

A universal protocol for bidirectional controlled teleportation with network coding

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Received 16 April 2021, revised 6 July 2021

Accepted for publication 6 July 2021

Published 3 September 2021



CrossMark

Abstract

We investigate bidirectional teleportation that works in a fair and efficient manner. Two explicit protocols are proposed to realize bidirectional teleportation with a controller. One is a symmetric protocol for two-qubit states. The other is an asymmetric protocol for single- and two-qubit states. We then devise a universal protocol for arbitrary n_1 - and n_2 -qubit states via a $(2n_1 + 2n_2 + 1)$ -qubit entangled state, where $n_1 \leq n_2$. The receiver only needs to perform the single-qubit recovery operation, which is derived by a general expression. Moreover, a $(2n_1 + 1)$ -bit classical communication cost can be saved within the controller's broadcast channel by the use of network coding technology.

Keywords: bidirectional controlled teleportation, network coding, projective measurement, recovery operation

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum entanglement has been exploited as a fundamental resource for the implementation of quantum information-processing tasks. For example, Gong *et al* [1] proposed a quantum network dialogue protocol based on continuous-variable Greenberger-Horne-Zeilinger (GHZ) states. Zhou *et al* [2] proposed a semi-quantum key distribution protocol with four-particle cluster states. Ma *et al* [3] investigated probabilistic quantum network coding over a butterfly network via non-maximal entanglement. In 1993, Bennett *et al* [4] first presented a remarkable protocol called quantum teleportation (QT) that securely transmits an unknown qubit state based on the dual use of classical and quantum channels. A different kind of QT scheme, controlled quantum teleportation (CQT) was introduced by Karlsson *et al* [5] to realize teleportation under the supervision of a controller. Although the controlling party does not own the teleported state, he decides whether the state can be recovered by the receiver. In addition, theoretical generalizations have been proposed, such as probabilistic QT, [6] hierarchical QT,

[7] QT in noisy environments, [8, 9] etc. Meanwhile, experimental implementations of QT have also been reported [10, 11].

In addition to one-way QT, for a practical quantum network that contains many processors working far apart, an exchange of information states between two parties may be required. This is possible simply by switching on two sets of independent QT equipment in opposite directions. Since standard QT needs to send a classical message about the measurement result from the sender to the receiver, a situation may arise in which Alice's state has been teleported to Bob, but Bob decides not to tell Alice about his measurement result. This leads to an insecure or dishonest quantum communication. Therefore, it is necessary to investigate bidirectional quantum teleportation that works in a fair manner. Mishra *et al* [12] presented a controlled communication scheme which can simultaneously exchange two unknown single-qubit states via a six-qubit entangled channel. Nevertheless, Alice and Bob cannot recover the target states in a deterministic way. Zha *et al* [13] investigated the bidirectional controlled quantum teleportation (BCQT) of arbitrary single-

qubit states using a five-qubit cluster state as the entangled resource with unit success probability. Because of its potential application in quantum network communication, BCQT has recently acquired a lot of attention. [14–27] Jiang et al [14] developed a deterministic BCQT scheme for single-qubit states utilizing a five-qubit non-maximally entangled channel. [17–19] accomplished the BCQT of two-qubit states via different entangled resources. Apart from the symmetric protocols, [14–19] there are many asymmetric BCQT protocols [20–25]. For example, [20–24] describe the BCQT of arbitrary single- and two-qubit states. Thapliya et al [26] gave a general method for selecting a quantum channel for BCQT. To our knowledge, few universal protocols have been described for arbitrary qubit states. Savaghad-Moghaddam et al [27] attempted to investigate the BCQT of m - and n -qubit states. Regretfully, they did not explicitly give the relationship between the recovery operations, the collapsed states, and the measurement results. A question may be naturally raised: is it possible to devise a universal and efficient BCQT protocol for arbitrary qubit states and give a general formula for the recovery operations?

In this paper, under the assumption that classical communication is not allowed between senders and that the controller has a broadcast channel, we propose a universal protocol to realize the two-way teleportation of arbitrary n_1 - and n_2 -qubit states with the help of the controller via a $(2n_1 + 2n_2 + 1)$ -qubit maximally entangled state. Without loss of generality, we assume $n_1 \leq n_2$. After the sender's projective measurement using the Bell basis and the controller's single-qubit measurement using the Z basis, the receiver performs single-qubit Pauli operations and gets the target state deterministically. Through rigorous deduction, we give the general expression for the recovery operators. Network coding, a new paradigm that allows information to be duplicated and mixed in the intermediate nodes, has demonstrated a significant throughput advantage over traditional routing algorithms in classical networks [28]. It is found that if network coding is applied at the controller's site, a $(2n_1 + 1)$ -bit classical communication cost (CCC) will be saved in the broadcast channel. This means that our protocol is more efficient than without network coding.

The remaining parts of this paper are arranged as follows. In section 2, an explicit symmetric BCQT protocol is proposed to realize the two-way teleportation of two-qubit states. In section 3, an asymmetric BCQT protocol is described for single- and two-qubit states. In section 4, we further extend our discussion to a universal BCQT protocol for arbitrary n_1 - and n_2 -qubit states by using a $(2n_1 + 2n_2 + 1)$ -qubit state as the entangled channel. Some discussions and comparisons are given in section 5. The last section offers the conclusions.

