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Bilinear forms through the binary Bell polynomials, *N* solitons and Bäcklund transformations of the Boussinesq–Burgers system for the shallow water waves in a lake or near an ocean beach

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Abstract

Water waves are one of the most common phenomena in nature, the studies of which help energy development, marine/offshore engineering, hydraulic engineering, mechanical engineering, etc. Hereby, symbolic computation is performed on the Boussinesq–Burgers system for shallow water waves in a lake or near an ocean beach. For the water-wave horizontal velocity and height of the water surface above the bottom, two sets of the bilinear forms through the binary Bell polynomials and *N*-soliton solutions are worked out, while two auto-Bäcklund transformations are constructed together with the solitonic solutions, where *N* is a positive integer. Our bilinear forms, *N*-soliton solutions and Bäcklund transformations are different from those in the existing literature. All of our results are dependent on the water-wave dispersive power.

Keywords: lakes and ocean beaches, shallow water waves, Boussinesq–Burgers system, symbolic computation, bilinear forms through the binary Bell polynomials, Bäcklund transformations, solitonic solutions

Water has been known as a constant reminder that life repeats and the only element that has a visible cycle [1]. The water cycle has been believed to result in the distribution of water on land surface, to purify water, to support plant growth, to facilitate agriculture and to sustain aquatic ecosystem [2]. Water waves have been seen as one of the most common phenomena in nature, the studies of which help the energy development, marine/offshore engineering, hydraulic engineering, mechanical engineering, etc [3–10]. On the water waves, each year, hundreds of papers have been published, a few of which, e.g. are [3–10] recently. More recent nonlinear-physics contributions have been seen as well [11–13].

For the propagation of shallow water waves in a lake or near an ocean beach, people have investigated the Boussinesq–Burgers system [14–19],

$$u_t = \frac{\beta - 1}{2} u_{xx} + 2uu_x + \frac{1}{2} v_x, \tag{1a}$$

$$v_t = \left(1 - \frac{\beta}{2}\right)\beta u_{xxx} + 2(uv)_x + \frac{1 - \beta}{2} v_{xx}, \tag{1b}$$

where x and t are the normalized space and time, the subscripts denote the partial derivatives, the differentiable function $u = u(x, t)$ represents the horizontal velocity, and the differentiable function $v = v(x, t)$ denotes the height of the water surface above the bottom level [16], while β is a real constant representing the dispersive power [15].

For system (1), finite-band solutions have been investigated with the help of the Lax representations of some stationary evolution equations [14], rational solutions have been obtained via the bilinear method and Kadomtsev–Petviashvili hierarchy reduction [15], non-auto-Bäcklund transformations, nonlocal symmetries, conservation laws and interaction solutions have been constructed [16], Darboux transformations and multi-soliton solutions have been presented [17, 18], bilinear form and multi-soliton solutions have been studied [19]. By the way, other issues related to system (1) have been seen [20–28].

However, to our knowledge, Bell-polynomial investigation on the bilinear forms and N solitons for system (1) has not been carried out, where N is a positive integer. Auto-Bäcklund-transformation work on system (1), which is different from that in [16], has not been discussed, either. In this paper, we will briefly review the Bell-polynomial concepts. With the binary Bell polynomials and symbolic computation [29] we will work out the bilinear forms for system (1), which are different from those in [15, 19], and N solitons for system (1), which are different from those in [15–19]. Also with symbolic computation, we will construct two auto-Bäcklund transformations with some soliton features. Conclusions will come out last.

Let us begin the work with the Bell-polynomial preliminary: Bell polynomials have been said to provide a relatively-direct way to get the bilinear forms for certain nonlinear evolution equations, instead of the dependent variable transformations [30–32]. References [30–32] have presented the following:

- The two-dimensional Bell polynomials:

$$Y_{mx,nt}(\theta) \equiv Y_{m,n}(\theta_{1,1},\cdots,\theta_{1,n},\cdots,\theta_{m,1},\cdots,\theta_{m,n}) = \mathrm{e}^{-\theta}\partial_x^m\partial_t^n\mathrm{e}^{\theta}, \tag{2}$$

where θ is a C^∞ function of x and t, $\theta_{k,l} = \partial_x^k\partial_t^l\theta$, $k = 0,\cdots,m$, $l = 0,\cdots,n$, with m and n being the nonnegative integers, e.g.

