity resonance and the soliton solution, we obtain soliton molecules and asymmetric solitons about the (2+1)-dimensional gKDKK equation. In section 3, by using the N-soliton solutions, velocity resonance, module resonance and long-wave limit, we study some novel hybrid solutions for the (2+1)-dimensional gKDKK equation, which consists of breather waves, lump waves and soliton molecules. We conclude this article in section 4.

N-soliton solutions of equation (1) can be written as [22]

$$u = 12 h_1 h_2^{-1} (\ln f)_{xx}, \tag{2}$$

$$f = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{N} \mu_i \mu_j A_{ij}\right), \tag{3}$$

where

$$\begin{aligned} &\eta_i = k_i x + p_i y + \omega_i t + \phi_i, \\ &e^{A_{ij}} = -\frac{-h_3(k_i - k_j)^6 - h_1(k_i - k_j)^4 - h_4(p_i - p_j)^2 - (k_i - k_j)(w_i - w_j) - (k_i - k_j)^3(p_i - p_j)}{-h_3(k_i + k_j)^6 - h_1(k_i + k_j)^4 - h_4(p_i + p_j)^2 - (k_i + k_j)(w_i + w_j) - (k_i + k_j)^3(p_i + p_j)}, \end{aligned} \tag{4}$$

with

$$\begin{aligned} \omega_i = &-(h_i k_i^3 + h_3 k_i^5 + h_5 k_i^2 p_i \\ &+ h_4 p_i^2 k_i^{-1}),\ (i, j = 1, 2, \cdots, N), \end{aligned} \tag{5}$$

where the parameters k_i and p_i are arbitrary constants.

2. Soliton molecules

To find soliton molecules for the gKDKK equation, we use a novel resonant conditions ($k_i \neq \pm k_j$, $p_i \neq \pm p_j$), the velocity resonance:

$$\frac{k_i}{k_j} = \frac{p_i}{p_j} = \frac{\omega_i}{\omega_j}. \tag{6}$$

Then we have the following expressions:

$$\begin{aligned} k_i &= \sqrt{-\frac{h_3 k_j^3 + h_1 k_j + h_5 p_j}{h_3 k_j}}, \\ p_i &= \frac{p_i}{k_j}\sqrt{-\frac{h_3 k_j^3 + h_1 k_j + h_5 p_j}{h_3 k_j}}, \end{aligned} \tag{7}$$

or

$$\begin{aligned} k_i &= -\sqrt{-\frac{h_3 k_j^3 + h_1 k_j + h_5 p_j}{h_3 k_j}}, \\ p_i &= -\frac{p_i}{k_j}\sqrt{-\frac{h_3 k_j^3 + h_1 k_j + h_5 p_j}{h_3 k_j}}. \end{aligned} \tag{8}$$

As we all know, for $N = 2$ in equation (3), we can get the two-soliton solution to the gKDKK equation. Under the resonance condition (6), select parameter:

$$\begin{aligned} &h_1 = -1,\ h_2 = -6,\ h_3 = -1,\ h_4 = -1,\ h_5 = -1, \\ &k_1 = -\frac{1}{3},\ p_1 = \frac{8}{5},\ k_2 = \frac{\sqrt{830}}{15},\ p_2 = -\frac{8\sqrt{830}}{25}, \\ &\phi_1 = 0,\ \phi_2 = -60. \end{aligned} \tag{9}$$

We can observe that the two-soliton solution behaves as one soliton molecule in figures 1(a) and (c). If the value of ϕ_1 and ϕ_2 changes, the distance between the solitons in the molecule will change. Figure 1(b) shows two solitons interact, and the symmetric solitons are transformed into asymmetric solitons at $\phi_2 = -10$.