2. The symmetric BCQT protocol for two-qubit states

In this section, we construct a controlled protocol that leads to the simultaneous exchange of arbitrary two-qubit states. There are three participants: Alice, Bob, and Charlie. Alice

and Bob are not only senders but also receivers, while Charlie is the controller.

Suppose Alice intends to teleport an arbitrary two-qubit state $|\varphi\rangle_{A_1A_2}$ to Bob. At the same time, Bob wishes to teleport an unknown two-qubit state $|\psi\rangle_{B_1B_2}$ to Alice's site.

$$\begin{aligned} |\varphi\rangle_{A_1A_2} &= (\alpha_0|00\rangle + \alpha_1|01\rangle \\ &\quad + \alpha_2|10\rangle + \alpha_3|11\rangle)_{A_1A_2}, \\ |\psi\rangle_{B_1B_2} &= (\beta_0|00\rangle + \beta_1|01\rangle \\ &\quad + \beta_2|10\rangle + \beta_3|11\rangle)_{B_1B_2}. \end{aligned} \tag{1}$$

Alice has the teleported state $|\varphi\rangle_{A_1A_2}$ and Bob owns $|\psi\rangle_{B_1B_2}$. Nothing is known about the complex coefficients, except that they satisfy the normalization condition $\sum_{j=0}^3 |\alpha_j|^2 = \sum_{j=0}^3 |\beta_j|^2 = 1$.

The quantum channel shared amongst the three participants is the nine-qubit entangled state

$$\begin{aligned} |C\rangle_{123456789} &= \frac{1}{\sqrt{2}}(|\Phi_0\rangle_{12}|\Phi_0\rangle_{34}|\Phi_0\rangle_{56}|\Phi_0\rangle_{78}|0\rangle_9 \\ &\quad + |\Phi_1\rangle_{12}|\Phi_1\rangle_{34}|\Phi_1\rangle_{56}|\Phi_1\rangle_{78}|1\rangle_9), \end{aligned} \tag{2}$$

where

$$\begin{aligned} |\Phi_0\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \\ |\Phi_1\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle). \end{aligned} \tag{3}$$

Qubits (1, 3, 5, 7) belong to Alice, qubits (2, 4, 6, 8) belong to Bob, and qubit 9 is held by Charlie.

Hence, the initial state of the whole system can be written as

$$|\varphi\rangle_{A_1A_2} \otimes |\psi\rangle_{B_1B_2} \otimes |C\rangle_{123456789}. \tag{4}$$

The complete process of the symmetric protocol is described as follows.

Step 1. Alice (Bob) performs joint projective measurements on her (his) qubits ($A_1, 1$) and ($A_2, 3$), ($B_1, 6$) and ($B_2, 8$) using the Bell basis

$$\begin{aligned} |\xi_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\xi_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \\ |\xi_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad |\xi_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \end{aligned} \tag{5}$$

After the measurements, Alice announces her measurement results $|\xi_{p_jq_j}\rangle$ to Charlie in the form of classical messages p_jq_j . Similarly, Bob sends four-bit classical information s_jt_j to Charlie corresponding to his measurement outcomes $|\xi_{s_jt_j}\rangle$. Here, $p_j, q_j, s_j, t_j = 0, 1, j = 1, 2$.

Based on the measurement bases, the joint system in equation (4) can be expressed as

$$\begin{aligned} &\prod_{j=1}^2 \left(\sum_{p_j, q_j, s_j, t_j=0}^1 |\xi_{p_jq_j}\rangle_{A_j, 2j-1} |\xi_{s_jt_j}\rangle_{B_j, 2j+4} \right) \\ &\left(\sum_{l=0}^1 |f_{p_1q_1p_2q_2}^l\rangle_{24} |g_{s_1t_1s_2t_2}^l\rangle_{57} |l\rangle_9 \right), \end{aligned} \tag{6}$$

where $\sum_{l=0}^1 |f_{p_1q_1p_2q_2}^l\rangle_{24} |g_{s_1t_1s_2t_2}^l\rangle_{57} |l\rangle_9$ is the collapsed state of qubits (2, 4, 5, 7, 9) if the measurement result is

Table 1. The expressions for $|f_{p_1q_1p_2q_2}^l\rangle$ and the recovery operations (RO).

