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Gravity-induced geometric spin Hall effect of freely falling quantum particle

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Abstract
We discuss a new gravitational effect that the wave packet of a free-fall quantum particle undergoes a spin-dependent transverse shift in Earth's gravitational field. This effect is similar to the geometric spin Hall effect (GSHE) (Aiello 2009 *et al Phys. Rev. Lett.* **103** 100401), and can be called gravity-induced GSHE. This effect suggests that the free-fall wave packets of opposite spin-polarized quantum particles can be split in the direction perpendicular to spin and gravity.

Keywords: gravity-induced geometric spin Hall effect, quantum particle, universality of free fall

(Some figures may appear in colour only in the online journal)

1. Introduction

The universality of free fall (UFF) is tested as a weak form of the Einstein equivalence principle, which is the most important guide when establishing Einstein's general relativity. The classical tests of UFF with macroscopic masses have achieved a high precision quantified by the Eötvös parameter η of about 10^{-13} [1, 2], and no violations were observed so far. To extend the domain of the test body, verifications of UFF based on microscopic particles in the quantum regime have been studied theoretically and experimentally since 1960s [3]. Recently, WEP-test experiments using atom interferometers were proposed to reach the level of $\eta \sim 10^{-15}$ [4, 5]. Quantum systems are advantageous in testing WEP with regard to fundamental properties such as charge [6], matter/anti-matter [7–9], spin [10–12] and internal structures. Possible violations of equivalence principle were discussed extensively, such as by spin-gravity coupling [13–16], by spin–torsion coupling [17–19], in extended or modified theories of gravity, and in almost all tentative theories to unify general relativity and the standard model of particle physics [20, 21].

Theoretical investigations have offered a wide variety of approaches to the WEP in the quantum domain [22–30]. However, quantum particles differ critically from classical point-like particles in many respects, because of their wave-like features and inherent spacial extension. Even the notion of WEP for quantum systems is not very clear and it may be different from the conventional WEP for classical systems [31–38]. In this paper, considering wave-like features and inherent spacial extension of quantum particles, we reveal an interesting phenomenon that the space-averaged free-fall point of quantum particles allows a spin-dependent transverse split in the gravitational field. Since such an effect is similar to the geometric spin Hall effect (GSHE) discussed in [39, 40], we call it gravity-induced GSHE.

For a light beam, the GSHE states that a spin-dependent transverse displacement of the light intensity centroid is observed in a plane tilted with respect to the propagation direction. Unlike the conventional spin Hall effect of light as a result of light-matter interaction [41–43], the GSHE of light is of purely geometric nature. Besides, it is distinct from the Relativistic Hall effect [44] and the Wigner translation of electromagnetic beams [45] both as an effect of Lorentz-boost-induced sideways shift of the energy-flux and energy-density centroids. Analogously, the gravity-induced GSHE reported here also differs from the so-called gravitational Hall effect presented in the literature [46–49] which describes a helicity-dependent geodesic deviation correction.

In this paper, we discuss the gravity-induced GSHE for spin-polarized Dirac particle beams. The paper is organized as follows. First, we derive an approximate wave-packet solution of the covariant Dirac equation in the Newtonian limit. With this solution, we demonstrate the gravity-induced GSHE of freely-falling Dirac particles. Next, we present an alternative derivation of the gravity-induced GSHE in a simple method without employing detailed knowledge of the

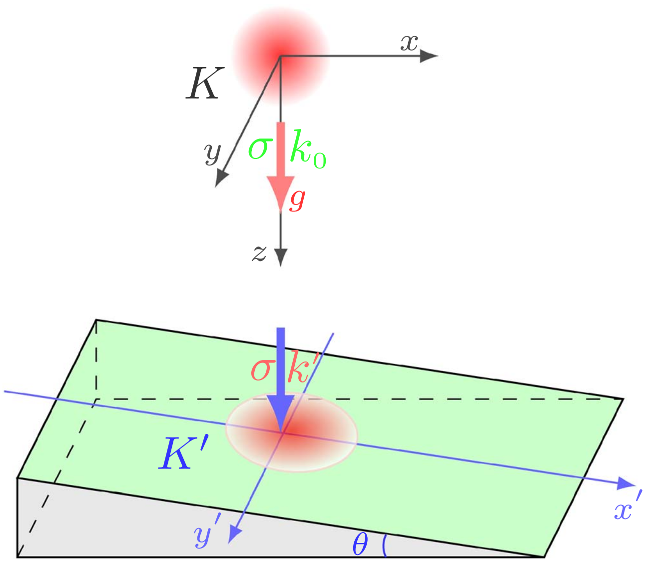

Figure 1. A schematic set-up to display gravity-induced GSHE. Dirac particles carrying spin and initial momentum along z axis are released to fall freely and hit a detection plane x'-y' tilted by an angle θ with respect to the horizontal plane. The detection and beam frames are denoted by K' and K, respectively. Just as an example, the spin orientation marked here is along the positive direction of z axis and $\sigma = 1/2$.

wave-function of the Dirac particle beam. Then, we go further to discuss another interesting configuration displaying gravity-induced GSHE. Finally, we give our conclusions.

