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Detecting entanglement of quantum channels

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Abstract

Entanglement is the crucial resource for different quantum information processing tasks. While conventional studies focus on the entanglement of bipartite or multipartite quantum states, recent works have extended the scenario to the entanglement of quantum channels, an operational quantification of the channel entanglement manipulation capability. Based on the recently proposed channel entanglement resource framework, here we study a further task of resource detection—witnessing entanglement of quantum channels. We first introduce the general framework and show how channel entanglement detection is related to the Choi state of the channel, enabling channel entanglement detection via conventional state entanglement detection methods. We also consider entanglement of multipartite quantum channels and use the stabilizer formalism to construct entanglement witnesses for circuits consisting of controlled-Z gates. We study the effectiveness of the proposed detection methods and compare their performance for several typical channels. Our work paves the way for systematic theoretical studies of channel entanglement and practical benchmarking of noisy intermediate scaled quantum devices.

Keywords: quantum entanglement, quantum channel, entanglement detection, entanglement witness

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement is a key feature of quantum physics [1–3], having wide applications in various quantum information processing tasks including quantum dense coding [4], quantum teleportation [5], quantum cryptography [6], quantum computation [7], etc. However, because almost every quantum system is noisy, how the noise affects the quantum system and whether entanglement survives under the noise is important for robust and reliable quantum information processing. It thus becomes one of the basic problems of quantum entanglement theory to check whether a quantum state is entangled or not. Entangled quantum states could be detected and analyzed by several theoretical and experimental tools. Many researchers have contributed to various separability standards and detection methods [8–16]. Notable approaches in the practical analysis of entanglement include the usage of positive but not completely positive maps [8] and entanglement witnesses [9–12].

Conventional entanglement theory focuses on the nonlocal correlation of quantum states, and quantum channels are used as their manipulation tool. Recent works have shown that quantum channel itself could be regarded as the resource object [17–22] and the entanglement of channels has be studied under the framework of quantum resource theories [23–28]. Analog to quantum states, quantum channels are also categorized into entangled and separable ones, with the amount of entanglement quantified via channel entanglement measures [23–25]. Since a positive channel entanglement measure generally implies the existence of entanglement, it serves as a natural way to detect channel entanglement. However, a channel entanglement measure generally requires full information of the process, hence demanding a cost that is exponential to the system size. Because the channel has both inputs and outputs, the cost increases even quadratically faster than the one of states. Efficient state entanglement detection methods exist when exploiting the specific structure of the target resource and allowing a certain amount of failure, whether we could analogously detect channel entanglement remains open.

Here, we address this problem by considering general approaches for detecting channel entanglement. We first review the framework of state and channel entanglement as well as the task of entanglement detection. Then we show that detecting the channel entanglement is equivalent to detecting the entanglement of the corresponding Choi state, the output state of the channel given the maximally entangled input. We give three general channel entanglement detection strategies based on conventional state entanglement detection methods—negative partial transpose (NPT) [29], the computable cross norm or realignment criterion (CCNR) criterion [30, 31], and entanglement witnesses. We further extend the discussion to multipartite quantum channels and show how the stabilizer formalism [32] helps in designing entanglement witnesses for circuits consisting of CZ gates. As examples, we consider noisy CNOT and SWAP channels and show the effectiveness of the propose methods.

2. Background

We first review the framework of entanglement for bipartite states and channels, and the task of entanglement detection. For a system A, we denote the corresponding Hilbert space as $\mathcal{H}_A$ and the set of state operators as $\mathcal{D}(A)$. Consider a bipartite system AB, we call the state σ_{AB} separable when it can be expressed as [33–35]

$$\sigma_{AB} = \sum_i \lambda_i \sigma_A^i \otimes \sigma_B^i \tag{1}$$

with $\lambda_i \geqslant 0$, $\sum_i \lambda_i = 1$, and local states σ_A, σ_B on quantum systems A and B, respectively. Otherwise the state is entangled. On the other hand, quantum channels are linear and completely positive and trace-preserving maps which act on quantum states [36]. A quantum channel $\mathcal{N}$ could be written in the Kraus form [37] as

$$\mathcal{N}(\rho) = \sum_{i=1}^{r} K_i \rho K_i^\dagger, \tag{2}$$

where the Kraus operators $\{K_i\}$ fulfil $\sum_{i=1}^{r} K_i^\dagger K_i = \mathbb{I}_A$. Let $\mathcal{N}_{AB} \in \mathrm{CPTP}(AB \to A'B')$ be a bipartite channel, it is called separable [23, 25] if it can be written as

$$\mathcal{N}_{AB}(\rho_{AB}) = \sum_i (X_A^i \otimes Y_B^i)\rho_{AB}(X_A^i \otimes Y_B^i)^\dagger, \tag{3}$$

where $\sum_i (X_A^i)^\dagger X_A^i \otimes (Y_B^i)^\dagger Y_B^i = \mathbb{I}_{AB}$, otherwise the channel is entangled. The entanglement of multipartite states and channels could be similarly defined.

Suppose a quantum information task requires to prepare an entangled state ψ_{AB} and the practically prepared state is ρ_{AB}. The task of entanglement detection is to determine whether ρ_{AB} is entangled. A general detection strategy corresponds to a function f, which distinguishes separable and entangled states. However, since the geometry of the set of separable states is complex, detecting the entanglement of an arbitrary state is a challenging task. A conventional entanglement detection strategy is to exploit a linear witness operator W such that $\mathrm{Tr}\,(W\sigma_{AB}) \geqslant 0$ for all separable states σ_{AB} and it detects the entanglement of ρ_{AB} with $\mathrm{Tr}\,(W\rho_{AB}) < 0$ [16]. Here we review three general strategies of constructing the witness operator, which will be extended to detecting entanglement of quantum channels in the next section.