p_1q_1, p_2q_2	l	p_1q_1, p_2q_2	l	$ f_{p_1q_1p_2q_2}^l\rangle$	RO
00,00	0	10,10	1	$\alpha_0 00\rangle + \alpha_1 01\rangle + \alpha_2 10\rangle + \alpha_3 11\rangle$	$I \otimes I$
00,01	0	10,11	1	$\alpha_0 01\rangle + \alpha_1 00\rangle + \alpha_2 11\rangle + \alpha_3 10\rangle$	$I \otimes X$
00,10	0	10,00	1	$\alpha_0 00\rangle - \alpha_1 01\rangle + \alpha_2 10\rangle - \alpha_3 11\rangle$	$I \otimes Z$
00,11	0	10,01	1	$\alpha_0 01\rangle - \alpha_1 00\rangle + \alpha_2 11\rangle - \alpha_3 10\rangle$	$I \otimes (ZX)$
01,00	0	11,10	1	$\alpha_0 10\rangle + \alpha_1 11\rangle + \alpha_2 00\rangle + \alpha_3 01\rangle$	$X \otimes I$
01,01	0	11,11	1	$\alpha_0 11\rangle + \alpha_1 10\rangle + \alpha_2 01\rangle + \alpha_3 00\rangle$	$X \otimes X$
01,10	0	11,00	1	$\alpha_0 10\rangle - \alpha_1 11\rangle + \alpha_2 00\rangle - \alpha_3 01\rangle$	$X \otimes Z$
01,11	0	11,01	1	$\alpha_0 11\rangle - \alpha_1 10\rangle + \alpha_2 01\rangle - \alpha_3 00\rangle$	$X \otimes (ZX)$
10,00	0	00,10	1	$\alpha_0 00\rangle + \alpha_1 01\rangle - \alpha_2 10\rangle - \alpha_3 11\rangle$	$Z \otimes I$
10,01	0	00,11	1	$\alpha_0 01\rangle + \alpha_1 00\rangle - \alpha_2 11\rangle - \alpha_3 10\rangle$	$Z \otimes X$
10,10	0	00,00	1	$\alpha_0 00\rangle - \alpha_1 01\rangle - \alpha_2 10\rangle + \alpha_3 11\rangle$	$Z \otimes Z$
10,11	0	00,01	1	$\alpha_0 01\rangle - \alpha_1 00\rangle - \alpha_2 11\rangle + \alpha_3 10\rangle$	$Z \otimes (ZX)$
11,00	0	01,10	1	$\alpha_0 10\rangle + \alpha_1 11\rangle - \alpha_2 00\rangle - \alpha_3 01\rangle$	$(ZX) \otimes I$
11,01	0	01,11	1	$\alpha_0 11\rangle + \alpha_1 10\rangle - \alpha_2 01\rangle - \alpha_3 00\rangle$	$(ZX) \otimes X$
11,10	0	01,00	1	$\alpha_0 10\rangle - \alpha_1 11\rangle - \alpha_2 00\rangle + \alpha_3 01\rangle$	$(ZX) \otimes Z$
11,11	0	01,01	1	$\alpha_0 11\rangle - \alpha_1 10\rangle - \alpha_2 01\rangle + \alpha_3 00\rangle$	$(ZX) \otimes (ZX)$

$|\xi_{p_1q_1}\rangle_{A_1}|\xi_{p_2q_2}\rangle_{A_2}|\xi_{s_1t_1}\rangle_{B_1}|\xi_{s_2t_2}\rangle_{B_2}$. The expressions for $|f_{p_1q_1p_2q_2}^l\rangle_{24}$ are listed in table 1. The expressions for $|g_{s_1t_1s_2t_2}^l\rangle_{57}$ can easily be obtained if α is replaced by β in table 1.

At this moment, Alice and Bob cannot obtain the original states without the controller's (Charlie's) help.

Step 2. If Charlie would like to provide assistance, he makes a single-qubit measurement of his qubit 9 using the Z basis $\{|0\rangle, |1\rangle\}$ and broadcasts a classical message to Alice and Bob.

It seems that Alice and Bob need to obtain the measurement results $s_1t_1s_2t_2l$ and $p_1q_1p_2q_2l$ to recover the teleported states, respectively. This means that Charlie has to broadcast nine-bit classical messages. Using the technology of network coding, Charlie just broadcasts the four-bit classical information $(p_j \oplus s_j \oplus l, q_j \oplus t_j)$ to Alice and Bob ($j = 1, 2$). Here \oplus means plus mod 2.

Step 3. According to the received classical messages and her (his) own measurement results $p_jq_j (s_jt_j)$, Alice calculates $[p_j \oplus (p_j \oplus s_j \oplus l), q_j \oplus (q_j \oplus t_j)]$ and gets $[s_j \oplus l, t_j]$. Bob computes $[s_j \oplus (p_j \oplus s_j \oplus l), t_j \oplus (q_j \oplus t_j)]$ and obtains $[p_j \oplus l, q_j]$. Then Alice performs the recovery operation,

$$R_A = Z_5^{s_1 \oplus l} X_5^{t_1} Z_7^{s_2 \oplus l} X_7^{t_2}, \tag{7}$$

on her qubits (5, 7) and gets $|\varphi\rangle$. Similarly, Bob carries out the appropriate operation,

$$R_B = Z_2^{p_1 \oplus l} X_2^{q_1} Z_4^{p_2 \oplus l} X_4^{q_2}, \tag{8}$$

on his qubits (2, 4) and recovers the two-qubit state $|\varphi\rangle$. Here X, Z are Pauli operations and X^0, Z^0 denote identity operations. In detail, Alice's and Bob's recovery operations conditioned on the measurement results are listed in table 1.

To illustrate the protocol more clearly, we assume that Alice's measurement results are $|\xi_{01}\rangle|\xi_{10}\rangle$ ($p_1q_1 = 01, p_2q_2 = 10$) and Bob's measurement results are $|\xi_{00}\rangle|\xi_{11}\rangle$ ($s_1t_1 = 00, s_2t_2 = 11$). If Charlie's measurement result is $|1\rangle$,

then the qubits (5, 7, 2, 4) collapse into:

$$|g_{0110}^1\rangle_{57}|f_{0011}^1\rangle_{24} = (\beta_0|01\rangle + \beta_1|00\rangle - \beta_2|11\rangle - \beta_3|10\rangle)_{57} \otimes (\alpha_0|10\rangle + \alpha_1|11\rangle - \alpha_2|00\rangle - \alpha_3|01\rangle)_{24}. \tag{9}$$