2. Gravity-induced GSHE

We consider a simple system of the Dirac particle originally carrying spin and momentum $\hbar k_0$ along the vertical direction (z-axis) and falling freely from a height h_g towards a tilted detection plane [see figure 1]. The particle's space-averaged point of free fall will be shifted along the y axis by an amount $\delta \sim (\lambda_g/4\pi)\sigma \tan\theta$ compared to its classical counterpart's. Here $\sigma = \pm 1/2$ is the initial spin polarization of the particle along the positive or negative direction of z-axis, and λ_g the *de Broglie* wavelength of the particle when it hitting the detector, with $\lambda_g \sim 2\pi\hbar/p_g$ where $p_g \sim \hbar k_0 + m\sqrt{2gh_g}$. Moreover, if $g \to 0$, the ordinary GSHE of Dirac particles is recovered. The displacement δ is an order of magnitude smaller than λ_g. This effect implies that Dirac or quantum particles with different spin orientations follow 'different paths'.

2.1. Heuristic result from GSHE

To calculate the space-averaged free-fall point of the Dirac particle, what we need to know is the spatial distribution of the particle beam's intensity in the detection plane. Following the method of Aiello *et al* [39], we choose the energy flux of the particle beam to represent its intensity and so consider the energy–momentum (E–M) tensor $T^{\mu\nu}$ of the particle beam. Thus, the space-averaged free-fall point of the particle can be calculated as the barycenter of the energy flux T'^{z0} across the tilted detection plane:

$$\langle y\rangle_g = \int y'\, T'^{z0}\, dx'dy' \Big/ \int T'^{z0}\, dx'dy'. \tag{1}$$

Note that the energy flux T'^{z0} is defined in the detection frame and not in the beam frame.

Before going into the detailed calculation, we first explain a heuristic way of understanding the gravity-induced GSHE, by making a close connection to the ordinary GSHE, which originates from a non-zero spin projection in the detection plane and has no relevance to gravity. To deal with the gravitational interactions, we follow [50–52] and adopt two reasonable approximations. First, the spin-precession effect can be safely neglected in our case from the result of gravity probe B (GP-B) [53] experiment. Second, the motion of the particle wave-packet's center can be approximately replaced by its classical trajectory. With these two approximations, the gravity-induced GSHE can be converted to a GSHE: gravity just induces a kinematic configuration that the particle can hit the detector with non-zero spin projection in the detection plane, and then the ordinary GSHE occurs. Considering the particles as set up in figure 1, we can directly quote the expression for ordinary GSHE as derived in [39], and write down our result for the gravity-induced GSHE:

$$\langle y\rangle_g = \frac{\lambda_g}{4\pi}\sigma\tan\theta. \tag{2}$$

Here $\lambda_g = 2\pi\hbar/p_g$ is the *de Broglie* wavelength with the momentum p_g when the particle arriving at the detection plane. For a non-relativistic particle and in the Newton's gravitation limit, we have $p_g \sim \hbar k_0 + m\sqrt{2gh_g}$. Strictly speaking, equation (2) is only the leading order effect. Since this effect is pretty small, throughout this paper we omit the discussion of high-order corrections such as from the angular spread of the beam.

2.2. Deriving Gravity-induced GSHE with wave-function

To convince the reader that our heuristic argument gives the correct result, we now make an explicit calculation of the particle's motion in the gravitational field. The dynamics of the particle in a gravitational field should be described by the covariate Dirac equation in a curved spacetime [54] ($\hbar = c = 1$):

$$(\mathrm{i}\gamma^a D_a - m)\psi = 0. \tag{3}$$

Here $\gamma^a (a = 0,\ldots,3)$ is the flat Dirac matrix, defined by $\gamma^a = e_\mu{}^a\gamma^\mu$ in the local tetrad frame. Hereafter the latin indices denote flat indices and the greek indices curved indices. The tetrad field can be defined by $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, and we adopt the flat metric $\eta_{ab} = \mathrm{diag}(+,-,-,-)$. D_a is the covariant derivative for the spinor field:

$$D_a = e_a^{\ \mu} D_\mu, \quad D_\mu = \partial_\mu - \frac{\mathrm{i}}{2}\omega_\mu{}^{ab} S_{ab},$$

where $S_{ab} = \mathrm{i}[\gamma_a, \gamma_b]/4$ and $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ is the spin connection which also can be expressed as

$$\omega_\mu{}^{ab} = e_\nu{}^a e^{\lambda b}\Gamma^\nu_{\mu\lambda} - e^{\lambda b}\partial_\mu e_\lambda{}^a,$$

in terms of the affine connection $\Gamma^\nu_{\mu\lambda}$ and the tetrad field.

Throughout this paper, we choose the Dirac representation of gamma matrices:

$$\gamma^{\hat{0}}=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\quad \gamma^{\hat{j}}=\begin{pmatrix} 0 & \sigma^{j} \\ -\sigma^{j} & 0 \end{pmatrix},\ j=1,2,3.$$

Hereafter, hatted specific indices denote flat indices and σ^j is the Pauli matrices.

We consider Earth's gravitational field near it's surface in the Newtonian limit, which is described by the metric (see, e.g. [47, 55–57])

$$\mathrm{d}s^2=A^2(z)\mathrm{d}t^2-\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r},\ A(z)=1+\zeta,\ \zeta=-gz. \tag{4}$$

Here we choose the gravitational acceleration g along the positive direction of z-axis, and the physical domain is limited by the scales $L_g=c^2/g$. This metric describes approximately the gravitational field in a small region on Earth's surface and so $|gz/c^2|=|z/L_g|\ll 1$ is necessary.