$$Y_{x,t} = \theta_{x,t} + \theta_x\theta_t,\quad Y_{2x,t} = \theta_{2x,t} + \theta_{2x}\theta_t + 2\theta_{x,t}\theta_x + \theta_x^2\theta_t,\quad \cdots. \tag{3}$$

- The binary Bell polynomials:

$$\mathscr{Y}_{mx,nt}(p, q) \equiv Y_{mx,nt}(\psi_{1,1},\cdots,\psi_{1,n},\cdots,\psi_{m,1},\cdots,\psi_{m,n})\Big|_{\psi_{k,l}=\begin{cases}p_{k,l}, & \text{if } k+l \text{ is odd,}\\ q_{k,l}, & \text{if } k+l \text{ is even,}\end{cases}} \tag{4}$$

where $p(x, t)$ and $q(x, t)$ are both the C^∞ functions of x and t, $\psi_{k,l}$'s are all the functions of p and q, $p_{k,l} = \partial_x^k\partial_t^l p$ and $q_{k,l} = \partial_x^k\partial_t^l q$, e.g.

$$\mathscr{Y}_x(p, q) = p_x,\quad \mathscr{Y}_{2x}(p, q) = q_{2x} + p_x^2,\quad \mathscr{Y}_{x,t}(p, q) = p_xp_t + q_{xt},\quad \mathscr{Y}_{3x}(p, q) = p_{3x} + 3p_xq_{2x} + p_x^3,\quad \cdots. \tag{5}$$

References [33–35] have linked the $\mathscr{Y}$ polynomials to the Hirota D operators as

$$\mathscr{Y}_{mx,nt}\left[p = \ln\left(\frac{f}{g}\right), q = \ln(fg)\right] = (fg)^{-1}D_x^mD_t^n f\cdot g, \tag{6}$$

where $f(x, t)$ and $g(x, t)$ are the C^∞ functions of x and t, while D_x and D_t are the Hirota D operators defined by

$$D_x^mD_t^n f(x, t)\cdot g(x, t) \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, t)g(x', t')\Big|_{x'=x,t'=t}, \tag{7}$$

with x' and t' being the formal variables.

Hereby, through the binary Bell polynomials, we need to work out the bilinear forms and N solitons for system (1).

To begin with, what is reported in [19] can be considered as a special case of our results in this part.

Similar to the work in [36, 37], the scaling transformation,

$$x \to \varphi x,\quad t \to \varphi^2 t,\quad \beta \to \varphi^0\beta,\quad u \to \varphi^{-1}u,\quad v \to \varphi^{-2}v,$$

leads to our assumptions

$$u(x, t) = \Upsilon_1\, p_x(x, t), \tag{8a}$$

$$v(x, t) = \Upsilon_2\, p_{xx}(x, t) + \Upsilon_3\, q_{xx}(x, t) + \sigma, \tag{8b}$$

where Υ_1, Υ_2, Υ_3 and σ are all the real constants. With the substitution of assumptions (8) and the assumptions that

$$\Upsilon_3 = 2\Upsilon_1^2,\quad \Upsilon_2 = (1 - \beta)\Upsilon_1 \quad\text{and}\quad \Upsilon_1 = \pm\frac{1}{2}$$

back into system (1), Bell-polynomial procedure and symbolic computation help us to reduce assumptions (8) to

$$u(x, t) = \pm\frac{1}{2}p_x(x, t), \tag{9a}$$

$$v(x, t) = \pm\frac{1 - \beta}{2}p_{xx}(x, t) + \frac{1}{2}q_{xx}(x, t) + \sigma, \tag{9b}$$

and then to convert system (1) into two sets of the coupled systems of the $\mathscr{Y}$ polynomials:

$$\pm\frac{1}{2}\mathscr{Y}_{2x}(p, q) - \mathscr{Y}_t(p) = 0, \tag{10a}$$

$$\pm 2\mathscr{Y}_{x,t}(p, q) - \mathscr{Y}_{3x}(p, q) - 4\sigma\mathscr{Y}_x(p) = 0\,. \tag{10b}$$

Further, with

$$p(x,t)=\ln\left[\frac{f(x,t)}{g(x,t)}\right], \tag{11a}$$

$$q(x,t)=\ln[f(x,t)g(x,t)], \tag{11b}$$

we transform system (1), through systems (10), into the following two branches of bilinear forms with the binary Bell polynomials:

$$\left(\pm\frac{1}{2}D_x^2-D_t\right)f\cdot g=0, \tag{12a}$$

$$(\pm 2D_xD_t-D_x^3-4\sigma D_x)f\cdot g=0\,. \tag{12b}$$

The reason for the existence of two branches of bilinear forms (12) is that there appear the '±' signs.