The first method is to exploit the positive but not completely positive map of transpose, or the NPT property of states. In particular, we always have $\sigma_{AB}^{\mathrm{T}_A} \geqslant 0$ when applying a partial transpose T_A on a separable state σ_{AB}, whereas we could have a negative partial transposed state $\rho_{AB}^{\mathrm{T}_A} < 0$ when ρ_{AB} is entangled. In this case, we can find the eigenvector $|\phi\rangle$ with a negative eigenvalue [38], i.e. $\rho_{AB}^{\mathrm{T}_A}|\phi\rangle = \lambda|\phi\rangle$ with $\lambda < 0$. Then, a linear witness operator can be constructed as follows [16, 29]

$$W = |\phi\rangle\langle\phi|^{\mathrm{T}_A}. \tag{4}$$

It is easy to verify that for any state with positive partial transpose, which include separable states σ_{AB}, we have

$$\mathrm{Tr}[W\sigma_{AB}] = \mathrm{Tr}[|\phi\rangle\langle\phi|^{\mathrm{T}_A}\sigma_{AB}] = \mathrm{Tr}[|\phi\rangle\langle\phi|\ \sigma_{AB}^{\mathrm{T}_A}] \geqslant 0, \tag{5}$$

and for the entangled state ρ_{AB},

$$\mathrm{Tr}\,(W\rho_{AB}) = \mathrm{Tr}(|\phi\rangle\langle\phi|^{\mathrm{T}_A}\rho_{AB}) = \mathrm{Tr}(|\phi\rangle\langle\phi|\rho_{AB}^{\mathrm{T}_A}) = \lambda < 0. \tag{6}$$

In practice, we may not know the density matrix ρ_{AB} and hence cannot get the eigenvectors with negative eigenvalues. Nevertheless, a realistic quantum protocol generally assumes an ideal pure state ψ_{AB}, and the partial transposed pure state $\psi_{AB}^{\mathrm{T}_A}$ could be used as the witness operator. Indeed, any partial transposed pure state could serve as the witness operator, although to be able to detect the entanglement of a given state, the operator should be accordingly chosen.

The second method is based on a Schimidt decomposition [39] of the density operator, namely the computable CCNR. Specifically, denote the set of Hermitian operators on $\mathcal{H}$ as $\mathcal{L}(\mathcal{H})$, which is a linear space with inner product $\langle V_1, V_2\rangle = \mathrm{Tr}[V_1^\dagger V_2]$, for any $V_1, V_2 \in \mathcal{L}(\mathcal{H})$. Then for any density matrix $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB}) \subset \mathcal{L}(\mathcal{H}_{AB})$, the Schmidt decomposition of ρ_{AB} is

$$\rho_{AB} = \sum_k \lambda_k V_k^A \otimes V_k^B, \tag{7}$$

where $\lambda_k \geqslant 0$, V_k^A and V_k^B are orthonormal bases in $\mathcal{L}(\mathcal{H}_A)$ and $\mathcal{L}(\mathcal{H}_B)$, respectively. Then we can construct the witness as follows [30, 31]

$$W = \mathbb{I}_A \otimes \mathbb{I}_B - \sum_k V_k^A \otimes V_k^B. \tag{8}$$

It is easy to verify that for any separable state

$\sigma_{AB} = \sum_i \tilde{\lambda}_i \sigma_A^i \otimes \sigma_B^i$, we have

$$\mathrm{Tr}[\sigma_{AB} W] = 1 - \sum_k \sum_i \tilde{\lambda}_i \mathrm{Tr}[(\sigma_A^i \otimes \sigma_B^i) \cdot (V_k^A \otimes V_k^B)] \geqslant 0. \tag{9}$$

Therefore, the state ρ_{AB} is entangled whenever $\mathrm{Tr}[\rho_{AB} W] < 0$. Again, the witness operator W could be constructed from any state, for example based on the ideal pure state ψ_{AB}.

In the third method, we construct the witness operator W based on any observable O as

$$W = \alpha \mathbb{I} - O, \tag{10}$$

where

$$\alpha = \max_{\sigma_{AB} \text{ is separable}} \mathrm{Tr}(\sigma_{AB} O). \tag{11}$$

Since $\mathrm{Tr}[\sigma_{AB} O] \leqslant \alpha$, we have $\mathrm{Tr}[W \sigma_{AB}] \geqslant 0$ for all separable states. In practice, we could choose $O = \psi_{AB}$ and the witness operator can effectively detect entanglement with white noise.

Given the witness operator, a practical scheme is to decompose the operator into local observables. This is particularly important for detecting multipartite entanglement because it is hard to measure a general multipartite observable. Efficient witnesses have been constructed for several typical classes of states including the W state and general graph states. In the next section, we show how similar strategies could be extended to efficient channel entanglement detection.

3. Channel entanglement detection

3.1. Bipartite channels

Since a quantum channel has both inputs and outputs, we first map a channel to a quantum state. In particular, consider a bipartite channel $\mathcal{N}_{AB}$ that maps AB to $A'B'$, we consider the maximally entangled input state $\Phi_{AA} \otimes \Phi_{BB}$ with $\Phi = \sum_{ij} |ii\rangle\langle jj| / d$ and d being the dimension of each subsystem [23–25]. The output state

$$\Phi^{\mathcal{N}}_{AA'B'B} = \mathcal{N}_{AB}(\Phi_{AA} \otimes \Phi_{BB}) \tag{12}$$

is called the Choi state which is a one-to-one map between states and channels. The entanglement of the bipartite channel can be now reformulated via the entanglement of the Choi state.

Lemma 1. *A bipartite channel $\mathcal{N}_{AB}$ is entangled if the Choi state $\Phi^{\mathcal{N}}_{AA'B'B}$ is entangled.*

Similar results have been discussed in several works [23–25] and we refer to appendix for the proof. Based on such a connection, we can now study the entanglement of channels via its Choi state. Focusing on entanglement detection, our task now becomes to find an observable W, such that $\mathrm{Tr}[W \Phi^{\mathcal{N}}_{AA'B'B}] \geqslant 0$ for all separable channels $\mathcal{N}$. Following the above results for quantum states, we can similarly introduces three types of witness operators for Choi states of channels.