When $N = 4$ in equation (3), k_1, p_1, ω_1 and k_2, p_2, ω_2 satisfy the velocity resonance condition (6), k_3, p_3, ω_3 and k_4, p_4, ω_4 satisfy the velocity resonance condition (6), the

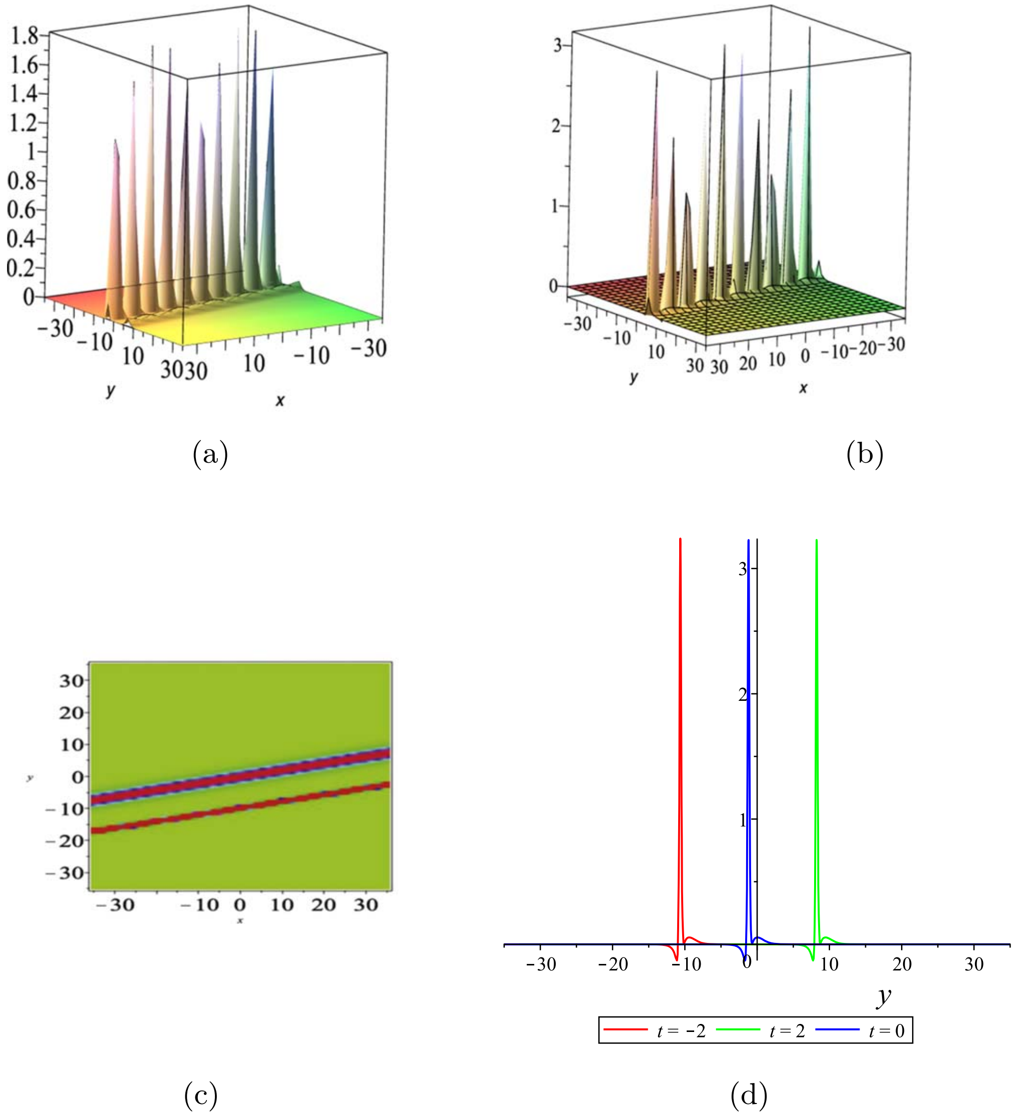


Figure 1. The solution u to equation (1) at $t = 0$. (a) and (c) Soliton molecule structure, (b) and (d) asymmetric soliton molecule.