With the metric, the tetrads and their inverse are

$$e_{\mu}^{\ a}=\begin{pmatrix} A & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},\qquad e^{\nu}_{\ b}=\begin{pmatrix} A^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Using the zero-torsion condition [58] or constructing the affine connection by the so-called Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}=g^{\lambda\sigma}(\partial_\mu g_{\nu\sigma}+\partial_\nu g_{\sigma\mu}-\partial_\sigma g_{\mu\nu})/2$ from the metric, one can derive the non-zero components of the spin connection $\omega_\mu^{\ ab}$:

$$\omega_0^{\ \hat{3}\hat{0}}=-\omega_0^{\ \hat{0}\hat{3}}=A'=-g.$$

Here the prime represents differentiation with respect to z. The covariant derivatives are thus written in explicit forms as

$$D_{\hat{j}}=e_{\hat{j}}^{\ \mu}D_\mu=D_j=\partial_j,$$
$$D_{\hat{0}}=e_{\hat{0}}^{\ \mu}D_\mu=A^{-1}D_0=A^{-1}(\partial_t+A'\gamma^{\hat{0}}\gamma^{\hat{3}}/2).$$

Now equation (3) can be changed into [59, 60]

$$(\mathrm{i}\gamma^{\hat{0}}A^{-1}\partial_t+\mathrm{i}\gamma^{\hat{j}}\partial_j+\frac{\mathrm{i}}{2}\gamma^{\hat{3}}A^{-1}A'-m)\psi=0. \tag{5}$$

It can be further rewritten in a Schrödinger-like form

$$\mathrm{i}\partial_t\psi=A(-\mathrm{i}\alpha^{\hat{j}}\partial_j-\mathrm{i}\frac{A'}{2A}\alpha^{\hat{3}}+\beta m)\psi=H\psi, \tag{6}$$

where $\alpha^{\hat{j}}=\gamma^{\hat{0}}\gamma^{\hat{j}}$ and $\beta=\gamma^{\hat{0}}$. The Hamiltonian H also can be written in an explicitly Hermitian form with the free-spacetime Hamiltonian $H_0=-\mathrm{i}\alpha^{\hat{j}}\partial_j+\beta m$:

$$H=\frac{1}{2}(AH_0+H_0A)=\frac{1}{2}\{H_0,A\}. \tag{7}$$

We turn now to the calculation of energy flux density. For the Dirac field, the familiar symmetric E–M tensor[3] [61] is

$$T_{ab}=\frac{\mathrm{i}}{4}\overline{\psi}(\gamma_a D_b+\gamma_b D_a)\psi+\mathrm{h.c.} \tag{8}$$

Here + h. c. indicates the addition of the Hermitian conjugate of the foregoing terms Recalling that $D_{\hat{j}}=\partial_j$ and $D_{\hat{0}}=A^{-1}(\partial_t+A'\alpha^{\hat{3}}/2)$, we can get the energy flux density

$$\begin{aligned} T^{j0}=e^{j\hat{a}}e^{0\hat{b}}T_{ab}&=\frac{1}{4A\mathrm{i}}\overline{\psi}(\gamma_{\hat{0}}D_{\hat{j}}+\gamma_{\hat{j}}D_{\hat{0}})\psi+\mathrm{h.c.}\\ &=\frac{1}{4A\mathrm{i}}\psi^\dagger\left(\partial_j-\frac{1}{A}\alpha^{\hat{j}}\partial_t-\frac{A'}{2A}\alpha^{\hat{j}}\alpha^{\hat{3}}\right)\psi+\mathrm{h.c.} \end{aligned} \tag{9}$$

To eliminate the time derivative in equation (9) by using equation (6), we obtain

$$T^{j0}=\frac{1}{2A\mathrm{i}}\{[\psi^\dagger\partial_j\psi-(\partial_j\psi^\dagger)\psi]+\mathrm{i}\varepsilon_{jkl}\partial_k(\psi^\dagger\Sigma^{\hat{l}}\psi)\}, \tag{10}$$

where $\Sigma^{\hat{k}}=-\mathrm{i}\varepsilon_{ijk}\alpha^{\hat{i}}\alpha^{\hat{j}}/4$.

As is known, the formal solution of equation (6) is given by

$$\psi(\boldsymbol{x},t)=\mathrm{e}^{-\mathrm{i}Ht}\psi_0, \tag{11}$$

as the initial wave function ψ_0 is known. For instance, consider a gaussian packet of half width d [62]:

$$\psi_0^{1,2}=\varphi(\boldsymbol{x})\mathcal{W}_0^{1,2},\quad \varphi(\boldsymbol{x})=\mathcal{N}\mathrm{e}^{-\frac{x^2+y^2}{2\mathrm{d}^2}}\mathrm{e}^{\mathrm{i}k_0z}, \tag{12}$$

where $\mathcal{N}$ is a normalization constant. It describes an particle of spin up or down with the expectation value of the initial momentum $\langle p\rangle_0=\hbar k_0$ along z axis for $\mathcal{W}_0^1=\{1,0,0,0\}^{\mathrm{T}}$ or $\mathcal{W}_0^2=\{0,1,0,0\}^{\mathrm{T}}$ in the non-relativistic limit ($\mathrm{d}\sim 1/k_0\gg 1/m$). Expanding the exponential operation in equation (11) to first order for small g, the approximation of the wave function $\psi(x,t)$ is as follows

$$\psi(\boldsymbol{x},t)\simeq\left(1-\mathrm{i}\zeta H_0t-\frac{gt^2}{2}\partial_z+\frac{gt}{2}\alpha^{\hat{3}}\right)\mathrm{e}^{-\mathrm{i}H_0t}\psi_0. \tag{13}$$