- Consider the eigenvector $|\phi\rangle$ with a negative eigenvalue of $\Phi^{\mathcal{N}}_{AA'B'B}$, the witness operators is $W = |\phi\rangle\langle\phi|^{\mathrm{T}_{AA'}}$.
- Suppose $\Phi^{\mathcal{N}}_{AA'B'B} = \sum_k \lambda_k V_k^{AA'} \otimes V_k^{B'B}$, the witness operator is $W = \mathbb{I} - \sum_k V_k^{AA'} \otimes V_k^{B'B}$.
- For any observable $O_{AA'B'B}$, we can construct a witness as $W = \alpha \mathbb{I} - O_{AA'B'B}$ with $\alpha = \max_{\Phi^{\mathcal{N}}_{AA'B'B}, \mathcal{N} \text{ is separable}} \mathrm{Tr}[\Phi^{\mathcal{N}}_{AA'B'B} O_{AA'B'B}]$.

We will shortly show that we can realize the witness with quantum games consisting of proper input states, measurements, and payoffs. Next, we extend the discussion to the multipartite scenario.

3.2. Multipartite channel entanglement

We consider the entanglement of multipartite channels [40, 41]. Consider a channel Θ_n that maps n systems $\{1, 2, \ldots, n\}$ to n systems $\{1', 2', \ldots, n'\}$, it is fully separable when

$$\Theta_n(\rho_{1,2,\ldots,n}) = \sum_j \left(\bigotimes_{k=1}^n X_k^j \right) \rho_{1,2,\ldots,n} \left(\bigotimes_{k=1}^n X_k^j \right)^\dagger, \tag{13}$$

where the Kraus operators satisfy $\sum_i (\bigotimes_{k=1}^n X_{A_k}^j)^\dagger \otimes (\bigotimes_{k=1}^n X_{A_k}^j) = \mathbb{I}_{1,2,\ldots,n}$. Otherwise, the channel is entangled. Furthermore, suppose we divide the n systems into two parts $\{S, \bar{S}\}$, the channel is called biseparable when it can be written as [11, 42]

$$\Theta_n(\rho_{1,2,\ldots,n}) = \sum_j (X_S^j \otimes X_{\bar{S}}^j) \rho_{1,2,\ldots,n} (X_S^j \otimes X_{\bar{S}}^j)^\dagger, \tag{14}$$

where $\sum_i (X_S^j \otimes X_{\bar{S}}^j)^\dagger \otimes (X_S^j \otimes X_{\bar{S}}^j) = \mathbb{I}$. When a channel is not biseparable under any bipartition, it is called genuinely entangled.

Based on the Choi state of the channel

$$\Phi^{\Theta}_{11', 22', \ldots, nn'} = \Theta_n(\Phi_{11'} \otimes \ldots \otimes \Phi_{nn'}), \tag{15}$$

we can similarly relate the entanglement of Θ_n to the entanglement of $\Phi^{\Theta}_{11', 22', \ldots, nn'}$.

Lemma 2. *A multipartite channel Θ_n is (genuinely) entangled if the Choi state $\Phi^{\Theta}_{11', 22', \ldots, nn'}$ (genuinely) entangled.*

Therefore, we could use techniques of detecting multipartite entanglement to detect channel entanglement. While the three entanglement detection methods work similarly for multipartite channels, how to make the detection to be efficient is in general a challenging task for multipartite channels.

3.3. Entanglement detection via quantum games

In the above discussion, the entanglement witness is constructed and applied with respect to the Choi state of the channel. While we could get the Choi state by inputting a maximally entangled state, we need to double the system size, making its implementation hard. Here, we show a different yet equivalent entanglement detection way via quantum games [43–46].

A quantum game $\mathcal{G}$ is defined by the tuple $\mathcal{G} = (\{\alpha_{ij}\}, \{\rho_i\}, \{O_j\})$, where ρ_i are input states, $\{O_j\}$ is a positive observable valued measures at the output, and $\alpha_{ij} \in \mathbb{R}$ are the real coefficients which define the particular game. The performance in the game $\mathcal{G}$ enabled by a channel $\mathcal{N}$ is quantified by the payoff function

$$\mathcal{P}(\mathcal{N}, \mathcal{G}) = \sum_{ij} \alpha_{ij} \operatorname{Tr}[O_j \mathcal{N}(\rho_i)]. \tag{16}$$

Consider the bipartite channel $\mathcal{N}_{AB}$ as an example, we have $\mathcal{N}(\rho_{i,AB}) = d\, \operatorname{Tr}_{AB}[\Phi^{\mathcal{N}}_{AA'B'B}(\rho^{\mathrm{T}}_{i,AB} \otimes I_{A'B'})]$ according to the Choi–Jamiolkowski isomorphism. Then we have

$$\begin{aligned}\mathcal{P}(\mathcal{N}, \mathcal{G}) &= \textstyle\sum_{i,j} \alpha_{i,j} d \operatorname{Tr}[\Phi^{\mathcal{N}}_{AA'B'B}(\rho^{\mathrm{T}}_{i,AB} \otimes O_{j,A'B'})],\\ &= d \operatorname{Tr}[\Phi^{\mathcal{N}}_{AA'B'B} W],\end{aligned} \tag{17}$$

where the W operator is

$$W = \sum_{i,j} \alpha_{i,j} \rho^{\mathrm{T}}_{i,AB} \otimes O_{j,A'B'}. \tag{18}$$

Therefore, for any witness operator W, we can decompose it as above and it corresponds to a quantum game [43]. Suppose the quantum game with a witness operator W is $\mathcal{G}(W)$, then we can show that for any separable channel N_{AB} we have

$$\mathcal{P}(\mathcal{N}_{AB}, \mathcal{G}(W)) \geqslant 0. \tag{19}$$

In practice, we can equivalently realize the witness via a quantum game. Instead of witnessing the entanglement of the Choi state, we can apply the channel to a set of input states, measure the output, and linearly combine the measurement outcomes. The game payoff function plays a similar role of detecting channel entanglement.

4. Example

Now we show entanglement detection for several typical channels. We also note the following fact that local unitary operations before and after the channel does not change the entanglement.