Charlie broadcasts the four-bit classical information $(p_1 \oplus s_1 \oplus l, p_2 \oplus s_2 \oplus l, q_1 \oplus t_1, q_2 \oplus t_2) = 1111$. Together with their own measurement results, Alice and Bob can obtain $(s_1 \oplus l, s_2 \oplus l, t_1, t_2) = 1001$ and $(p_1 \oplus l, p_2 \oplus l, q_1, q_2) = 1010$, respectively. Alice performs a Z_5X_7 operation on her collapsed qubits $(\beta_0|01\rangle + \beta_1|00\rangle - \beta_2|11\rangle - \beta_3|10\rangle)_{57}$ and Bob carries out $(ZX)_{24}$ on his collapsed qubits $(\alpha_0|10\rangle + \alpha_1|11\rangle - \alpha_2|00\rangle - \alpha_3|01\rangle)_{24}$. As a result, the desired states $|\varphi\rangle, |\psi\rangle$ are simultaneously teleported.

3. The asymmetric BCQT protocol for single- and two-qubit states

In this section, we demonstrate an asymmetric BCQT protocol. Alice wants to teleport an arbitrary single-qubit state $|\varphi\rangle_A$ to Bob, while Bob hopes to transmit an arbitrary two-qubit state $|\psi\rangle_{B_1B_2}$ to Alice.

$$|\varphi\rangle_A = (\alpha_0|0\rangle + \alpha_1|1\rangle)_A, \\ |\psi\rangle_{B_1B_2} = (\beta_0|00\rangle + \beta_1|01\rangle + \beta_2|10\rangle + \beta_3|11\rangle)_{B_1B_2}. \tag{10}$$

The coefficients of the teleported states are all complex numbers, and satisfy the normalization condition $\sum_{j=0}^1 |\alpha_j|^2 = 1, \sum_{j=0}^3 |\beta_j|^2 = 1$.

A seven-qubit entangled state,

$$|C\rangle_{1234567} = \frac{1}{\sqrt{2}}(|\Phi_0\rangle_{12}|\Phi_0\rangle_{34}|\Phi_0\rangle_{56}|0\rangle_7 \\ + |\Phi_1\rangle_{12}|\Phi_1\rangle_{34}|\Phi_1\rangle_{56}|1\rangle_7), \tag{11}$$

serves as the quantum channel. Alice has qubits (1, 3, 5) and Bob processes qubits (2, 4, 6), while qubit 7 is possessed by Charlie.

Table 2. Expressions for $|f_{p_1q_1}^l\rangle$ and the RO.

p_1q_1	l	p_1q_1	l	$ f_{p_1q_1}^l\rangle$	RO
00	0	10	1	$\alpha_0 0\rangle + \alpha_1 1\rangle$	I
01	0	11	1	$\alpha_0 1\rangle + \alpha_1 0\rangle$	X
10	0	00	1	$\alpha_0 0\rangle - \alpha_1 1\rangle$	Z
11	0	01	1	$\alpha_0 1\rangle - \alpha_1 0\rangle$	ZX

The compound state is

$$|\varphi\rangle_A \otimes |\psi\rangle_{B_1B_2} \otimes |C\rangle_{1234567}. \quad (12)$$

The detailed process of the asymmetric protocol is shown below.

Step 1. Alice (Bob) carries out a Bell basis measurement on her (his) qubits (A, 1) ((B₁, 4) and (B₂, 6)). If Alice's (Bob's) measurement result is $|\xi_{p_1q_1}\rangle$ ($|\xi_{s_jt_j}\rangle$, $j = 1, 2$), she (he) sends the classical bits p_1q_1 (s_jt_j) to Charlie through the classical channel.

The whole system in equation (12) can be rewritten as

$$\left(\sum_{p_1,q_1=0}^1 |\xi_{p_1q_1}\rangle_{A1} \prod_{j=1}^2 \left(\sum_{s_j,t_j=0}^1 |\xi_{s_jt_j}\rangle_{B_j,2j+2} \right) \right) \otimes \left(\sum_{l=0}^1 |f_{p_1q_1}^l\rangle_2 |g_{s_1t_1s_2t_2}^l\rangle_{35} |l\rangle_7 \right). \quad (13)$$

After the measurements, qubits (2, 3, 5, 7) collapse into $\sum_{l=0}^1 |f_{p_1q_1}^l\rangle_2 |g_{s_1t_1s_2t_2}^l\rangle_{35} |l\rangle_7$. The detailed expression for $|f_{p_1q_1}^l\rangle_2$ is given in table 2, while $|g_{s_1t_1s_2t_2}^l\rangle$ is same as that in section 2.

Step 2. Charlie measures his qubit 7 in the Z basis. In order to recover the teleported state, Alice and Bob need to obtain the measurement results $s_1t_1s_2t_2l$ and p_1q_1l , respectively. With the aid of network coding, Charlie broadcasts the four-bit classical information ($s_1 \oplus p_1 \oplus l$, $t_1 \oplus q_1$, $s_2 \oplus l$, t_2).