four-soliton solution can represent two soliton molecules. We set:

$$\begin{aligned}&h_1 = -1,\ h_2 = -6,\ h_3 = -1,\ h_4 = -1,\ h_5 = -1,\\ &k_1 = -\frac{1}{3},\ p_1 = \frac{8}{5},\ k_2 = \frac{\sqrt{830}}{15},\ p_2 = -\frac{8\sqrt{830}}{25},\\ &k_3 = -\frac{1}{2},\ p_3 = 1,\ k_4 = \frac{\sqrt{3}}{2},\ p_4 = -\sqrt{3},\\ &\phi_1 = 10,\ \phi_2 = -20,\ \phi_3 = -20,\ \phi_4 = 40.\end{aligned} \tag{10}$$

Figure 2 shows the elastic interaction characteristics of two soliton molecules for the gKDKK equation.

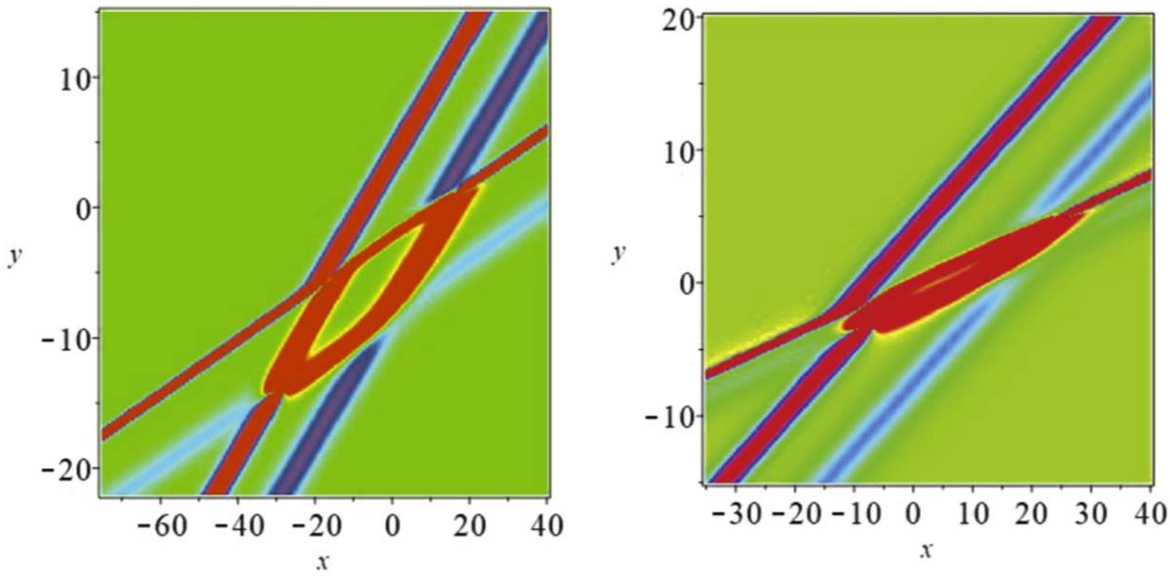
3. Some novel hybrid solutions

In this section, we investigate the interaction solutions to soliton molecules with breather and lump through velocity resonance, module resonance, and long-wave limit method. As far as we know, solutions to the interaction of soliton molecules of the gKDKK equation with breather and lump have not been studied.

When $N = 4$ in equation (3), k_1, p_1, ω_1 and k_2, p_2, ω_2 accord with velocity resonance condition, η_3, η_4 accord with module resonance condition $\eta_3 = \overline{\eta_4}$, we select this parameters as follows:

$$\begin{aligned}&h_1 = -1,\ h_2 = -6,\ h_3 = -1,\ h_4 = -1,\ h_5 = -1,\\ &k_1 = -\frac{1}{3},\ p_1 = \frac{8}{5},\ k_2 = \frac{\sqrt{830}}{15},\ p_2 = -\frac{8\sqrt{830}}{25},\\ &k_3 = \frac{1}{7} + \frac{1}{7}\mathrm{i},\ p_3 = \frac{1}{8} - \frac{1}{2}\mathrm{i},\ k_4 = \frac{1}{7} - \frac{1}{7}\mathrm{i},\ p_4 = \frac{1}{8} + \frac{1}{2}\mathrm{i},\\ &\phi_1 = 0,\ \phi_2 = -60,\ \phi_3 = 0,\ \phi_4 = 0.\end{aligned} \tag{11}$$