Lemma 3. *Given a multipartite channel Θ_n with local unitary $U_1, U_2, \ldots, U_n$ and $V_1, V_2, \ldots, V_n$, Θ_n is entangled if $(U_1 \otimes U_2 \otimes \ldots \otimes U_n) \circ \Theta_n \circ (V_1 \otimes V_2 \otimes \ldots \otimes V_n)$ is entangled.*

Therefore, any entanglement witness for a channel $\mathcal{N}$ works similarly to other channels $\mathcal{V} \circ \mathcal{N} \circ \mathcal{U}$ that are equivalent to $\mathcal{N}$ under local unitary operations $\mathcal{U}$ and $\mathcal{V}$.

4.1. Bipartite channels

We first consider bipartite channels, specifically, the CNOT and SWAP gate under local depolarizing noise, as shown in figure 1. In particular the CNOT and SWAP gates are unitary

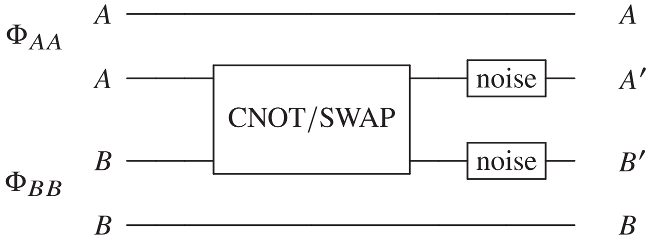

Figure 1. The CNOT or SWAP gate with local depolarizing noise. Given maximally entangled input states $\Phi_{AA} = |\Phi\rangle\langle\Phi|_{AA}$ and $\Phi_{BB} = |\Phi\rangle\langle\Phi|_{BB}$, the output state corresponds to the Choi state of the noise channel. The task is to detect the entanglement of the output state between the bipartition between AA' and $B'B$.

$$U_{\mathrm{CNOT}} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0 \end{bmatrix}, \; U_{\mathrm{SWAP}} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

and the local depolarizing noise is

$$\mathcal{D}_p(\rho) = (1-p)\rho + p\mathbb{I}/2. \tag{21}$$

The noisy CNOT/SWAP channel is

$$\mathcal{N}_{p,AB}(\rho_{AB}) = \mathcal{D}_{p,A'} \otimes \mathcal{D}_{p,B'}(U(\rho_{AB})U^{\dagger}), \tag{22}$$

where U is either U_{CNOT} or U_{SWAP}. Here we explicitly show the three entanglement witnesses for the CNOT gate and the result works analogy for the SWAP gate.

The Choi state of the CNOT gate is

$$\begin{aligned}|\Phi^{\mathrm{CNOT}}_{AA'B'B}\rangle &= U^{AB}_{\mathrm{CNOT}}|\Phi_{AA}\rangle|\Phi_{BB}\rangle,\\ &= \frac{1}{2}(|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle),\end{aligned} \tag{23}$$

and the density matrix is

$$\Phi^{\mathrm{CNOT}}_{AA'B'B} = |\Phi^{\mathrm{CNOT}}_{AA'B'B}\rangle\langle\Phi^{\mathrm{CNOT}}_{AA'B'B}|. \tag{24}$$

The three witness operators could then be constructed accordingly. Consider the eigenvector with a negative eigenvalue of $\Phi^{\mathrm{CNOT}}_{AA'B'B}$, the witness operators of the first method is $W_{\mathrm{CNOT},1} = |\phi\rangle\langle\phi|^{\mathrm{T}_{AA'}}$ with

$$|\phi\rangle = (-|0001\rangle - |0010\rangle + |1100\rangle + |1111\rangle)/2. \tag{25}$$

Suppose $\Phi^{\mathrm{CNOT}}_{AA'B'B} = \sum_k \lambda_k V^{AA'}_k \otimes V^{B'B}_k$, the witness operator of the second method is $W_{\mathrm{CNOT},2} = \mathbb{I} - \sum_{k,\lambda_k \neq 0} V^{AA'}_k \otimes V^{B'B}_k$. We have four nonzero eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0.5$ with

$$\begin{aligned}
&V^{AA'}_1 = (-\mathbb{I} - \sigma_z\sigma_z)/\sqrt{2}, \; V^{B'B}_1 = (-\mathbb{I} - \sigma_x\sigma_x)/\sqrt{2},\\
&V^{AA'}_2 = (\mathbb{I}\sigma_z + \sigma_z\mathbb{I})/\sqrt{2}, \; V^{B'B}_2 = (-\sigma_y\sigma_y + \sigma_z\sigma_z)/\sqrt{2},\\
&V^{AA'}_3 = (\sigma_x\sigma_x - \sigma_y\sigma_y)/\sqrt{2}, \; V^{B'B}_3 = (\mathbb{I}\sigma_x + \sigma_x\mathbb{I})/\sqrt{2},\\
&V^{AA'}_4 = (\sigma_x\sigma_y + \sigma_y\sigma_x)/\sqrt{2}, \; V^{B'B}_4 = (-\sigma_y\sigma_z - \sigma_z\sigma_y)/\sqrt{2}.
\end{aligned} \tag{26}$$

For any observable $O_{AA'B'B}$, we can construct a third witness as $W_{\mathrm{CNOT},3} = \frac{1}{2}\mathbb{I} - \Phi^{\mathrm{CNOT}}_{AA'B'B}$.

Table 1. Quantum game with $W_{\mathrm{CNOT},1}$ for the noisy CNOT gate.

α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$	α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$
1	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	1	$\mathbb{I}$	σ_x	$\mathbb{I}$	σ_x
1	$\mathbb{I}$	σ_y	σ_z	σ_y	-1	$\mathbb{I}$	σ_z	σ_z	σ_z
-1	σ_x	σ_x	σ_x	$\mathbb{I}$	-1	σ_x	$\mathbb{I}$	σ_x	σ_x
1	σ_x	σ_z	σ_y	σ_y	1	σ_x	σ_y	σ_y	σ_z
1	σ_y	σ_z	σ_x	σ_y	1	σ_y	σ_y	σ_x	σ_z
1	σ_y	σ_x	σ_y	$\mathbb{I}$	1	σ_y	$\mathbb{I}$	σ_y	σ_x
1	σ_z	σ_y	$\mathbb{I}$	σ_y	-1	σ_z	σ_z	$\mathbb{I}$	σ_z
1	σ_z	$\mathbb{I}$	σ_z	$\mathbb{I}$	1	σ_z	σ_x	σ_z	σ_x

Table 2. Quantum game with $W_{\mathrm{CNOT},2}$ for the noisy CNOT gate.