Step 3. According to the received classical messages and her (his) own measurement result p_1q_1 (s_1t_1 , s_2t_2), Alice gets [$s_1 \oplus l$, t_1 , $s_2 \oplus l$, t_2] by calculating [$p_1 \oplus (s_1 \oplus p_1 \oplus l)$, $q_1 \oplus (t_1 \oplus q_1)$]. Similarly, Bob computes [$s_1 \oplus (s_1 \oplus p_1 \oplus l)$, $t_1 \oplus (t_1 \oplus q_1)$] and acquires [$p_1 \oplus l$, q_1]. Alice applies the recovery operation,

$$R_A = Z_3^{s_1 \oplus l} X_3^{t_1} Z_5^{s_2 \oplus l} X_5^{t_2}, \quad (14)$$

on her qubits (3, 5) and gets $|\psi\rangle$ in equation (10). Bob can recover the teleported state $|\varphi\rangle$ by performing the unitary operation,

$$R_B = Z_2^{p_1 \oplus l} X_2^{q_1}, \quad (15)$$

on his collapsed particle 2.

An example is given below to clarify the protocol. Suppose that Alice's measurement outcome is $|\xi_{01}\rangle$ ($p_1q_1 = 01$) and Bob's measurement outcomes are $|\xi_{01}\rangle|\xi_{10}\rangle$ ($s_1t_1 = 01$, $s_2t_2 = 10$). If Charlie's measurement result is $|0\rangle$, he broadcasts the four-bit classical information ($s_1 \oplus p_1 \oplus l$, $t_1 \oplus q_1$, $s_2 \oplus l$, t_2) = 0010. Alice and Bob can then get ($s_1 \oplus l$, t_1 , $s_2 \oplus l$, t_2) = 0110 and ($p_1 \oplus l$, q_1) = 01, respectively. Alice and Bob separately perform the recovery operations X_3Z_5 and

X_2 on the collapsed state:

$$|g_{0110}^1\rangle_{35} |f_{01}^1\rangle_2 = (\beta_0|10\rangle - \beta_1|11\rangle + \beta_2|00\rangle - \beta_3|01\rangle)_{35} \otimes (\alpha_0|1\rangle + \alpha_1|0\rangle)_2. \quad (16)$$

In the end, the desired states are two-way teleported with unit success probability.

4. The universal BCQT protocol

In this section, we generalize the two explicit protocols given above to a universal BCQT protocol that can realize the simultaneous teleportation of arbitrary n_1 - and n_2 -qubit states.

Alice (Bob) has an unknown n_1 (n_2)-qubit state in this form:

$$|\varphi\rangle_{A_1 \dots A_{n_1}} = \sum_{a_1, \dots, a_{n_1}=0}^1 \alpha_a |a_1 \dots a_{n_1}\rangle_{A_1 \dots A_{n_1}},$$

$$(|\psi\rangle_{B_1 \dots B_{n_2}} = \sum_{b_1, \dots, b_{n_2}=0}^1 \beta_b |b_1 \dots b_{n_2}\rangle_{B_1 \dots B_{n_2}}), \quad (17)$$

and wants to transmit it to Bob (Alice). The subscripts a and b individually represent the decimal forms of binary strings $a_1 \dots a_{n_1}$ and $b_1 \dots b_{n_2}$; α_a and β_b are complex numbers that satisfy $\sum_{a=0}^{2^{n_1}-1} |\alpha_a|^2 = 1$, $\sum_{b=0}^{2^{n_2}-1} |\beta_b|^2 = 1$.

We take a $(2n_1 + 2n_2 + 1)$ -qubit entangled state,

$$|C\rangle_{1 \dots 2n_1+2n_2+1} = \frac{1}{\sqrt{2}} \sum_{l=0}^1 |\Phi_l\rangle_{12}$$

$$\dots |\Phi_l\rangle_{2n_1+2n_2-1, 2n_1+2n_2} |l\rangle_{2n_1+2n_2+1}, \quad (18)$$

as the quantum channel. The construction process of the channel is similar to that given in [29]. The qubits (1, \dots , $2n_1 - 1$, $2n_1 + 1$, \dots , $2n_1 + 2n_2 - 1$) belong to Alice, the qubits (2, \dots , $2n_1$, $2n_1 + 2$, \dots , $2n_1 + 2n_2$) belong to Bob, and the controller Charlie possesses the qubit $2n_1 + 2n_2 + 1$.

The initial state of the whole system is given by

$$|\varphi\rangle_{A_1 \dots A_{n_1}} \otimes |\psi\rangle_{B_1 \dots B_{n_2}} \otimes |C\rangle_{1 \dots 2n_1+2n_2+1}. \quad (19)$$

In order to complete the task, the three participants are required to perform appropriate measurements and corresponding recovery operations. The process of our universal BCQT protocol is shown in figure 1.

The sender's operation. Alice (Bob) carries out projective measurements on her (his) qubits (A_1 , 1), \dots , (A_{n_1} , $2n_1 - 1$) ((B_1 , $2n_1 + 2$), \dots , (B_{n_2} , $2n_1 + 2n_2$)) using the Bell basis and gets the measurement results $|\xi_{p_iq_i}\rangle$, $i = 1, \dots, n_1$ ($|\xi_{s_jt_j}\rangle$, $j = 1, \dots, n_2$).

Then, Alice (Bob) sends the classical bits p_iq_i (s_jt_j) to Charlie, representing the Bell-state measurement outcomes.