We can obtain the interaction between one soliton molecule and one breather wave as show in figure 3. By changing the value of ϕ_1, ϕ_2, we can obtain the solution to the interaction between asymmetric soliton molecules and breather wave in figure 3(b). We can observe that they are elastic collisions from figure 3.



(a) (b)

Figure 2. Two soliton molecules structure for equation (1) at $t = 0$.

Based on the N-soliton solutions, we can obtain the interaction between the soliton molecule and a lump wave by using the long-wave limit method. For example, when $N = 4$, we taking a long-wave limit $k_3, k_4, p_3, p_4(\epsilon \to 0)$, k_1, p_1, k_2, p_2 satisfy the velocity resonance condition, we set:

$$\begin{aligned}
&h_1 = -1,\ h_2 = -6,\ h_3 = -1,\ h_4 = -1,\ h_5 = -1,\\
&k_1 = -\frac{1}{3},\ p_1 = \frac{8}{5},\ k_2 = \frac{\sqrt{830}}{15},\ p_2 = -\frac{8\sqrt{830}}{25},\\
&k_3 = \left(\frac{1}{2} + \frac{4}{7}\mathrm{i}\right)\epsilon,\ p_3 = 2\epsilon,\ k_4 = \left(\frac{1}{2} - \frac{4}{7}\mathrm{i}\right)\epsilon,\ p_4 = 2\epsilon,\\
&\phi_1 = -10,\ \phi_2 = -60,\ \phi_3 = \pi\mathrm{i},\ \phi_4 = \pi\mathrm{i}.
\end{aligned} \tag{12}$$

Figure 4 exhibit the elastic collision of one soliton molecule and one lump wave. From figure 5, we get the asymmetric soliton molecule and one lump wave elastic collision by changing the value of ϕ_1, ϕ_2.

For the more generalized case, in order to obtain hybrid solutions consisting of m soliton molecules, h breather waves and s lump waves, we can make the following restrictions on the parameters,

$$\begin{aligned}
&\frac{k_1}{k_2} = \frac{p_1}{p_2} = \frac{\omega_1}{\omega_2},\ \cdots,\frac{k_{2m-1}}{k_{2m}} = \frac{p_{2m-1}}{p_{2m}} = \frac{\omega_{2m-1}}{\omega_{2m}},\\
&\eta_{2m+1} = \overline{\eta_{2m+2}},\ \cdots,\eta_{2m+2h-1} = \overline{\eta_{2m+2h}},\\
&k_{2m+2h+1} = K_{2m+2h+1}\epsilon,\ k_{2m+2h+2} = \overline{K_{2m+2h+1}\epsilon},\\
&p_{2m+2h+1} = P_{2m+2h+1}\epsilon,\ p_{2m+2h+1} = \overline{P_{2m+2h+1}\epsilon},\ \cdots\\
&k_{2m+2h+2s-1} = K_{2m+2h+2s-1}\epsilon,\ k_{2m+2h+2s}\\
&\quad = \overline{K_{2m+2h+2s-1}\epsilon},\ \cdots,\\
&p_{2m+2h+2s-1} = P_{2m+2h+2s}\epsilon,\ p_{2m+2h+2s}\\
&\quad = \overline{P_{2m+2h+2s-1}\epsilon},\ \epsilon \to 0,\\
&\phi_{2m+2h+1} = \pi\mathrm{i},\ \cdots,\phi_{2m+2h+2s} = \pi\mathrm{i}.
\end{aligned} \tag{13}$$