α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$	α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$
14	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	-2	$\mathbb{I}$	σ_x	$\mathbb{I}$	σ_x
2	$\mathbb{I}$	σ_y	σ_z	σ_y	-2	$\mathbb{I}$	σ_z	σ_z	σ_z
-2	σ_x	σ_x	σ_x	$\mathbb{I}$	-2	σ_x	$\mathbb{I}$	σ_x	σ_x
2	σ_x	σ_z	σ_y	σ_y	2	σ_x	σ_y	σ_y	σ_z
2	σ_y	σ_z	σ_x	σ_y	2	σ_y	σ_y	σ_x	σ_z
2	σ_y	σ_x	σ_y	$\mathbb{I}$	2	σ_y	$\mathbb{I}$	σ_y	σ_x
2	σ_z	σ_y	$\mathbb{I}$	σ_y	-2	σ_z	σ_z	$\mathbb{I}$	σ_z
-2	σ_z	$\mathbb{I}$	σ_z	$\mathbb{I}$	-2	σ_z	σ_x	σ_z	σ_x

Given the witness operator, we can now construct the quantum game by decomposing the witness in the Pauli basis. In particular, we have the quantum games for $W_{\mathrm{CNOT},1}$ and $W_{\mathrm{CNOT},2}$ as shown in tables 1 and 2, respectively. We note that interestingly, even though the two witness operators $W_{\mathrm{CNOT},2}$ and $W_{\mathrm{CNOT},3}$ are constructed differently, we do have $W_{\mathrm{CNOT},2} = 2 * W_{\mathrm{CNOT},3}$. We note that in the quantum games for $W_{\mathrm{CNOT},1}$ and $W_{\mathrm{CNOT},2}$, we used Pauli matrix as input states for simplicity, which is not exactly correct. Nevertheless, we can solve the problem by decomposing each Pauli matrix as a linear combination of pure states. Furthermore, there may exist better decomposition of the witness operator when we consider general input states and measurements [44, 47].

For the SWAP gate, its Choi state is

$$\begin{aligned} |\Phi_{AA'B'B}^{\mathrm{SWAP}}\rangle &= U_{\mathrm{SWAP}}^{AB}|\Phi_{AA'}\rangle|\Phi_{B'B}\rangle, \\ &= \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle), \end{aligned} \tag{27}$$

and the density matrix is

$$\Phi_{AA'B'B}^{\mathrm{SWAP}} = |\Phi_{AA'B'B}^{\mathrm{SWAP}}\rangle\langle\Phi_{AA'B'B}^{\mathrm{SWAP}}|. \tag{28}$$

The three witness operators could then be constructed accordingly. Consider the eigenvector with a negative eigenvalue of $\Phi_{AA'B'B}^{\mathrm{SWAP}}$, the witness operators of the first method is $W_{\mathrm{SWAP},1} = \frac{1}{2}(-|0010\rangle + |1000\rangle)(-\langle 0010| + \langle 1000|)^{\mathrm{T}_{AA'}}$. Interestingly, for the second and third method, we again have the same witness (with different but irrelevant normalization factor). In particular, we have $W_{\mathrm{SWAP},2} = 6W_{\mathrm{SWAP},3}$ with $W_{\mathrm{SWAP},3} = \frac{1}{4}\mathbb{I} - \Phi_{AA'B'B}^{\mathrm{SWAP}}$. We also convert the witness into quantum games. In particular, we have the quantum games for $W_{\mathrm{SWAP},1}$ and $W_{\mathrm{SWAP},2}$ as shown in table 3 and 4, respectively.

Consider a noisy CNOT and SWAP gate with different noise ratio, we show the three entanglement witness values in figure 2. Since the second and third methods give basically the same witness, we have normalized the witness value. For the noisy CNOT gate, the second and the third methods outperform the first method, whereas for the noisy SWAP gate, have the same effect under normalization.

Table 3. Quantum game with $W_{\mathrm{SWAP},1}$ for the noisy SWAP gate.

α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$	α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$
1	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	1	$\mathbb{I}$	σ_z	$\mathbb{I}$	$\mathbb{I}$
1	$\mathbb{I}$	$\mathbb{I}$	σ_z	$\mathbb{I}$	1	$\mathbb{I}$	σ_z	σ_z	$\mathbb{I}$
-1	σ_x	$\mathbb{I}$	$\mathbb{I}$	σ_x	-1	σ_x	σ_z	$\mathbb{I}$	σ_x
-1	σ_x	$\mathbb{I}$	σ_z	σ_x	-1	σ_x	σ_z	σ_z	σ_x
1	σ_y	$\mathbb{I}$	$\mathbb{I}$	σ_y	1	σ_y	σ_z	$\mathbb{I}$	σ_y
1	σ_y	$\mathbb{I}$	σ_z	σ_y	1	σ_y	σ_z	σ_z	σ_y
-1	σ_z	$\mathbb{I}$	$\mathbb{I}$	σ_z	-1	σ_z	σ_z	$\mathbb{I}$	σ_z
-1	σ_z	$\mathbb{I}$	σ_z	σ_z	-1	σ_z	σ_z	σ_z	σ_z

Table 4. Quantum game with $W_{\mathrm{SWAP},2}$ for the noisy SWAP gate.