The compound system in equation (19) can be rewritten as

$$\prod_{i=1}^{n_1} \left(\sum_{p_i,q_i=0}^1 |\xi_{p_iq_i}\rangle \right) \prod_{j=1}^{n_2} \left(\sum_{s_j,t_j=0}^1 |\xi_{s_jt_j}\rangle \right)$$

$$\sum_{l=0}^1 |f_{p_1q_1 \dots p_{n_1}q_{n_1}}^l\rangle |g_{s_1t_1 \dots s_{n_2}t_{n_2}}^l\rangle |l\rangle. \quad (20)$$

$\sum_{l=0}^1 |f_{p_1q_1 \dots p_{n_1}q_{n_1}}^l\rangle |g_{s_1t_1 \dots s_{n_2}t_{n_2}}^l\rangle |l\rangle$ is the collapsed state if Alice's and Bob's measurement results are $|\xi_{p_1q_1}\rangle, \dots, |\xi_{p_{n_1}q_{n_1}}\rangle$ and $|\xi_{s_1t_1}\rangle, \dots, |\xi_{s_{n_2}t_{n_2}}\rangle$, respectively.

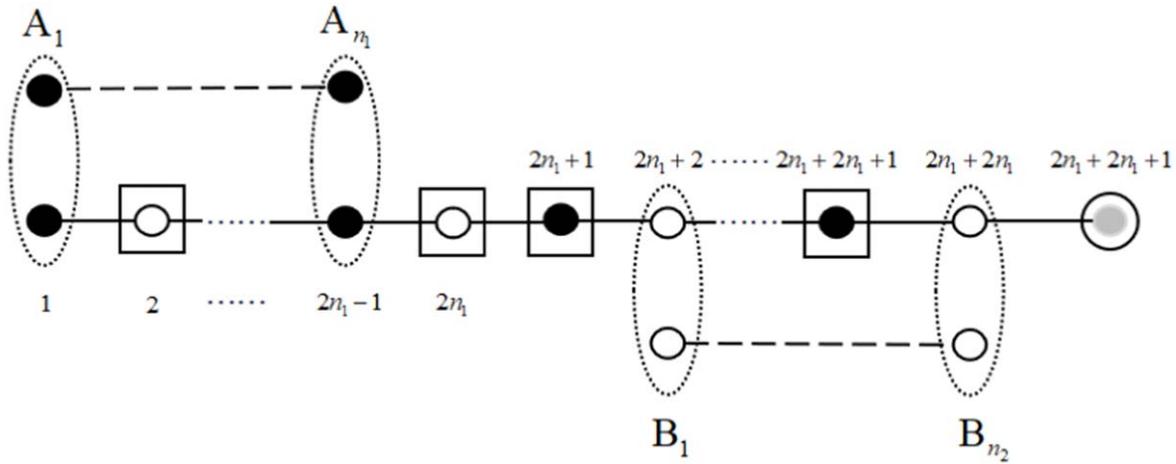


Figure 1. Process of our universal BCQT protocol. Black solid points represent Alice’s particles, white points represent Bob’s particles, and the gray point represents Charlie’s particle. The dotted ellipses indicate Bell-state measurements, and the solid circle is a single-qubit measurement. The solid rectangle indicates the recovery operation.

Since

$$|kh\rangle = \frac{1}{\sqrt{2}} \sum_{n=0}^1 (-1)^{kn} |\xi_{n,k\oplus h}\rangle, \quad (21)$$

one can get

$$|f_{p_1 q_1 \dots p_{n_1} q_{n_1}}^l\rangle = (-1)^{(q_1 \oplus \dots \oplus q_{n_1})l} \sum_{a_1, \dots, a_{n_1}=0}^1 \alpha_a \prod_{j=1}^{n_1} (-1)^{a_j (p_j \oplus l)} |a_j \oplus q_j\rangle, \quad (22)$$

and

$$|g_{s_1 t_1 \dots s_{n_2} t_{n_2}}^l\rangle = (-1)^{(t_1 \oplus \dots \oplus t_{n_2})l} \sum_{b_1, \dots, b_{n_2}=0}^1 \beta_b \prod_{j=1}^{n_2} (-1)^{b_j (s_j \oplus l)} |b_j \oplus t_j\rangle. \quad (23)$$

At this time, Alice and Bob are still unable to complete the task without Charlie’s assistance, since their qubits are entangled with Charlie’s qubit $2n_1 + 2n_2 + 1$.

The controller’s operation. If Charlie consents to help them, he executes a single-qubit projective measurement using the Z basis.

If Charlie’s measurement result is $|l\rangle$ ($l = 0, 1$), Alice’s qubits ($2n_1 + 1, 2n_1 + 3, \dots, 2n_1 + 2n_2 - 1$) and Bob’s qubits ($2, 4, \dots, 2n_1$) collapse into a product state $|f_{p_1 q_1 \dots p_{n_1} q_{n_1}}^l\rangle |g_{s_1 t_1 \dots s_{n_2} t_{n_2}}^l\rangle$. In order to recover the teleported states, Alice and Bob need to know the measurement results $s_1 t_1 \dots s_{n_2} t_{n_2} l$ and $p_1 q_1 \dots p_{n_1} q_{n_1} l$, respectively. It seems that Charlie needs to broadcast $(2n_1 + 2n_2 + 1)$ -bit classical information, including $p_1 q_1 \dots p_{n_1} q_{n_1}, s_1 t_1 \dots s_{n_2} t_{n_2}$ he received and l corresponding to his own measurement result. Instead, with the aid of network coding, Charlie just broadcasts $2n_2$ bits, as follows:

$$s_1 \oplus p_1 \oplus l, \dots, s_{n_1} \oplus p_{n_1} \oplus l, s_{n_1+1} \oplus l, \dots, s_{n_2} \oplus l, \quad (24)$$

$$t_1 \oplus q_1, \dots, t_{n_1} \oplus q_{n_1}, t_{n_1+1}, \dots, t_{n_2}. \quad (25)$$

This means that a $(2n_1 + 1)$ -bit CCC is saved.