For the detailed calculations of the above results, see the Appendix. Using the non-relativistic limit $\mathrm{d}\sim 1/k_0\gg 1/m$, we have

$$\begin{aligned}(-\mathrm{i}H_0t)^2\psi_0&=(\partial_x^2+\partial_y^2+\partial_z^2-m^2)t^2\psi_0\\ &\simeq-(k_0^2+m^2)t^2\psi_0=-\omega^2t^2\psi_0.\end{aligned} \tag{14}$$

With equation (14) in hand, it is possible to conveniently express the part of exponential operator in equation (13) in the form

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}H_0t}\psi_0&=\sum_{n=0}^{\infty}\frac{(-\mathrm{i}H_0t)^n}{n!}\psi_0\\ &\simeq\sum_{k=0}^{\infty}\left[\frac{(-\omega^2t^2)^k}{(2k)!}-\frac{(-\omega^2t^2)^k}{(2k+1)!}\mathrm{i}H_0t\right]\psi_0\\ &=\left[\cos(\omega t)-\frac{\sin(\omega t)}{\omega t}\mathrm{i}H_0t\right]\psi_0.\end{aligned} \tag{15}$$

Applying equations (13)–(15) and the non-relativistic limit $\omega\sim m\gg k_0\sim 1/d$ and neglecting the high-order terms of g, we can obtain the wave functions $\psi^1(\boldsymbol{x},t)$ and $\psi^2(\boldsymbol{x},t)$ of spin-

[3] It should be noted that the E–M tensors have various versions, e.g. the canonical one and symmetric one. It may be tricky to pick out a proper one in actual application. Fortunately, the use of different E–M tensors does not change qualitatively the key features of the GSHE.

up and spin-down particles, respectively

$$\psi^1 \simeq \begin{pmatrix} (1+\mathrm{i}mgtz-\mathrm{i}gt^2k_0/2)\mathrm{e}^{-\mathrm{i}mt} \\ 0 \\ gt\mathrm{e}^{-\mathrm{i}mt}/2+\mathrm{i}gtzk_0\cos(mt) \\ -gtz(x+\mathrm{i}y)\cos(mt)/d^2 \end{pmatrix}\varphi(\boldsymbol{x}), \tag{16}$$

$$\psi^2 \simeq \begin{pmatrix} 0 \\ (1+\mathrm{i}mgtz-\mathrm{i}gt^2k_0/2)\mathrm{e}^{-\mathrm{i}mt} \\ -gtz(x-\mathrm{i}y)\cos(mt)/d^2 \\ -gt\mathrm{e}^{-\mathrm{i}mt}/2-\mathrm{i}gtzk_0\cos(mt) \end{pmatrix}\varphi(\boldsymbol{x}). \tag{17}$$

Plugging equations (16) and (17) into the energy flux density (10) and computing the first-order approximation with respect to g, we finally obtain

$$T^{x0}(\boldsymbol{x},t) = -\frac{\sigma y}{d^2}(1+gz)|\varphi|^2, \tag{18}$$

$$T^{z0}(\boldsymbol{x},t) = [(1+gz)k_0+mgt]|\varphi|^2, \tag{19}$$

where $\sigma=\pm 1/2$ for spin-up and -down particles. Returning to equation (1), we now should notice the connection between the beam and detector frames (K and K' frames). These two frames are connected by $x'^a = \Lambda^a_{\ b}(x^b-\epsilon^b)$, with the constant vector $\epsilon^a=(0,0,0,h_g)$ and the transformation matrix

$$\Lambda^a_{\ b} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{20}$$

Now we use the map $T'^{z0}(x') = \Lambda^z_{\ a}\Lambda^0_{\ b}T^{ab}(\Lambda^{-1}x'+\epsilon)$, and then have $T'^{z0}(x',y',0,t) = T^{z0}(x'\cos\theta, y', x'\sin\theta + h_g, t)\cos\theta - T^{x0}(x'\cos\theta, y', x'\sin\theta + h_g, t)\sin\theta$ on the detection plane $z'=0$ in K' frame. Thus, with equations (18) and (19) the resulting expression for the energy flux density is given by

$$\begin{aligned} T'^{z0}(x') = {} & ([1+g(x'\sin\theta+h_g)]k_0+mgt)|\varphi(x')|^2\cos\theta \\ & + \frac{\sigma y'}{d^2}[1+g(x'\sin\theta+h_g)]|\varphi(x')|^2\sin\theta, \end{aligned} \tag{21}$$

where $|\varphi(x')|^2 = \mathcal{N}^2\ \mathrm{Exp}[-(x'^2\cos^2\theta+y'^2)/d^2]$. Finally, inserting equation (21) into equation (1) and restoring explicit factors $\hbar$ and c leads to

$$\begin{aligned} \langle y\rangle_g &= \frac{\hbar\sigma\tan\theta}{2[\hbar k_0+mgt/(1+gh_g/c^2)]} \\ &\simeq \frac{\hbar\sigma\tan\theta}{2(\hbar k_0+mgt)} = \frac{\lambda_g}{4\pi}\sigma\tan\theta. \end{aligned} \tag{22}$$

Here $gh_g/c^2 \ll 1$ and so $\lambda_g = 2\pi\hbar/p_g$ with $p_g=\hbar k_0+mgt$. Notice that when the particles hit the detector $h_g = gt^2/2$ and $p_g = \hbar k_0 + mgt = \hbar k_0 + m\sqrt{2gh_g}$ in the weak-field approximation. We therefore show the previous result of equation (2) and confirm the heuristic interpretation.