In order to show their interaction better, let us take $N = 6$ case as an example. When $N = 4$ in equation (3), k_1, p_1, ω_1 and k_2, p_2, ω_2 satisfy velocity resonance condition, taking a long-wave limit k_3, k_4, p_3, $p_4(\epsilon \to 0)$ and η_5, η_6, satisfy module resonance condition, we set

$$\begin{aligned}
&h_1 = -1,\ h_2 = -6,\ h_3 = -1,\ h_4 = -1,\ h_5 = -1,\\
&k_1 = -\frac{1}{3},\ p_1 = \frac{8}{5},\ k_2 = \frac{\sqrt{830}}{15},\ p_2 = -\frac{8\sqrt{830}}{25},\\
&k_3 = \left(\frac{1}{2} + \frac{4}{7}\mathrm{i}\right)\epsilon,\ p_3 = 2\epsilon,\ k_4 = \left(\frac{1}{2} - \frac{4}{7}\mathrm{i}\right)\epsilon,\ p_4 = 2\epsilon,\\
&k_5 = \frac{1}{7} + \frac{1}{7}\mathrm{i},\ p_5 = \frac{1}{8} - \frac{1}{2}\mathrm{i},\ k_6 = \frac{1}{7} - \frac{1}{7}\mathrm{i},\ p_6 = \frac{1}{8} + \frac{1}{2}\mathrm{i},\\
&\phi_1 = -10,\ \phi_2 = -60,\ \phi_3 = \pi\mathrm{i},\ \phi_4 = \pi\mathrm{i},\ \phi_5 = 0,\ \phi_6 = 0.
\end{aligned} \tag{14}$$

We can get a solution for the interaction between one soliton molecule, one breather wave and one lump wave in figure 6(a). In figure 6(b), we can observe the interaction of one asymmetric soliton molecule, one breather wave and one lump wave.

4. Conclusions

In summary, we studied soliton molecules, asymmetric solitons and some hybrid solutions for the gKDKK equation. By selecting the appropriate ϕ_i, soliton molecules become asymmetric soliton molecules. Based on velocity resonance and module resonance conditions, we can obtain hybrid solutions of soliton molecules and breathers waves. Based on the resonance condition and long-wave limit method, we can obtain the elastic collision of one asymmetric soliton molecule and one lump wave. Using velocity resonance, module resonance, and long-wave limit methods, we can get some