It is noted that bilinear forms (12) are different from those in [15, 19].

Expanding $f(x,t)$ and $g(x,t)$ in bilinear forms (12) with respect to a small parameter ϵ as

$$f(x,t)=1+\sum_{\varsigma=1}^{N}\epsilon^{\varsigma}f_{\varsigma}(x,t), \tag{13a}$$

$$g(x,t)=1+\sum_{\varpi=1}^{N}\epsilon^{\varpi}g_{\varpi}(x,t), \tag{13b}$$

and then setting $\epsilon=1$, we obtain the N-soliton solutions for system (1) as

$$\left.\begin{matrix}u(x,t)=\\ v(x,t)=\end{matrix}\right\}\text{ expressions (9) and (11), with} \tag{14a}$$

$$f(x,t)=\sum_{\mu_i,\mu_j=0,1}\exp\left[\sum_{i=1}^{N}\mu_i(\lambda_i x+\tau_i t+\omega_i)+\sum_{1\leqslant i<j}^{(N)}\mu_i\mu_j\delta_{ij}\right], \tag{14b}$$

$$g(x,t)=\sum_{\mu_i,\mu_j=0,1}\exp\left[\sum_{i=1}^{N}\mu_i(\lambda_i x+\tau_i t+\kappa_i)+\sum_{1\leqslant i<j}^{(N)}\mu_i\mu_j\delta_{ij}\right], \tag{14c}$$

$$\mathrm{e}^{\delta_{ij}}=\left(\frac{\sqrt{a_ib_j}-\sqrt{a_jb_i}}{\sqrt{a_ia_j}-\sqrt{b_ib_j}}\right)^2, \tag{14d}$$

$$\mathrm{e}^{\omega_i}=a_i, \tag{14e}$$

$$\mathrm{e}^{\kappa_i}=b_i, \tag{14f}$$

$$\tau_i=\pm\frac{(a_i+b_i)\lambda_i^2}{2(a_i-b_i)}, \tag{14g}$$

$$\lambda_i=(a_i-b_i)\sqrt{\frac{\sigma}{a_ib_i}}, \tag{14h}$$

ς, ϖ, i and j being the positive integers with $\varsigma\leqslant N$, $\varpi\leqslant N$, $i\leqslant N$ and $j\leqslant N$, ω_i's and κ_i's denoting the real constants, $f_{\varsigma}(x,t)$'s and $g_{\varpi}(x,t)$'s being all the real differentiable functions of x and t, a_i and b_i representing the parameters characterizing the i-th soliton, $a_i\,\sigma>0$, $b_i\,\sigma>0$, the sum $\sum_{\mu_i,\mu_j=0,1}$ taken over all the possible combinations of $\mu_j=0,1$ while $\sum_{1\leqslant i<j}^{(N)}$ being the summation over all the possible pairs chosen from the N elements under the condition $i<j$.

It is noted that there exist two branches of N-soliton solutions (14) because of the '±' signs. Both of the branches are dependent on β, the water-wave dispersive power.

It is also noted that N-soliton solutions (14) are different from those in [15–19].

Our next goal should be auto-Bäcklund transformations (17)–(25), which are different from those reported in [16], and the relevant soliton features for system (1), to be seen below.

For system (1), we need to consider the Painlevé expansions in the form of the generalized Laurent series [38, 39] (and references therein), i.e.

$$u(x,t)=\phi^{-\Xi}(x,t)\sum_{\xi=0}^{\infty}u_{\xi}(x,t)\phi^{\xi}(x,t), \tag{15a}$$

$$v(x,t)=\phi^{-\mathcal{X}}(x,t)\sum_{\chi=0}^{\infty}v_{\chi}(x,t)\phi^{\chi}(x,t), \tag{15b}$$

and to balance the powers of ϕ at the lowest orders, so as to get

$$\mathcal{X}=2,\qquad \Xi=1, \tag{16}$$

where Ξ and $\mathcal{X}$ are the natural numbers, u_{ξ}'s, v_{χ}'s and ϕ are all the analytic functions with $u_0\neq 0$, $v_0\neq 0$ and $\phi_x\neq 0$.