A few parenthetical remarks are in order. First, this effect is purely the result of matter-wave effect of quantum particle because it would be vanished as $\hbar\to 0$. However, even in flat spacetime $g\to 0$, it just gives rise to the ordinary GSHE of the particle and of course does not completely disappear. Second, although this effect originates from the wave feature of quantum particle, the split is not due to the geodesic deviation out of the inherent spacial extension of wave, because the above derivation is acted in a uniform gravitational field and the tidal effect does not exist. Thus, this effect differs radically from the gravitational Hall effect [46–49] out of a helicity-dependent geodesic deviation correction and as an genuine gravitational effect.

2.3. Deriving Gravity-induced GSHE without wave-function

The *simple* spin-dependent result of equation (22) obtained by a lengthy calculation is not accidental. In fact, assuming the wave-function with cylindrical symmetry, we can derive the previous result by a clever method without more detailed knowledge of the particle wave-function. From equation (21), equation (1) can be re-expressed by

$$\begin{aligned} \langle y\rangle_g &= \frac{\int y'[T^{z0}(x)\cos\theta - T^{x0}(x)\sin\theta]dx'dy'}{\int [T^{z0}(x)\cos\theta - T^{x0}(x)\sin\theta]\mathrm{d}x'\mathrm{d}y'} \\ &= \frac{\int y[T^{z0}(x)\cos\theta - T^{x0}(x)\sin\theta]\mathrm{d}x\mathrm{d}y}{\int [T^{z0}(x)\cos\theta - T^{x0}(x)\sin\theta]\mathrm{d}x\mathrm{d}y}. \end{aligned} \tag{23}$$

In the last step, expressing the area element from K' frame to K frame does not change the main result. It might be more convenient to deduce the previous result again in K frame.

One use of the symmetric E–M tensor is to construct a conserved angular momentum tensor:

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}. \tag{24}$$

In K frame, T^{x0} can be ignored compared to T^{z0} because the beam mainly carries energy along the propagation direction. Thus, the T^{x0} term can be ignored for a small tilted angle θ in the denominator of equation (23), and the remaining T^{z0} term can be computed via the sum rule of energy:

$$\int T^{z0}\,\mathrm{d}x\mathrm{d}y = K_g^z \simeq n\varepsilon_g, \tag{25}$$

where n is the particle number per unit time across the plane x-y, namely the particle number flux. Thus, $\mathcal{K}_g^z$ denotes the energy per unit time across the plane x-y. Notice that the particle's energy $\varepsilon_g = (m^2+p_g^{\,2})^{1/2} \simeq m$ in the non-relativistic limit. Due to the axial symmetry of the beam around its beam axis (z-axis), T^{z0} should be even function of x and y. Thus, the integral of $yT^{z0}\cos\theta$ would vanish and only the integral of $yT^{x0}\sin\theta$ was left in the numerator of equation (23). Again, due to the axial symmetry of the beam, we have an angular momentum sum rule as follows

$$\begin{aligned} \int -yT^{x0}\mathrm{d}x\mathrm{d}y &= \int xT^{y0}\mathrm{d}x\mathrm{d}y = \int \frac{xT^{y0}-yT^{x0}}{2}\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\int M^{012}\mathrm{d}x\mathrm{d}y = \frac{1}{2}N\sigma\hbar. \end{aligned} \tag{26}$$

Here N is the particle number per unit length along the direction of propagation and we have $n = v_g N = Np_g/\varepsilon_g$, where v_g is the particle speed when hitting the detector.

Substituting equations (25) and (26) into equation (23), we can verify again that the barycenter of a spin-polarized beam's energy flux

$$\langle y \rangle_g = \frac{N\sigma\hbar\sin\theta}{2n\varepsilon_g\cos\theta} = \frac{\lambda_g}{4\pi}\sigma\tan\theta. \tag{27}$$

We can offer a simple explanation of equation (27). For equation (1), the denominator is related to the component of the particle's energy flux or momentum normal to the detection plane, i.e. the longitudinal energy flux or momentum $p_g\cos\theta$; the numerator corresponds to the projection of the particle's spin in the detection plane, i.e. the transverse spin angular momentum $\sigma\hbar\sin\theta$. Hence, we get immediately the result $\langle y \rangle_g \propto (\lambda_g/2\pi)\sigma\tan\theta$. This means that the particles display spin-dependent pattern in the tilted detection plane and their barycenter of the free-fall point changes with their spin orientations. Additionally, in comparison with the classical particle, the quantum particle with spin polarization is able to fall freely in a different 'path structure'.

2.4. Another configuration of Gravity-induced GSHE

There is another possible configuration of gravity-induced GSHE. Figure 2 depicts the configuration that the particles with horizontal spin polarization as well as momentum along the longitudinal direction are released into free fall. Interestingly, the detection plane x'-y' in K' frame is not tilted with respect to the x'-y' plane in K frame in this configuration. Indeed, the 'geometric' factor here originates from the configuration of transverse spin polarization that the spin orientation is initially perpendicular to the momentum. Repeating the above analysis and computation, we can get the result of the gravity-induced GSHE for figure 2:

$$\langle y \rangle_g = \frac{N\sigma\hbar}{2n\varepsilon_g} = \frac{\lambda_g}{4\pi}\sigma, \tag{28}$$

Here $\lambda_g \sim 2\pi\hbar/p_g$ with $p_g \sim \hbar k_0 + m\sqrt{2gh_g}$ again and $\sigma = \pm 1/2$ is the initial spin polarization of the particle along the positive or negative direction of x-axis. This effect suggests that the free-fall wave packets of opposite spin-polarized quantum particles can be split in the direction perpendicular to spin and gravity. More interestingly, unlike the GSHE of light reported previously and the configuration we discuss above, the gravity-induced GSHE in such configuration can be observed in a non-tilted beam-detector system.