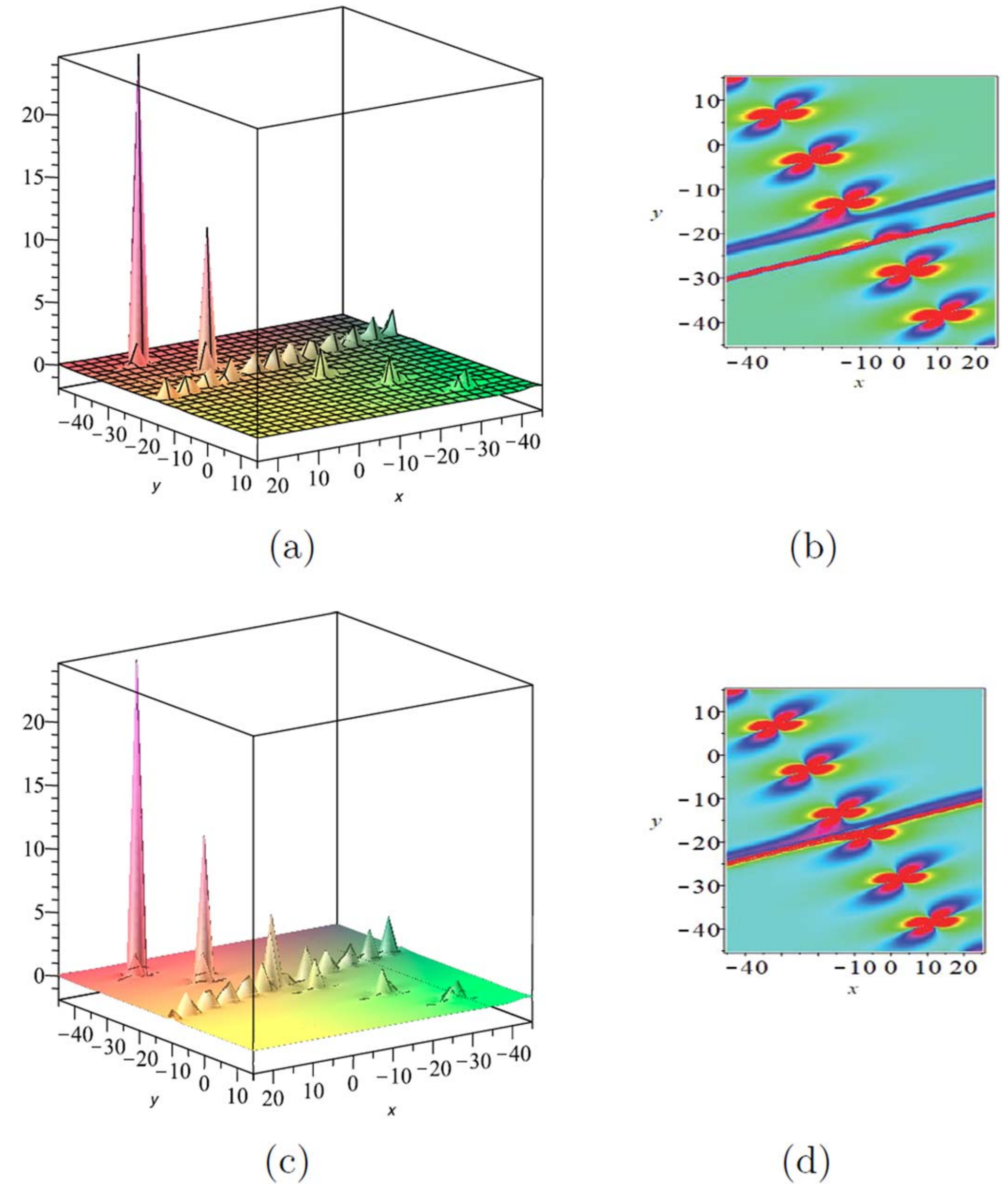


Figure 3. (a) and (b) Interaction of one soliton molecule and one breather wave for equation (1) at $t = -3$, (c) and (d) interaction of asymmetric soliton molecule and one breather wave at $t = -3$, $\phi_2 = -10$.