With symbolic computation, we will truncate Painlevé expansions (15) at the constant level terms [38, 39], as

$$u(x,t)=\frac{u_0(x,t)}{\phi(x,t)}+u_1(x,t), \tag{17a}$$

$$v(x,t)=\frac{v_0(x,t)}{\phi^2(x,t)}+\frac{v_1(x,t)}{\phi(x,t)}+v_2(x,t), \tag{17b}$$

and substitute expansions (17) into system (1). Then, we require that the coefficients of like powers of ϕ vanish, and see the Painlevé–Bäcklund equations:

$$v_0=\frac{\pm(\beta-1)-1}{2}\phi_x^2, \tag{18}$$

$$u_0=\pm\frac{\phi_x}{2}, \tag{19}$$

$$v_1=\pm\frac{2\phi_t-4\phi_xu_1-(\beta-1)\phi_{xx}}{2}, \tag{20}$$

$$\pm\phi_{xx}+4\phi_xu_1-2\phi_t=0, \tag{21}$$

$$\begin{aligned}&\pm 4\beta\phi_x^2u_{1,x}\mp 4\phi_x^2u_{1,x}\pm 16\phi_x\phi_tu_1\\&\mp 16\phi_x^2u_1^2\pm 4\phi_x^2v_2\mp 4\phi_t^2\pm 3\phi_{xx}^2\\&\pm 4\phi_x\phi_{xxx}+12\phi_x^2u_{1,x}+24\phi_x\phi_{xx}u_1\\&-4\phi_{xx}\phi_t-8\phi_x\phi_{xt}=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&4\beta\phi_{xx}u_{1,x}+4\beta\phi_xu_{1,xx}\\&\quad-32\phi_xu_{1,x}u_1+8\phi_xu_{1,t}+8\phi_tu_{1,x}-4\phi_{xx}u_{1,x}\\&\quad-4\phi_xu_{1,xx}-16\phi_{xx}u_1^2+16\phi_{xt}u_1+4\phi_xv_{2,x}\\&\quad+4\phi_{xx}v_2-4\phi_{tt}+\phi_{xxxx}=0,\end{aligned} \tag{23}$$

$$u_{1,t} = \frac{\beta - 1}{2}u_{1,xx} + 2u_1 u_{1,x} + \frac{1}{2}v_{2,x}, \tag{24}$$

$$v_{2,t} = \left(1 - \frac{\beta}{2}\right)\beta u_{1,xxx} + 2(u_1 v_2)_x + \frac{1-\beta}{2}v_{2,xx}. \tag{25}$$

Hereby, $u_1(x, t)$ and $v_2(x, t)$ can be treated as the seed solutions for system (1) [38, 39]. The sets of equations (17)–(25) constitute two auto-Bäcklund transformations, since there exist the '$\pm$' signs, and the whole sets are mutually consistent, or, explicitly solvable with respect to $\phi(x, t)$, $u_0(x, t)$, $u_1(x, t)$, $v_0(x, t)$, $v_1(x, t)$ and $v_2(x, t)$, to be seen right below.

Each of auto-Bäcklund transformations (17)–(25) works as a system of equations relating a set of the solutions of system (1), e.g. one set of solutions (27) or (28), to another set of the solutions of system (1) itself. Therefore, we could, in principle at least, be able to progressively construct more and more complicated solutions of system (1).

Each of auto-Bäcklund transformations (17)–(25) is dependent on β, the water-wave dispersive power.

We will try to find the explicitly-solvable soliton examples, assuming that

$$\begin{cases} \phi(x, t) = e^{\zeta_1(t)x + \zeta_2(t)} + 1, \\ u_1(x, t) = \zeta_3(t) + \zeta_4(t)x, \\ v_2(x, t) = \zeta_5(t) + \zeta_6(t)x + \zeta_7(t)x^2, \end{cases} \tag{26}$$

where $\zeta_l(t)$'s ($l = 1, ..., 7$) are the real differentiable functions, with $\zeta_1(t) \neq 0$ since $\phi_x \neq 0$.

With symbolic computation, we substitute assumptions (26) into equations (18)–(25), and work out

$$\zeta_7(t) = \frac{[\zeta_1{}'(t) - 2\zeta_1(t)\zeta_4(t)]^2}{\zeta_1(t)^2}$$

with the "$'$" sign denoting the derivative, with respect to t. There exist two cases, $\zeta_1(t) \neq$ constant or $\zeta_1(t) =$ constant.

