

Analytical approximations to a generalized forced damped complex Duffing oscillator: multiple scales method and KBM approach

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Abstract

In this investigation, some different approaches are implemented for analyzing a generalized forced damped complex Duffing oscillator, including the hybrid homotopy perturbation method (H-HPM), which is sometimes called the Krylov-Bogoliubov-Mitropolsky (KBM) method and the multiple scales method (MSM). All mentioned methods are applied to obtain some accurate and stable approximations to the proposed problem without decoupling the original problem. All obtained approximations are discussed graphically using different numerical values to the relevant parameters. Moreover, all obtained approximate solutions are compared with the 4th-order Runge-Kutta (RK4) numerical approximation. The maximum residual distance error (MRDE) is also estimated, in order to verify the high accuracy of the obtained analytic approximations.

Keywords: complex Duffing oscillators, forced damped complex oscillator, multiple scales method, KBM method

(Some figures may appear in colour only in the online journal)

1. Introduction

It is worth noting that all natural phenomena behave nonlinearly. Therefore, to accurately describe these phenomena, many researchers have modeled them based on many nonlinear differential equations [1, 2]. For instance, the averaging method was applied for studying some different examples of autonomous systems [3]: $\ddot{x} + \alpha x + \beta x^3 + \varepsilon g(x, \dot{x}) = 0$ for small ε and β . Also, the forced/unforced damped/undamped parametric pendulum oscillators [4]: $\ddot{\theta} + 2\beta\dot{\theta} + (\omega_0^2 - Q_0 \cos(\gamma t)) \sin \theta = f(t)$ and some other equations related to this oscillator have been analyzed and investigated using some different effective

analytical and numerical techniques, such as the ansatz method [4], He's frequency-amplitude principle [4], He's homotopy perturbation method (HPM) [4], the Krylov-Bogoliubov-Mitropolsky (KBM) method [4], the 4th-order Runge Kutta (RK4), the hybrid Padé-finite difference method [4], the Chebyshev collocation method (CCM) [5], the Galerkin method [6], the ansatz method (AM) and He's frequency formulation [6]. Moreover, the AM and the HPT with the extended KBM were used in the study of the damped cubic nonlinearity Duffing-Mathieu-type oscillator [7]. Since many different types of oscillators are associated with several potential engineering and physical applications and many other complicated nonlinear phenomena, many researchers have devoted considerable effort to analyze these oscillators to arrive at an accurate interpretation

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that serves both engineering and physical applications. Consequently, nonlinear oscillation is one of the most popular and widely researched fields due to its diverse applications in automobiles, sensing, micro- and nano-scale, fluid and solid interaction, nonlinear oscillations in plasma physics, bioengineering, and nonlinear oscillations in optics. There are many different and various equations of motion that were used for modeling several nonlinear oscillations in different physical and engineering systems [8–11]. Most published papers about the nonlinear oscillatory equations focused on the one-dimensional oscillatory differential equations that succeeded in explaining many different oscillations in different engineering, physical systems (especially in plasma physics), and statistical mechanics such as the Duffing oscillatory equation (DOE), the damped DOE [12], the forced damping DOE [13], the fifth-order DOE [14], the damped Helmholtz oscillator equation [15], the damped/undamped Helmholtz-Duffing oscillatory equation [16], the Helmholtz-Fangzhu oscillator equation [17], the damped pendulum oscillator equation [18], and many others. On the other hand, there are some studies that have been conducted on the complex/coupled system of oscillatory differential equations [19–22]. A complex Duffing oscillator (CDO) is an example of nonlinear differential equations with complex variables [23, 24]. Also, this equation has many applications, including rotor dynamics, plasma waves and oscillations, vibrations of the nonlinear Jeffcott rotor [24], etc. In order to analyze and solve this type of differential equations (DEs), they separate into two components of the real and imaginary parts, which leads to a coupled system of DEs. For example, Cveticanin [23] studied the coupled system of third and fifth-order nonlinearity Duffing oscillators by using the hybrid elliptic Krylov-Bogolubov method (eKBM) with the power series method. Also, Cveticanin [24] used the eKBM for investigating the following complex Duffing-type oscillator,