Although the gravity-induced GSHE presented here is tiny, it could be detectable. May be the ultra-cold neutron beam and the neutron detector of high-resolution are suitable for the measurements of this effect. For instance, for an ultra-cold neutron beam with a neutron kinetic-energy of 41 neV or a neutron velocity of 1.0 m s^{-1}, its *de Broglie* wavelength is 400 nm [63]. If it is released with $h_g = 4$ cm, the final velocity $v_g \sim 1.3$ m/s and $\lambda_g \sim 300$ nm. In such case, we could rudely estimate the scale of the shift $\langle y \rangle_g \sim 12$ nm, which is possible tested by a neutron detector of high spatial resolution about 10 nm [64].

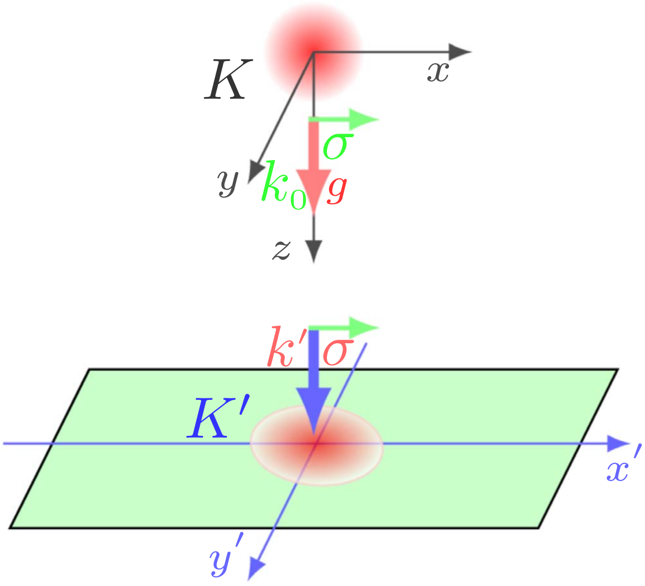

Figure 2. A sketch of particles initially carrying spin along the horizontal direction (x-axis) and momentum along the longitudinal direction (z-axis), and falling freely to a horizontal detection plane x'-y'. Just as an example, the spin orientation marked here is along the positive direction of x axis and $\sigma = 1/2$.

3. Discussion and summary

In conclusion, we revealed a nontrivial phenomenon called gravity-induced GSHE containing simultaneously quantum and gravitational effects. Though we considered an ideal metric describing a uniform gravitational field in a sufficiently small region, it does recover the Newtonian limit with neglecting terms of order $(gz/c^2)^2$ or higher. Such a new effect suggests that the 'free-fall points' of quantum particles (or matter waves) vary with their spin polarization. In comparison, the atom in the tests of UFF using atomic interferometers is also quantum matter, but treated as a classical point particle in the interaction with the gravitational field and only as matter wave in the interaction with the probing light pulse. Such special treatment is valid when the *de Broglie* wave-length of the atom is sufficiently small. This effect can be mainly interpreted as the result of the matter-wave effect of quantum particles in a gravitational field, seen from the following two respects: One can predict the ordinary GSHE of quatum particles or matter waves in the absence of gravitational field ($g \to 0$); and it is treated as an effect of quantum mechanics because it would vanish as $\hbar \to 0$.

The measurement of this effect will be of great interest and importance. As mentioned above, the displacement is an order of magnitude smaller than the *de Broglie* wavelength. Thus, this effect is extremely small for the spatial resolution of conventional detectors, considering the common particle beams such as electron, neutron and atom beams. However, the effect would perhaps be observed if the particle is extremely slow and the detector realizes a higher spatial resolution. To test the gravity-induced GSHE might be as a new probe of UFF of quantum particles, so as to clarify the notion of quantum WEP.

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Appendix : first-order approximation of exponential operator

For simplicity, we put $B = \mathrm{i}H_0 t$, $C = \{\zeta, B\}/2$ and $\zeta = -gz$, then $\mathrm{i}Ht = B + C$. Let us consider the expansion:

$$\mathrm{e}^{-\mathrm{i}Ht} = \sum_{n=0}^{\infty} \frac{1}{n!}(-B - C)^n \simeq \mathrm{e}^{-B} + F. \tag{A1}$$

It is rather complicated to expand the exponential function of two *noncommutative* operators B and C [65]. However, to the first order in g, we can arrive at

$$F = \sum_{n=1}^{\infty}\sum_{k=1}^{n} \frac{(-1)^n}{n!} B^{n-k} C B^{k-1}.$$

Noting $[B^2, \alpha^{\hat{3}}] = 0$, $\{B, \alpha^{\hat{3}}\} = 2t\partial_z$ and $C = \zeta B - gt\alpha^{\hat{3}}/2$ with $[B, \zeta] = -gt\alpha^{\hat{3}}$, we can derive

$$F = (-\zeta B - \frac{gt}{2} t\partial_z + \frac{gt}{2}\alpha^{\hat{3}})\mathrm{e}^{-B}, \tag{A2}$$

and

$$\mathrm{e}^{-\mathrm{i}Ht} \simeq (1 - \zeta B - \frac{gt}{2} t\partial_z + \frac{gt}{2}\alpha^{\hat{3}})\mathrm{e}^{-B}. \tag{A3}$$