Case (1): $\zeta_1(t) \neq$ **constant**

$$\begin{aligned} \zeta_1(t) &= \zeta_8\, e^{2\int \zeta_4(t)\, dt}, \\ \zeta_4(t) &= \frac{1}{\zeta_9 - 2t} \text{ with } \zeta_9 - 2t \neq 0, \\ \zeta_6(t) &= 0, \\ \zeta_3(t) &= \frac{\rho_1}{\zeta_9 - 2t}, \\ \zeta_5(t) &= \frac{\rho_2}{\zeta_9 - 2t}, \\ \zeta_2(t) &= \frac{\zeta_8(4\rho_1 \pm \zeta_8)}{4(\zeta_9 - 2t)} + 2\rho_3, \\ \rho_2 &= 1 - \beta \pm 1, \end{aligned}$$

where $\zeta_8 \neq 0$, ζ_9, ρ_1, ρ_2 and ρ_3 are all the real constants.

Computing with expressions (17) as well, we can obtain the following β-dependent soliton solutions of system (1):

$$\begin{aligned} u(x, t) = &\pm \frac{\zeta_8}{4(\zeta_9 - 2t)} \\ &\times \text{Tanh}\left[\frac{\zeta_8 x}{2(\zeta_9 - 2t)} + \frac{\zeta_8(4\rho_1 \pm \zeta_8)}{8(\zeta_9 - 2t)} + \rho_3\right] \\ &+ \frac{x}{\zeta_9 - 2t} + \frac{4\rho_1 \pm \zeta_8}{4(\zeta_9 - 2t)}, \end{aligned} \tag{27a}$$

$$\begin{aligned} v(x, t) = &\frac{\zeta_8^2(\pm\beta \mp 1 - 1)}{8(\zeta_9 - 2t)^2} \\ &\times \text{Tanh}^2\left[\frac{\zeta_8 x}{2(\zeta_9 - 2t)} + \frac{\zeta_8(4\rho_1 \pm \zeta_8)}{8(\zeta_9 - 2t)} + \rho_3\right] \\ &+ \frac{\zeta_8^2(1 \mp \beta \pm 1) + 8(1 - \beta \pm 1)(\zeta_9 - 2t)}{8(\zeta_9 - 2t)^2}. \end{aligned} \tag{27b}$$

There exist two branches of those solutions because of the '$\pm$' signs.

Case (2): $\zeta_1(t) = \eta_1 =$ **non-zero constant**

Similarly, we can obtain the following β-dependent solitonic solutions of system (1):

$$\begin{aligned} u(x, t) = &\pm \frac{\eta_1}{4}\text{Tanh}\left[\frac{\eta_1 x}{2} + \frac{\eta_1(4\eta_2 \pm \eta_1)t}{4} + \frac{\eta_3}{2}\right] \\ &+ \eta_2 \pm \frac{\eta_1}{4}, \end{aligned} \tag{28a}$$

$$\begin{aligned} v(x, t) = &\frac{\eta_1^2(\pm\beta \mp 1 - 1)}{8} \\ &\times \text{Tanh}^2\left[\frac{\eta_1 x}{2} + \frac{\eta_1(4\eta_2 \pm \eta_1)t}{4} + \frac{\eta_3}{2}\right] \\ &- \frac{\eta_1^2(\pm\beta \mp 1 - 1)}{8}, \end{aligned} \tag{28b}$$

where η_2 and η_3 are also the real constants. There exist two branches of those solutions because of the '$\pm$' signs.

Right now, let us finish up the paper. Water has been known as a constant reminder that life repeats and the only element that has a visible cycle. The water cycle has been believed to result in the distribution of water on land surface, to purify water, to support plant growth, to facilitate agriculture and to sustain aquatic ecosystem. Water waves have been seen as one of the most common phenomena in nature, the studies of which help the energy development, marine/offshore engineering, hydraulic engineering, mechanical engineering, etc. Hereby, on system (1), the Boussinesq–Burgers system for the shallow water waves in a lake or near an ocean beach, symbolic computation has been performed. For $u(x, t)$, the water-wave horizontal velocity, and $v(x, t)$, the height of the water surface above the bottom, two sets of the bilinear forms through the binary Bell polynomials, i.e. bilinear forms (12), and N-soliton solutions (14) have been worked out, while two sets of auto-Bäcklund transformations (17)–(25) have been constructed together with solitonic solutions (27) and (28). It has been noted that bilinear forms (12), *N*-soliton solutions (14) and Bäcklund transformations (17)–(25) are different from those in the existing literatures. All of our results have been shown to be dependent on β, the water-wave dispersive power.