α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$	α	ρ_A^{T}	ρ_B^{T}	$O_{A'}$	$O_{B'}$
12	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	$\mathbb{I}$	-4	$\mathbb{I}$	σ_x	σ_x	$\mathbb{I}$
4	$\mathbb{I}$	σ_y	σ_y	$\mathbb{I}$	-4	$\mathbb{I}$	σ_z	σ_z	$\mathbb{I}$
-4	σ_x	$\mathbb{I}$	$\mathbb{I}$	σ_x	-4	σ_x	σ_x	σ_x	σ_x
4	σ_x	σ_y	σ_y	σ_x	-4	σ_x	σ_z	σ_z	σ_x
4	σ_y	$\mathbb{I}$	$\mathbb{I}$	σ_y	4	σ_y	σ_x	σ_x	σ_y
-4	σ_y	σ_y	σ_y	σ_y	4	σ_y	σ_z	σ_z	σ_y
-4	σ_z	$\mathbb{I}$	$\mathbb{I}$	σ_z	-4	σ_z	σ_x	σ_x	σ_z
4	σ_z	σ_y	σ_y	σ_z	-4	σ_z	σ_z	σ_z	σ_z

4.2. Multipartite channels

Up to now, we have studied the effectiveness of the proposed detection methods and compare their performance for several typical bipartite channels. Here we consider the entanglement of multipartite quantum channels, and use stabilizer witness to detect multipartite channels consisting of noisy CZ gates [48]. We consider a special circuit with two CZ gates in figure 3(a) and the results could be similarly generalized to larger circuits with more CZ gates. Using lemma 3, the Choi state $\Phi_{AA_1BB_1CC_1}$ of the circuit could be mapped to a graph state $|G_{\mathrm{CZ}}\rangle$ (see figure 3(b)) with white noise on local system [49]. Therefore, we can use stabilizer witness to detect the entanglement of the graph state $|G_{\mathrm{CZ}}\rangle$ to study the entanglement of the noisy CZ multipartite channel [50]. The elements of the stabilizer for

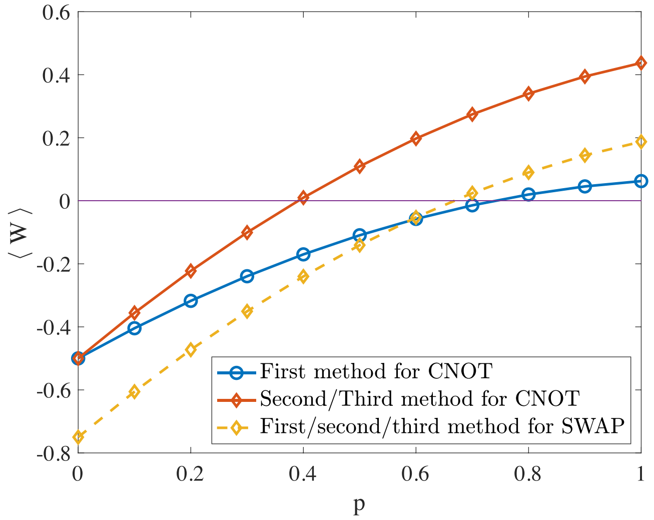

Figure 2. Entanglement witness values for the noisy CNOT and SWAP gates as a function of the noise parameter p. Since the witnesses have different normalization, we only plot different witness values under normalization. In particular, for the noisy CNOT gate, the second and third witnesses have the same value after proper normalization. For the noisy SWAP gate, they all have the same value after normalization.

graph state $|G_{CZ}\rangle$ are products of the operators

$$\begin{aligned} &g_1^{(|G_{CZ}\rangle)} := X_1Z_2,\ g_2^{(|G_{CZ}\rangle)} := Z_1X_2Z_4,\\ &g_3^{(|G_{CZ}\rangle)} := X_3Z_4,\ g_4^{(|G_{CZ}\rangle)} := Z_2Z_3X_4Z_6,\\ &g_5^{(|G_{CZ}\rangle)} := X_5Z_6,\ g_6^{(|G_{CZ}\rangle)} := Z_4Z_5X_6. \end{aligned} \quad (29)$$

Then, we can consider the following stabilizer witness

$$W_{|G_{CZ}\rangle} := 3\mathbb{I} - 2\left[\prod_{k=1,4,5} \frac{g_k^{(|G_{CZ}\rangle)} + \mathbb{I}}{2} + \prod_{k=2,3,6} \frac{g_k^{(|G_{CZ}\rangle)} + \mathbb{I}}{2}\right] \quad (30)$$

for the graph state to detect the genuine entanglement of the channel. We can similarly convert the witness into a quantum game according to section 3.3. In particular, we apply the Hadamard gate on qubit 2, 4, 6 and map the Pauli measurements on qubit 1, 3, 5 as input states. The entanglement of the noisy CZ multipartite channel is shown in figure 4. We can clearly observe the existence of genuine entanglement whenever $p < 0.2$. We note that the method could be extended to quantum circuits consisting of multiple CZ gates. In the general case, we can similarly construct the stabilizer witness.

5. Conclusion

In this work, we have studied entanglement detection of quantum channels. By relating the channel entanglement to the entanglement of the corresponding Choi state, we exploit state entanglement detection methods for witnessing channel entanglement. Using the language of quantum games, we can further convert the witness operator as a quantum game of the channel. Based on the general result, we proposed three methods to construct witnesses that allow to detect entanglement of bipartite channels and compare their performance for noisy CNOT and SWAP channels. From these results, the three methods could become convenient tools for routine performance detection of bipartite quantum channels. We also introduced the definition of mulitipartite channel entanglement, noting that Choi state of quantum channels with CZ gates correspond to graph states. We can then use graph state witnesses to detect multipartite circuits consisting of CZ gates.