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References

[1] Williams J G, Turyshev S G and Boggs D H 2004 Progress in lunar laser ranging tests of relativistic gravity *Phys. Rev. Lett.* **93** 261101
[2] Schlamminger S, Choi K-Y, Wagner T A, Gundlach J H and Adelberger E G 2008 Test of the equivalence principle using a rotating torsion balance *Phys. Rev. Lett.* **100** 041101
[3] For a review see, for example Ni W-T 2010 Searches for the role of spin and polarization in gravity *Rep. Prog. Phys.* **73** 056901
[4] Aguilera D N *et al* 2014 STE-QUEST—test of the universality of free fall using cold atom interferometry *Class. Quantum Grav.* **31** 115010
[5] Williams J, Chiow S, Yu N and Müller H 2016 Quantum test of the equivalence principle and space-time aboard the International Space Station *New J. Phys.* **18** 025018
[6] Dittus H, Lämmerzahl C and Selig H 2004 Testing the universality of free fall for charged particles in space *Gen. Rel. Gravit* **36** 571
[7] (The ALPHA Collaboration), Charman A E 2013 Description and first application of a new technique to measure the gravitational mass of antihydrogen *Nat. Commun.* **4** 1785
[8] Hohensee M A, Müller H and Wiringa R B 2013 Equivalence principle and bound kinetic energy *Phys. Rev. Lett.* **111** 151102
[9] Hamilton P, Zhmoginov A, Robicheaux F, Fajans J, Wurtele J S and Müller H 2014 Antimatter interferometry for gravity measurements *Phys. Rev. Lett.* **112** 121102
[10] Tarallo M G, Mazzoni T, Poli N, Sutyrin D V, Zhang X and Tino G M 2014 Test of einstein equivalence principle for 0-spin and half-integer-spin atoms: search for spin-gravity coupling effects *Phys. Rev. Lett.* **113** 023005
[11] Duan X-C *et al* 2016 Test of the universality of free fall with atoms in different spin orientations *Phys. Rev. Lett.* **117** 023001
[12] Roura A 2017 Circumventing Heisenberg's uncertainty principle in atom interferometry tests of the equivalence principle *Phys. Rev. Lett.* **118** 160401
[13] Mashhoon B 2000 Gravitational couplings of intrinsic spin *Class. Quantum Grav.* **17** 2399
[14] Silenko A J and Teryaev O V 2007 Equivalence principle and experimental tests of gravitational spin effects *Phys. Rev.* D **76** 061101
[15] Obukhov Y N, Silenko A J and Teryaev O V 2009 Spin dynamics in gravitational fields of rotating bodies and the equivalence principle *Phys. Rev.* D **80** 064044
[16] Obukhov Y N, Silenko A J and Teryaev O V 2016 Spin-gravity interactions and equivalence principle *Int. J. Mod. Phys. Conf. Ser.* **40** 1660081
[17] Hehl F W 1973 Spin and torsion in general relativity: I. Foundations *Gen. Relat. Grav.* **4** 333
Hehl F W 1974 Spin and torsion in general relativity II: Geometry and field equations *Gen. Relat. Grav.* **5** 491
Hehl F W, von der Heyde P, Kerlick G D and Nester J M 1976 General relativity with spin and torsion: Foundations and prospects *Rev. Mod. Phys.* **48** 393
[18] Shapiro I L 2002 Physical aspects of the space–time torsion *Phys. Rep.* **357** 113
[19] Obukhov Y N, Silenko A J and Teryaev O V 2014 Spin-torsion coupling and gravitational moments of Dirac fermions: Theory and experimental bounds *Phys. Rev.* D **90** 124068
[20] Damour T 2012 Theoretical aspects of the equivalence principle *Class. Quantum Grav.* **29** 184001
[21] Lämmerzahl C 2003 *Quantum Gravity* (Berlin, Heidelberg: Springer) pp 367–94
[22] Colella R, Overhauser A and Werner S 1975 Observation of gravitationally induced quantum interference *Phys. Rev. Lett.* **34** 1472
[23] Alvarez C and Mann R B 1997 Testing the equivalence principle in the quantum regime *Gen. Relativ. Gravit.* **29** 245
[24] Viola L and Onofrio R 1997 Testing the equivalence principle through freely falling quantum objects *Phys. Rev.* D **55** 455
[25] Chowdhury P *et al* 2012 Strong quantum violation of the gravitational weak equivalence principle by a non-Gaussian wave packet *Class. Quantum Grav.* **29** 025010
[26] Rosi G *et al* 2017 Quantum test of the equivalence principle for atoms in coherent superposition of internal energy states *Nat. Commun.* **8** 15529
[27] Orlando P J, Mann R B, Modi K and Pollock F A 2016 A test of the equivalence principle(s) for quantum superpositions *Class. Quantum Grav.* **33** 19LT01
[28] Anastopoulos C and Hu B-L 2018 Equivalence principle for quantum systems: dephasing and phase shift of free-falling particles *Class. Quantum Grav.* **35** 035011
[29] Schwartz P K and Giulini D 2019 Post–Newtonian corrections to Schrödinger equations in gravitational fields *Class. Quantum Grav.* **36** 095016