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References

[1] Cooper E J *et al* 2020 *Northwest Sci.* **94** 70
Alejandra Moreno M *et al* 2020 *Food Hydrocolloid.* **107** 105973
Su J J, Gao Y T, Deng G F and Jia T T 2019 *Phys. Rev.* E **100** 042210
Deng G F, Gao Y T, Su J J and Ding C C 2019 *Appl. Math. Lett.* **98** 177
Jia T T, Gao Y T, Deng G F and Hu L 2019 *Nonlinear Dyn.* **98** 269
Feng Y J, Gao Y T, Li L Q and Jia T T 2020 *Applicable Anal.* accepted (https://doi.org/10.1080/00036811.2019.1652734)

[2] Hethnawi A *et al* 2020 *Colloids Surf.* A **597** 124814
Mhawej M, Fadel A and Faour G 2020 *Int. J. Remote Sen.* **41** 5321
Wang M, Tian B, Sun Y and Zhang Z 2020 *Comput. Math. Appl.* **79** 576
Hu C C, Tian B, Wu X Y, Yuan Y Q and Du Z 2018 *Eur. Phys. J. Plus* **133** 40
Chen Y Q, Tian B, Qu Q X, Li H, Zhao X H, Tian H Y and Wang M 2020 *Mod. Phys. Lett.* B accepted (https://doi.org/10.1142/S0217984920502875)
Chen Y Q, Tian B, Qu Q X, Li H, Zhao X H, Tian H Y and Wang M 2020 *Int. J. Mod. Phys.* B accepted JPB20076387

[3] Sowmya G, Gireesha B J, Sindhu S and Prasannakumara B C 2020 *Commun. Theor. Phys.* **72** 025004
Jia T T, Gao Y T, Feng Y J, Hu L, Su J J, Li L Q and Ding C C 2019 *Nonlinear Dyn.* **96** 229
Hu S H, Tian B, Du X X, Liu L and Zhang C R 2019 *Pramana —J. Phys.* **93** 38

[4] Khater M M, Lu D C, Attia R A and Inc M 2019 *Commun. Theor. Phys.* **71** 1267
Feng Y J, Gao Y T, Jia T T and Li L Q 2019 *Mod. Phys. Lett.* B **33** 1950354
Yuan Y Q, Tian B, Qu Q X, Zhao X H and Du X X 2020 *Z. Angew. Math. Phys.* **71** 46

[5] Kumar R, Kumar R, Koundal R, Shehzad S A and Sheikholeslami M 2019 *Commun. Theor. Phys.* **71** 779
Hu L, Gao Y T, Jia S L, Su J J and Deng G F 2019 *Mod. Phys. Lett.* B **33** 1950376
Su J J, Gao Y T and Ding C C 2019 *Appl. Math. Lett.* **88** 201

[6] Bona J L, Colin T and Guillope C 2019 *Discrete Cont. Dyn. Syst.* **39** 5543
Ding C C, Gao Y T and Deng G F 2019 *Nonlinear Dyn.* **97** 2023
Deng G F, Gao Y T, Su J J, Ding C C and Jia T T 2020 *Nonlinear Dyn.* **99** 1039

[7] Congy T, El G A and Hoefer M A 2019 *J. Fluid Mech.* **875** 1145

[8] Roy R, De S and Mandal B N 2019 *Appl. Math. Comput.* **355** 458
Roy R, De S and Mandal B N 2019 *Fluid Dyn. Res.* **51** 045508

[9] Hu C C, Tian B, Yin H M, Zhang C R and Zhang Z 2019 *Comput. Math. Appl.* **78** 166
Wang M, Tian B, Qu Q X, Du X X, Zhang C R and Zhang Z 2019 *Eur. Phys. J. Plus* **134** 578

[10] Eeltink D, Armaroli A, Ducimetiere Y M, Kasparian J and Brunetti M 2019 *Phys. Rev.* E **100** 013102