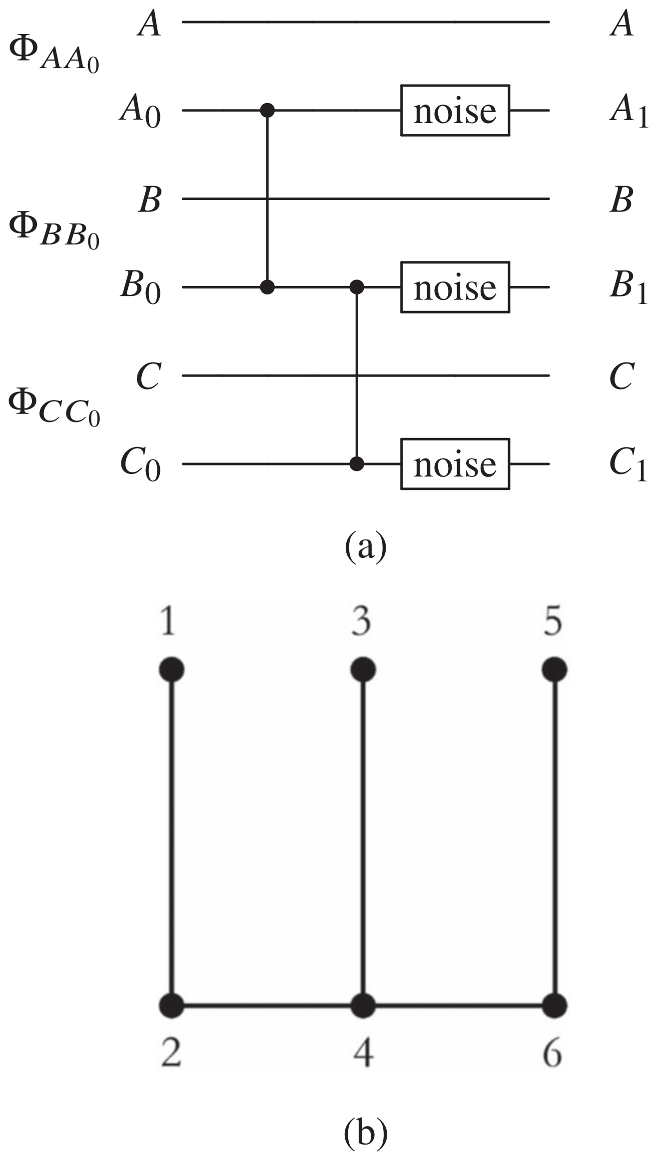

Figure 3. (a) An example quantum circuit consisting of two CZ gates with local depolarizing noise. The Choi state of (a) is equivalent to (b) the graph state $|G_{CZ}\rangle$ up to local unitary rotation.

Acknowledgments

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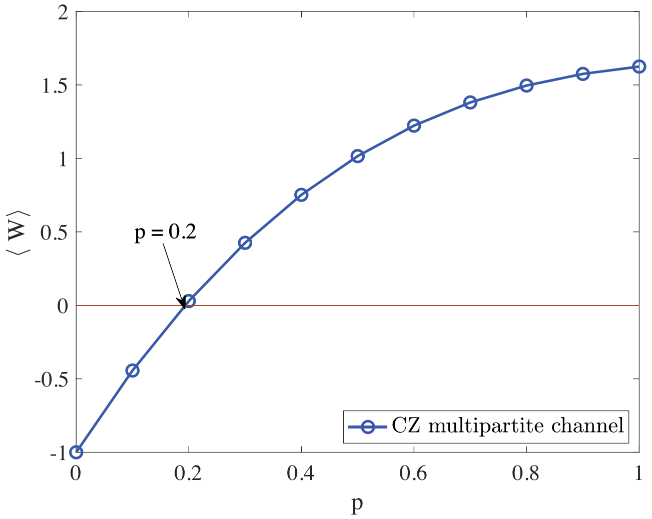

Figure 4. Stabilizer witness value $\langle W_{|G_{CZ}\rangle}\rangle$ as a function about the noise parameter p.

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Appendix A. Proof of lemma 1

Here we prove that a bipartite channel is separable if its Choi state is separable. First, let $\mathcal{N}\in \mathrm{CPTP}(A_0B_0\to A_1B_1)$ be a bipartite channel. If $\mathcal{N}$ is separable, it can be expressed as

$$\mathcal{N}(\rho_{A_0B_0})=\sum_i (X^i_{A_0}\otimes Y^i_{B_0})\rho_{A_0B_0}(X^i_{A_0}\otimes Y^i_{B_0})^\dagger, \tag{A1}$$

where $\sum_i (X^i_{A_0}\otimes Y^i_{B_0})^\dagger (X^i_{A_0}\otimes Y^i_{B_0})=I_{A_0B_0}$. We prove the two directions using the following two lemmas.

Lemma 4. *For a bipartite channel $\mathcal{N}\in \mathrm{CPTP}(A_0B_0\to A_1B_1)$, its Choi state is separable if $\mathcal{N}$ is a separable channel.*

Proof. If a bipartite channel $\mathcal{N}\in \mathrm{CPTP}(A_0B_0\to A_1B_1)$ is separable. Then, we have

$$\begin{aligned}
&\mathcal{N}(\Phi_{AA_0}\otimes\Phi_{BB_0})\\
&=\sum_i (X^i_{A_0}\otimes Y^i_{B_0})\Phi_{AA_0}\otimes\Phi_{BB_0}(X^i_{A_0}\otimes Y^i_{B_0})^\dagger\\
&=\sum_i (X^i_{A_0}\Phi_{AA_0}\otimes Y^i_{B_0}\Phi_{BB_0})(X^i_{A_0}\otimes Y^i_{B_0})^\dagger\\
&=\sum_i (X^i_{A_0}\Phi_{AA_0}(X^i_{A_0})^\dagger)\otimes(Y^i_{B_0}\Phi_{BB_0}(Y^i_{B_0})^\dagger).
\end{aligned} \tag{A2}$$

So Choi state of the bipartite channel $\mathcal{N}$ is separable. □

If Choi state $\Phi_{AA_1BB_1}$ of a bipartite channel $\mathcal{N}$ is separable, it can be expressed as

$$\Phi_{AA_1BB_1}=\sum_i p_i\,\varrho_i^{AA_1}\otimes\varrho_i^{BB_1}. \tag{A3}$$

we assume that $\varrho_i^{AA_1}$ and $\varrho_i^{BB_1}$ are pure states. Hence

$$\begin{aligned}
\varrho_i^{AA_1}&=|\psi_i\rangle\langle\psi_i|_{AA_1},\\
\varrho_i^{BB_1}&=|\psi_i\rangle\langle\psi_i|_{BB_1},
\end{aligned} \tag{A4}$$

where $|\psi_i\rangle_{AA_1}=\sum_j\lambda_{ij}\,|\alpha_{ij}\rangle_A|\beta_{ij}\rangle_{A_1}$ and we can define $|\psi_i\rangle_{AA_1}$ similarly.