[30] Flores P C M and Galapon E A 2019 Quantum free-fall motion and quantum violation of the weak equivalence principle *Phys. Rev.* A **99** 042113
[31] Davies P C W and Fang J 1982 Quantum theory and the equivalence principle *Proc. R. Soc. Lond.* A **381** 469
[32] Ali M M, Majumdar A S, Home D and Pan A K 2006 On the quantum analogue of Galileo's leaning tower experiment *Class. Quantum Grav.* **23** 6493
[33] Accioly A and Paszko R 2008 Quantum mechanics versus equivalence principle *Phys. Rev.* D **78** 064002
[34] Herrmann S, Dittus H and Lämmerzahl C 2012 Testing the equivalence principle with atomic interferometry *Class. Quantum Grav.* **29** 184003
[35] Mousavi S V, Majumdar A S and Home D 2015 Effect of quantum statistics on the gravitational weak equivalence principle *Class. Quantum Grav.* **32** 215014
[36] Zych M and Brukner Č 2018 Quantum formulation of the Einstein equivalence principle *Nature Phys.* **14** 1027
Zych M 2017 *Quantum Systems under Gravitational Time Dilation* (Cham: Springer)
[37] Seveso L and Paris M G A 2017 Can quantum probes satisfy the weak equivalence principle? *Ann. Phys.* **380** 213
[38] Anastopoulos C and Hu B L 2020 Quantum superposition of two gravitational cat states *Class. Quantum Grav.* **37** 235012
[39] Aiello A, Lindlein N, Marquardt C and Leuchs G 2009 Transverse angular momentum and geometric spin hall effect of light *Phys. Rev. Lett.* **103** 100401
[40] Korger J *et al* 2014 Observation of the geometric spin hall effect of light *Phys. Rev. Lett.* **112** 113902
[41] Onoda M, Murakami S and Nagaosa N 2004 Hall effect of light *Phys. Rev. Lett.* **93** 083901
[42] Hosten O and Kwiat P 2008 Observation of the spin hall effect of light via weak measurements *Science* **319** 787
[43] Bliokh K Y and Aiello A 2013 Goos–Hanchen and Imbert–Fedorov beam shifts: an overview *J. Opt.* **15** 014001
[44] Bliokh K Y and Nori F 2012 Relativistic Hall effect *Phys. Rev. Lett.* **108** 120403
[45] Stone M, Dwivedi V and Zhou T 2015 Wigner translations and the observer dependence of the position of massless spinning particles *Phys. Rev. Lett.* **114** 210402
[46] Gosselin P, Bérard A and Mohrbach H 2007 Spin Hall effect of photons in a static gravitational field *Phys. Rev.* D **75** 084035
[47] Hacyan S 2012 Physical optics in a uniform gravitational field *Phys. Rev.* D **85** 024035
[48] Yamamoto N 2018 Spin Hall effect of gravitational waves *Phys. Rev.* D **98** 061701
[49] Oancea M A *et al* 2020 Gravitational spin Hall effect of light *Phys. Rev.* D **102** 024075
[50] Obukhov Y N 2001 Spin, gravity, and inertia *Phys. Rev. Lett.* **86** 192
[51] Silenko A J and Teryaev O V 2005 Semiclassical limit for Dirac particles interacting with a gravitational field *Phys. Rev.* D **71** 064016
[52] Mashhoon B and Obukhov Y N 2013 Spin precession in inertial and gravitational fields *Phys. Rev.* D **88** 064037
[53] Everitt C W F *et al* 2011 Gravity probe b: final results of a space experiment to test general relativity *Phys. Rev. Lett.* **106** 221101
[54] Birrell N D and Davies P C W 1984 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[55] Rindler W 2001 *Relativity: Special, General, and Cosmological* II edn (Oxford: Oxford University Press)
[56] Rohrlich F 1963 The principle of equivalence *Ann. Phys. (N.Y.)* **22** 169
[57] Muñoz G and Jones P 2010 The equivalence principle, uniformly accelerated reference frames, and the uniform gravitational field *Am. J. Phys.* **78** 377
[58] Carroll S M 2004 *Spacetime and Geometry: An Introduction to General Relativity* (San Francisco: Addison Wesley)
[59] Boulanger N, Spindel P and Buisseret F 2006 Bound states of Dirac particles in gravitational fields *Phys. Rev.* D **74** 125014
[60] Crispino L C B, Higuchi A and Matsas G E A 2008 The Unruh effect and its applications *Rev. Mod. Phys.* **80** 787
[61] Wang Z-L and Chen X-S 2019 Anomalous geometric spin Hall effect of light *Phys. Rev.* A **99** 063832
[62] Itzykson C and Zuber J-B 1980 *Quantum Field Theory* (New York: McGraw-Hill)
Ohanian H C 1986 What is spin? *Am. J. Phys.* **54** 500
[63] Santoro V *et al* 2002 Development of high intensity neutron source at the European Spallation Source *J. Neutron Res.* **22** 209
[64] Naganawa N *et al* 2018 A cold/ultracold neutron detector using fine-grained nuclear emulsion with spatial resolution less than 100 nm *Eur. Phys. J.* C **78** 959
[65] Casas F, Murua A and Nadinic M 2012 Efficient computation of the Zassenhaus formula *Comput. Phys. Commun.* **183** 2386