[11] Yin H M, Tian B and Zhao X H 2020 *Appl. Math. Comput.* **368** 124768
Chen S S, Tian B, Sun Y and Zhang C R 2020 *Ann. Phys.* **531** 1900011
Ding C C, Gao Y T and Li L Q 2019 *Chaos Solitons Fractals* **120** 259

[12] Du Z, Tian B, Chai H P and Zhao X H 2020 *Appl. Math. Lett.* **102** 106110
Du X X, Tian B, Qu Q X, Yuan Y Q and Zhao X H 2020 *Chaos Solitons Fractals* **134** 109709
Li L Q, Gao Y T, Hu L, Jia T T, Ding C C and Feng Y J 2020 *Nonlinear Dyn.* **100** 2729

[13] Zhang C R, Tian B, Qu Q X, Liu L and Tian H Y 2020 *Z. Angew. Math. Phys.* **71** 18
Hu S H, Tian B, Du X X, Du Z and Wu X Y 2019 *J. Comput. Nonlin. Dyn.* **14** 111001
Yuan Y Q, Tian B, Qu Q X, Zhang C R and Du X X 2020 *Nonlinear Dyn.* **99** 3001

[14] Geng X G and Wu Y T 1999 *J. Math. Phys.* **40** 2971

[15] Li M, Hu W K and Wu C F 2018 *Nonlinear Dyn.* **94** 1291

[16] Dong M J, Tian S F, Yan X W and Zhang T T 2019 *Nonlinear Dyn.* **95** 273

[17] Xu R 2008 *Commun. Theor. Phys.* **50** 579

[18] Mei J and Ma Z 2013 *Appl. Math. Comput.* **219** 6163

[19] Zhang C C and Chen A H 2016 *Appl. Math. Lett.* **58** 133

[20] Kupershmidt B A 1985 *Commun. Math. Phys.* **99** 51

[21] Kawamoto S 1984 *J. Phys. Soc. Jpn.* **53** 469

[22] Wazwaz A M 2017 *Filomat* **31** 831

[23] Hu J, Xu Z W and Yu G F 2016 *Appl. Math. Lett.* **62** 76

[24] Rady A S, Osman E S and Khalfallah M 2010 *Commun. Nonlinear Sci. Numer. Simul.* **15** 1172

[25] Gu Z Q 1990 *J. Math. Phys.* **31** 1374

[26] Wang P, Tian B, Liu W J, Lv X and Jiang Y 2011 *Appl. Math. Comput.* **218** 1726

[27] Wang Y H 2014 *Appl. Math. Lett.* **38** 100

[28] Roshid H O and Rahman M A 2014 *Results Phys.* **4** 150
Zhang Y F and Feng B L 2011 *Commun. Nonlinear Sci. Numer. Simul.* **16** 3045

[29] Chen S S, Tian B, Liu L, Yuan Y Q and Zhang C R 2019 *Chaos Solitons Fractals* **118** 337
Du Z, Tian B, Qu Q X, Wu X Y and Zhao X H 2020 *Appl. Numer. Math.* **153** 179
Yin H M, Tian B and Zhao X C 2020 *J. Magn. Magn. Mater.* **495** 165871
Du X X, Tian B, Yuan Y Q and Du Z 2019 *Ann. Phys.* (Berlin) **531** 1900198
Zhang C R, Tian B, Y. Sun and Yin H M 2019 *EPL* **127** 40003
Zhao X, Tian B, Qu Q X, Yuan Y Q, Du X X and Chu M X 2020 *Mod. Phys. Lett.* B accepted (https://doi.org/10.1142/S0217984920502826)

[30] Bell E T 1934 *Ann. Math.* **35** 258

[31] Lambert F, Loris I, Springael J and Willox R 1994 *J. Phys. A: Math. Gen.* **27** 5325

[32] Wang Y F, Tian B and Jiang Y 2017 *Appl. Math. Comput.* **292** 448

[33] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)

[34] Wadati M 1975 *J. Phys. Soc. Jpn.* **38** 673

[35] Caruello F and Tabor M 1989 *Physica* D **39** 77

[36] Lambert F and Springael J 2001 *Chaos Solitons Fractals* **12** 2821

[37] Lambert F and Springael J 2008 *Acta Appl. Math.* **102** 147

[38] Gao X Y, Guo Y J and Shan W R 2020 *Appl. Math. Lett.* **104** 106170

[39] Gao X Y 2019 *Appl. Math. Lett.* **91** 165