Lemma 5. *For a given bipartite channel $\mathcal{N}\in \mathrm{CPTP}(A_0B_0\to A_1B_1)$. $\mathcal{N}$ is separable if its Choi state is separable.*

Proof. We first assume that the input state is a product state, i.e. $\rho_{AB}=\rho_A\otimes\rho_B$. According to the Choi–Jamiolkowski isomorphism, we have

$$\begin{aligned}
\mathcal{N}(\rho_{AB})&=\mathrm{Tr}_{\mathrm{A\,B}}[\Phi_{AA_1BB_1}(\rho^{\mathrm{T}}_{AB}\otimes\mathcal{I}_{A_1B_1})]\\
&=\mathrm{Tr}_{\mathrm{A\,B}}\Big[\textstyle\sum_i p_i\,\varrho_i^{AA_1}\otimes\varrho_i^{BB_1}(\varrho_A^{\mathrm{T}}\otimes\varrho_B^{\mathrm{T}}\otimes\mathcal{I}_{A_1B_1})\Big]\\
&=\textstyle\sum_i p_i\,\mathrm{Tr}_{\mathrm{A\,B}}[(\varrho_i^{AA_1}\varrho_A^{\mathrm{T}})\otimes(\varrho_i^{BB_1}\varrho_B^{\mathrm{T}})]\\
&=\textstyle\sum_i p_i\,\mathrm{Tr}_{\mathrm{A}}[\varrho_i^{AA_1}\varrho_A^{\mathrm{T}}]\otimes\mathrm{Tr}_{\mathrm{B}}[\varrho_i^{BB_1}\varrho_B^{\mathrm{T}}].
\end{aligned} \tag{A5}$$

Note that $\mathrm{Tr}_{\mathrm{A}}[\varrho_i^{AA_1}\varrho_A^{\mathrm{T}}]$ can be written as

$$\begin{aligned}
&\mathrm{Tr}_{\mathrm{A}}[\varrho_i^{AA_1}\varrho_A^{\mathrm{T}}]=\mathrm{Tr}_{\mathrm{A}}[|\psi_i\rangle\langle\psi_i|_{AA_1}\varrho_A^{\mathrm{T}}]\\
&=\textstyle\sum_{jj'}\lambda_{ij}\lambda^*_{ij'}\mathrm{Tr}_{\mathrm{A}}[|\alpha_{ij}\rangle\langle\alpha_{ij'}|_A\varrho_A^{\mathrm{T}}]|\beta_{ij}\rangle\langle\beta_{ij'}|_{A_1}\\
&=\textstyle\sum_{jj'}\lambda_{ij}\lambda^*_{ij'}\langle\alpha_{ij'}|\,\varrho_A^{\mathrm{T}}\,|\alpha_{ij}\rangle_A|\beta_{ij}\rangle\langle\beta_{ij'}|_{A_1}\\
&=\textstyle\sum_{jj'}\lambda_{ij}\lambda^*_{ij'}\varrho^A_{ijij'}|\beta_{ij}\rangle\langle\beta_{ij'}|_{A_1},
\end{aligned} \tag{A6}$$

where $\langle\alpha_{ij'}|\,\varrho_A^{\mathrm{T}}\,|\alpha_{ij}\rangle_A=\varrho^A_{ijij'}$. We denote that $X^i_A=\sum_j\lambda_{ij}|\beta_{ij}\rangle\langle\alpha_{ij}|$ and $(X^i_A)^+=\sum_{j'}\lambda^*_{ij'}|\alpha_{ij'}\rangle\langle\beta_{ij'}|$. Hence

$$\begin{aligned}
X^i_A\varrho(X^i_A)^\dagger&=\sum_{jj'}\lambda_{ij}\lambda^*_{ij'}|\beta_{ij}\rangle\langle\alpha_{ij}|\,\varrho|\alpha_{ij'}\rangle\langle\beta_{ij'}|\\
&=\sum_{jj'}\lambda_{ij}\lambda^*_{ij'}\varrho_{ijij'}|\beta_{ij}\rangle\langle\beta_{ij'}|,
\end{aligned} \tag{A7}$$

where $\langle\alpha_{ij}|\,\varrho|\alpha_{ij'}\rangle=\varrho_{ijij'}$. Therefore we have $\mathrm{Tr}_{\mathrm{A}}[\varrho_i^{AA_1}\varrho_A^{\mathrm{T}}]=X^i_A\varrho(X^i_A)^\dagger$ and similarly $\mathrm{Tr}_{\mathrm{B}}[\varrho_i^{BB_1}\varrho_B^{\mathrm{T}}]=X^i_B\varrho(X^i_B)^\dagger$ when we define X^i_B accordingly. Finally we obtain

$$\begin{aligned}
\mathcal{N}(\rho_{AB})&=\sum_i p_i(X^i_A\rho_A(X^i_A)^\dagger)\otimes(X^i_B\rho_B(X^i_B)^\dagger)\\
&=\sum_i p_i(X^i_A\otimes X^i_B)\rho_{AB}(X^i_A\otimes X^i_B)^\dagger.
\end{aligned} \tag{A8}$$

For a general input state, we can always decompose it as a linear combination of product states (possibly with negative coefficients). Together with the linearity of the above equation, it is not hard to see that it also hold for any bipartite quantum state. □

Appendix B. Sketch proof of lemma 2

The proof of lemma 2 follows naturally with the help lemma 1. First, we show that a separable multipartite channel is equivalent to a separable Choi state. In this case, we divide the parties into two partitions and regard the system as a bipartite one. Then we can exploit lemma 1 to prove the equivalence. The equivalence for genuine entanglement follow naturally.

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