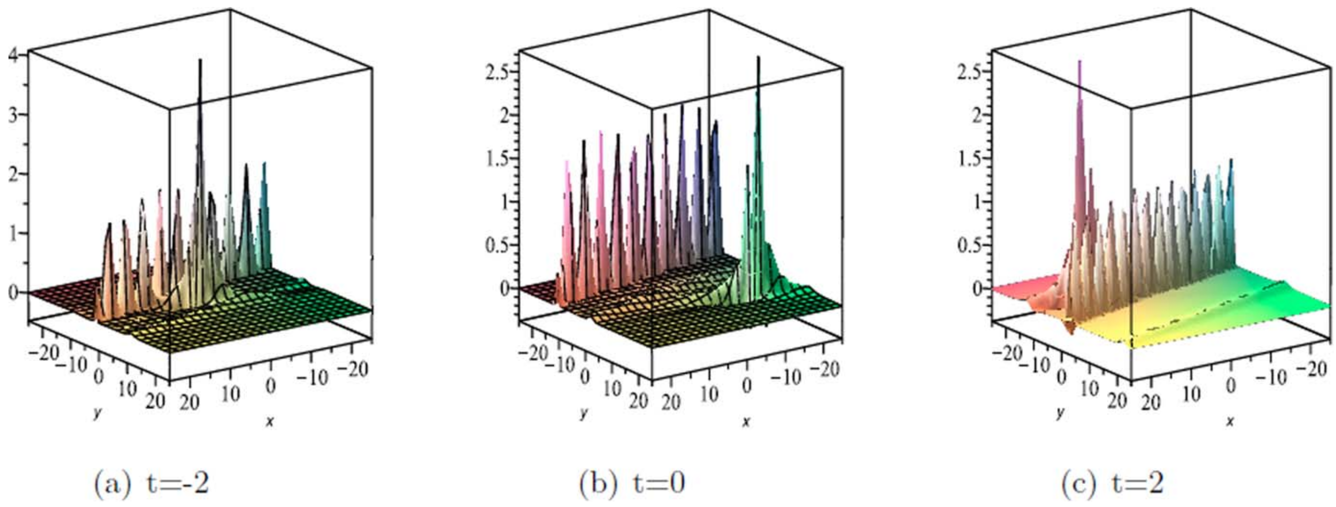


Figure 4. The elastic collision of one soliton molecule and one lump wave for equation (1).

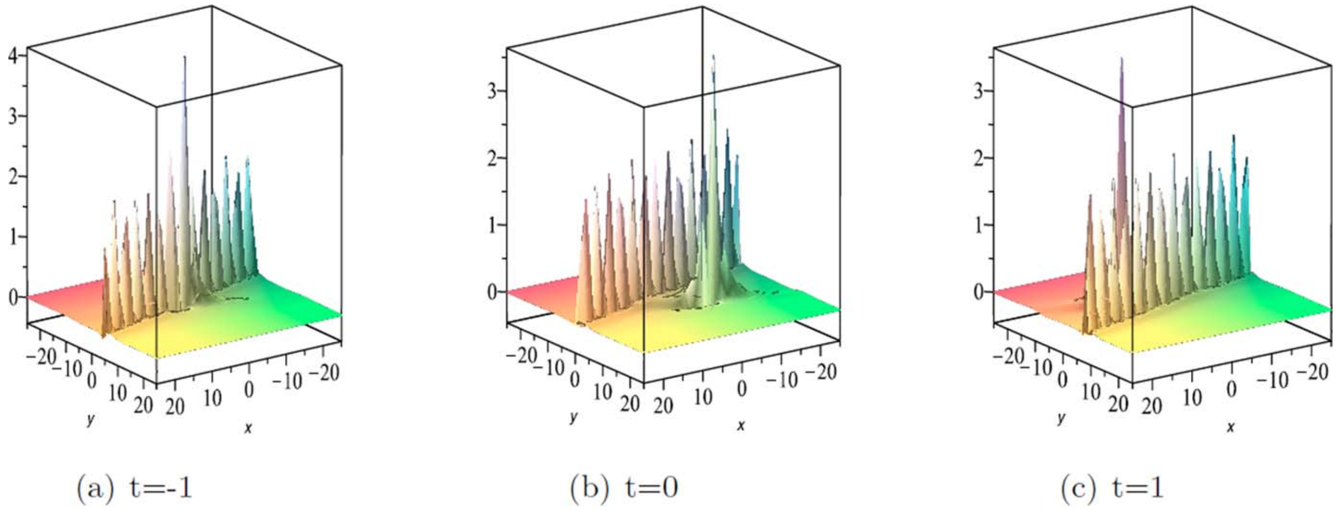


Figure 5. The elastic collision of one asymmetric soliton and one lump wave for equation (1).