Table 3. Comparison of the symmetric protocol for two-qubit states ($n_1 = n_2 = 2$).

Protocol	QC	BO	CCC	NQT	η
[17]	10-ES	BSM, SQM, RUO	10	4	$\frac{1}{5} = 20\%$
[18]	9-ES	BSM, SQM, RUO	9	4	$\frac{2}{9} \approx 22.22\%$
[19]	9-ES	BSM, SQM, RUO	9	4	$\frac{2}{9} \approx 22.22\%$
Ours	9-ES	BSM, SQM, RUO	12	4	$\frac{4}{21} \approx 19.05\%$

The receiver’s operation. According to the classical messages received from Charlie and her (his) own measurement results, Alice calculates

$$p_i \oplus (s_i \oplus p_i \oplus l), q_i \oplus (q_i \oplus t_i), \quad i = 1 \dots n_1. \quad (26)$$

Together with $s_{n_1+1} \oplus l, \dots, s_{n_2} \oplus l, t_{n_1+1}, \dots, t_{n_2}$, Alice gets $[s_j \oplus l, t_j], j = 1, \dots, n_2$. Bob computes

$$s_i \oplus (p_i \oplus s_i \oplus l), t_i \oplus (q_i \oplus t_i), \quad (27)$$

and gains $[p_i \oplus l, q_i], i = 1 \dots n_1$. Using equations (22) and (23), Alice and Bob perform the recovery operations R_A and R_B on their qubits, which can be summarized as

$$R_A = Z_{2n_1+1}^{s_1 \oplus l} X_{2n_1+1}^{t_1} \dots Z_{2n_1+2n_2-1}^{s_{n_2} \oplus l} X_{2n_1+2n_2-1}^{t_{n_2}}, \quad (28)$$

$$R_B = Z_2^{p_1 \oplus l} X_2^{q_1} \dots Z_{2n_1}^{p_{n_1} \oplus l} X_{2n_1}^{q_{n_1}}. \quad (29)$$

Accordingly, the BCQT is achieved with a success probability of 100%.

5. Discussion and comparisons

In this section, we first discuss the intrinsic efficiency [18] and the necessary operations of our universal protocol. Then, some comparisons with other protocols are given.

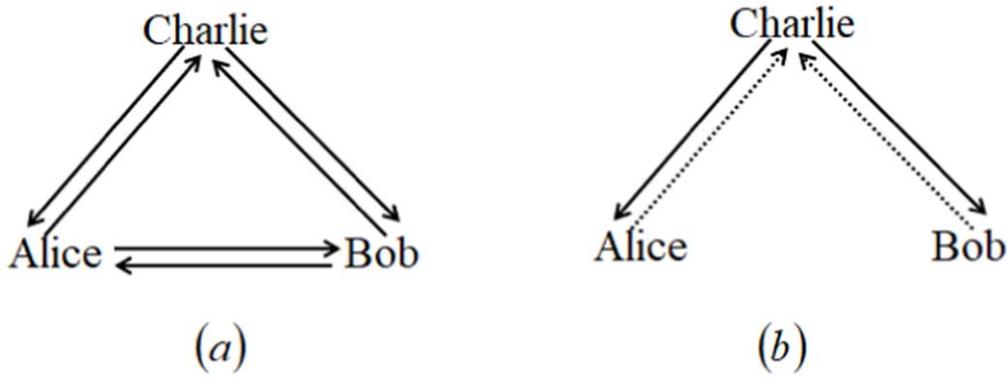


Figure 2. Classical channel models. (a) The usual model; (b) our model. A solid line with one arrow denotes a one-way broadcast channel, while a dotted line with one arrow denotes a one-way classical channel.

Table 4. Comparison of the asymmetric BCQT protocol for single- and two-qubit states ($n_1 = 1, n_2 = 2$).

Protocol	QC	BO	CCC	NQT	η
[20]	8-ES	UO, BSM, FQM, TQM, RUO	8	3	$\frac{3}{16} = 18.75\%$
[21]	7-ES	BSM, SQM, CNOT, RUO	7	3	$\frac{3}{14} \approx 21.43\%$
[22]	7-ES	BSM, SQM, RUO	7	3	$\frac{3}{14} \approx 21.43\%$
[23]	7-ES	BSM, SQM, RUO	7	3	$\frac{3}{14} \approx 21.43\%$
[24]	7-ES	BSM, SQM, RUO	7	3	$\frac{3}{14} \approx 21.43\%$
Ours	7-ES	BSM, SQM, RUO	10	3	$\frac{3}{17} \approx 17.65\%$

The intrinsic efficiency is an important factor for evaluating the performance of a protocol, which is defined by

$$\eta = \frac{q_s}{q_u + b_t}, \tag{30}$$

where q_s represents the number of qubits to be transmitted, q_u is the number of particles used in the quantum channel, and b_t represents the classical bits that need to be transmitted.