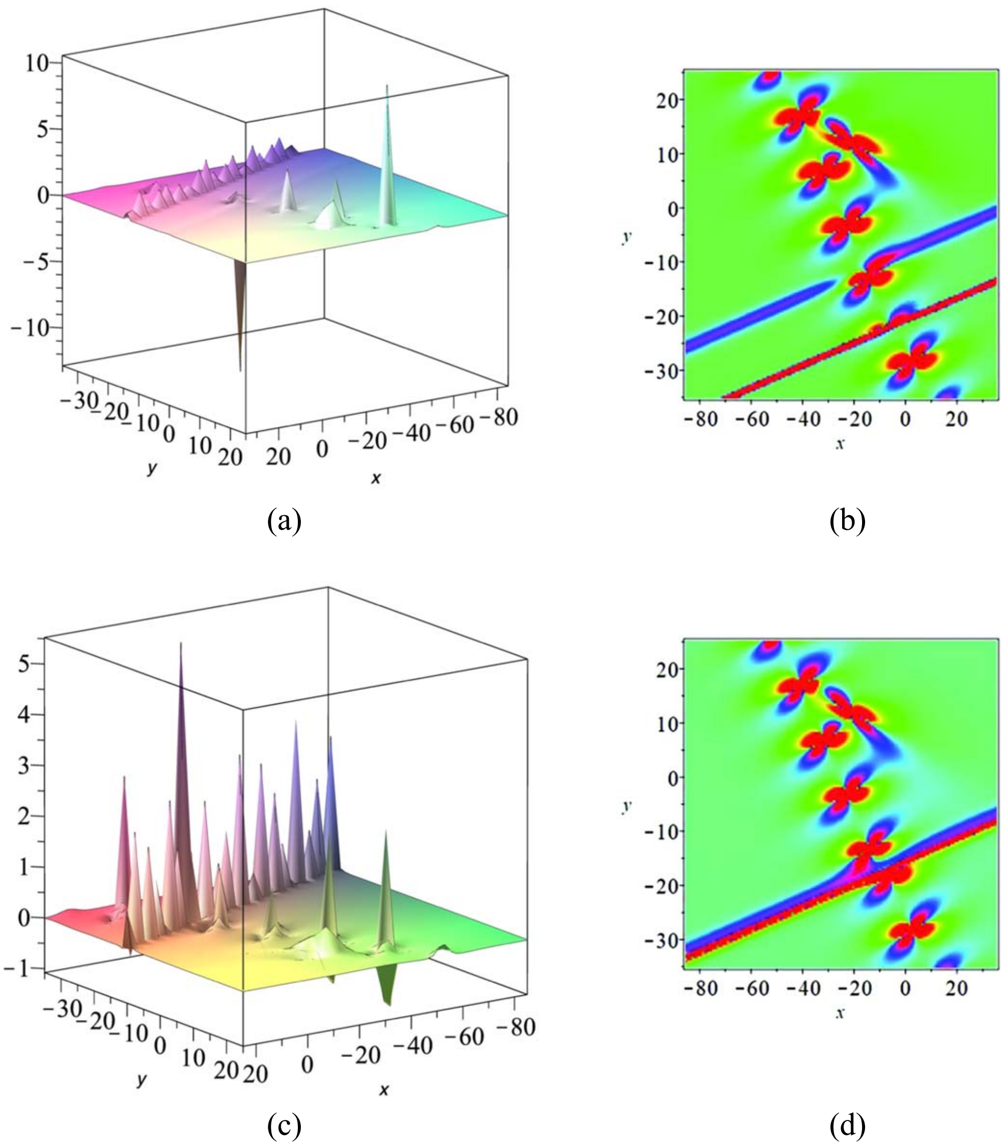


Figure 6. (a) and (b) Elastic collision of one soliton molecule, one breather wave and one lump wave for equation (1) at $t = -3$, (c) and (d) elastic collision of one asymmetric soliton molecule, one breather wave and one lump wave at $t = -3$, $\phi_1 = 0$, $\phi_2 = -10$.

hybrid solutions, which is composed of soliton molecules, asymmetric solitons, lump waves and breather waves. They are also elastic collisions. At the same time, we find out the generalized constraints for obtaining these hybrid solutions, which include m soliton molecules, h breather waves and s lump waves. We hope that the results of this paper can provide valuable information about the study of mathematical physics.

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