As mentioned above, we propose a universal BCQT protocol for arbitrary n_1 - and n_2 - qubit states, which can be realized deterministically. The $(2n_1 + 2n_2 + 1)$ -qubit entangled state is chosen as the quantum channel. Alice and Bob respectively transmit $2n_1$ and $2n_2$ bits to the controller Charlie. Charlie encodes his classical information corresponding to his measurement result with the classical messages he received, and then broadcasts the encoded result to Alice and Bob. The CCC in the controller's broadcast channel is $2n_2$ bits. The total CCC is $2n_1 + 4n_2$ bits. Therefore, the intrinsic efficiency of our universal protocol is

$$\begin{aligned} \eta &= \frac{n_1 + n_2}{(2n_1 + 2n_2 + 1) + (2n_1 + 4n_2)} \\ &= \frac{n_1 + n_2}{4n_1 + 6n_2 + 1}. \end{aligned} \tag{31}$$

In our symmetric BCQT protocol for arbitrary two-qubit states ($n_1 = n_2 = 2$), the intrinsic efficiency is $\eta = \frac{4}{21} \approx 19.05\%$. In our asymmetric BCQT protocol for arbitrary single- and two-qubit states ($n_1 = 1, n_2 = 2$), the intrinsic efficiency is $\eta = \frac{3}{17} \approx 17.65\%$.

If network coding is not applied, the total CCC is $4n_1 + 4n_2 + 1$ bits. The intrinsic efficiency of the universal protocol will be

$$\begin{aligned} \eta &= \frac{n_1 + n_2}{(2n_1 + 2n_2 + 1) + (4n_1 + 4n_2 + 1)} \\ &= \frac{n_1 + n_2}{6n_1 + 6n_2 + 2}, \end{aligned} \tag{32}$$

which is smaller than the intrinsic efficiency in equation (31). In this case, the intrinsic efficiency of the symmetric BCQT protocol is $\eta = \frac{4}{26} \approx 15.39\%$. When $n_1 = 1$ and $n_2 = 2$, the intrinsic efficiency is $\eta = \frac{3}{20} \approx 15\%$. This reveals that the use of network coding technology can reduce the CCC and improve the intrinsic efficiency.

In the following, we give some comparisons with other BCQT protocols. To make the comparison convective, we only choose some specific values of n_1, n_2 . The results of comparisons with previous symmetric and asymmetric protocols are given in tables 3 and 4, respectively. A detailed explanation of the abbreviations is as follows: QC (quantum channel), ES (entangled state), BO (basic operation), UO (unitary operation), RUO (recovery unitary operation), BSM (Bell-state measurement), SQM (single-qubit measurement), TQM (two-qubit measurement), FQM (four-qubit measurement), CNOT (controlled-NOT), NQT (number of the teleported qubits).

From tables 3 and 4, one can see that the intrinsic efficiencies of [17–24] are all greater than ours. The reason for this is that the classical channel model they use is different

from ours. Classical channel models are clearly shown in figure 2, where (a) represents the model of [17–24], and (b) represents our model. As well as the unfavorable aspects, our scheme also has favorable aspects. The classical channel models in [17–24] assume that each sender broadcasts her measurement result. In other words, everyone has a broadcast channel. This may reduce the CCC and improve the intrinsic efficiency; however, the establishment of a broadcast channel is much more difficult than that of a one-way classical channel. For an actual communication network with multiple users and a control center, it is not realistic to set up a broadcast channel for each user. In our model, each sender only has one classical channel with the controller, which greatly reduces the requirement for classical channels. Moreover, our model may be more applicable and safer in the case that direct classical communication is not allowed between different users, and the role of the controller is to be a transmission relay or trusted third party.

6. Conclusions

Assuming that the two senders cannot transmit classical messages to each other and that the controller has a broadcast channel, we have devised a universal protocol for the two-way teleportation of arbitrary n_1 - and n_2 -qubit states. The proposed scheme has the following advantages. (i) Through strict derivation, the recovery operator is given by a general formula which clearly reveals the relationship with the measurement results. (ii) Network coding is performed at the site of the controller. It is easy to see if the network coding is not applied; a $(2n_1 + 2n_2 + 1)$ -bit CCC is required for Charlie's broadcast channel. Therefore, a $(2n_1 + 1)$ -bit CCC is saved. It may be helpful to realize secure quantum distributed transmission tasks with a reduced CCC. We also believe that network coding may be useful for saving resources in various quantum network information-processing tasks.

The construction of the quantum channel only needs H and CNOT gate operations on the auxiliary particles with an initial state of $|00 \cdots 0\rangle_{1 \dots 2n_1 + 2n_2 + 1}$. The process of the universal protocol requires Bell-basis measurement, Z-basis measurement, and single-qubit local Pauli operations. These are feasible with the current experimental technology. It should be noted that quantum entanglement is indispensable in most existing BCQT schemes. However, from a practical point of view, entanglements are often very vulnerable and may suffer from destructive influences, or even disappear completely. Therefore, the effects of noise should be taken into account. Similarly to [8, 9], we can envisage the proposed scheme in a noisy environment. Fortunately, some mature anti-noise techniques, such as entanglement purification and entanglement concentration, can be employed to improve the fidelity of the transmitted quantum signals and reduce the effects of noise.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Grant Nos. 61 201 253, 61 572 246), the Open Foundation of State Key Laboratory of Networking and Switching Technology (Beijing University of Posts and Telecommunications) (Grant No. SKLNST-2020-2-02), and the Open Foundation of Guangxi Key Laboratory of Trusted Software (Grant No. KX202040).

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