$$\ddot{z} + c_1 z + 3c_3 z |z|^2 - c_3 \bar{z}^3 = \epsilon G(z, \dot{z}, cc), \quad (1)$$

where $z \equiv z(t) = x + iy$ is a complex function and \bar{z} is the conjugate of z , whereas $x \equiv x(t)$ and $y \equiv y(t)$ indicate the real and imaginary parts, respectively, $z\bar{z} = |z|^2$ and $G(z, \dot{z}, cc)$ represent a complex deflection, “cc” indicates the complex conjugate function, and $i = \sqrt{-1}$. Cveticanin [24] derived an analytic approximation in the form of the Jacobi elliptic functions for the unforced case. Also, Cveticanin [24] tested the accuracy of the obtained approximations by comparing them with the RK numerical approximation and found that there is a good agreement between them. Moreover, some coupled nonlinear differential equations related to an enzymatic reaction have been solved using the HPM [25]. In [2], the complex Duffing system with damping term has been studied using the Wiener-Hermite expansion. Furthermore, a complex Duffing system with a nonstationary random excitation has been investigated using the Wiener-Hermite expansion method [26]. In [27], the authors analyzed the dynamical behavior of the following new form to a complex Duffing equation (CDE): $\ddot{z} + k\dot{z} - z + \beta z |z|^2 = \gamma \exp(it)$. The authors [27] used the later equation for detecting complex signals in noise. However, in our investigation, we will

consider the following generalized forced damped CDO [23],

$$\ddot{z} + 2\epsilon \dot{z} + \alpha z + \beta z |z|^2 + \gamma \bar{z}^3 = F(t), \quad (2)$$

where $F(t) = f_1(t) + if_2(t)$ represents the excited periodic force in which both real and imaginary components ($f_1(t), f_2(t)$) may take the following values: $f_1(t) = \gamma_1 \cos(\omega_1 t)$ and $f_2(t) = \gamma_2 \cos(\omega_2 t)$ or any other time-dependent function. In equation (2), the coefficients ($\epsilon, \alpha, \beta, \gamma$) are real constants. More details about the derivation and potential applications of equation (2) can be found in [23]. However, in [23], the author did not investigate the forced case. Therefore, the main objective of this study is to analyze and discuss both unforced ($F(t) = 0$) and forced ($F(t) \neq 0$) general damped CDO (2) and find some approximation for them using two different approaches, including the hybrid HPM (H-HPM), sometimes also called the KBM method [28–30], and the multiple scales method (MSM) [31, 32].

The rest of this paper consists of the following parts: in section 2, the mathematical methods for analyzing the generalized forced damped CDO are introduced. In this section, two different approaches including both the H-HPM and MSM are implemented to find some approximations to the generalized forced damped CDO. In section 3, all of the obtained approximations are discussed in detail. Some numerical examples are considered for studying the impact of different parameters on the profile of the oscillator. Finally, the most significant findings we make are outlined in section 4.

2. Mathematical methods for analyzing the generalized forced damped CDO

Let us write the general forced damped complex Duffing (2) in the following initial value problem (i.v.p.):

$$\begin{cases} \ddot{z} + 2\epsilon \dot{z} + \alpha z + \beta z |z|^2 + \gamma \bar{z}^3 = F(t), & \alpha > 0, \\ \ddot{z} + 2\epsilon \dot{z} + \alpha z + \beta z |z|^2 + \gamma \bar{z}^3 = F(t), \\ z(0) = x_0 + iy_0 \text{ and } \dot{z}(0) = \dot{x}_0 + i\dot{y}_0. \end{cases} \quad (3)$$

Here, the coefficients ($\epsilon, \alpha, \beta, \gamma$) are real constants. For analyzing this problem, we first should reduce it to the standard two forced damped Duffing oscillators. Using the relation $z = x + iy$ in the i.v.p. (3), we have

$$\begin{aligned} \mathbb{k}_1 &\equiv \ddot{x} + 2\epsilon \dot{x} + x(\alpha + (\beta - 3\gamma)y^2) \\ &+ (\beta + \gamma)x^3 - f_1(t) = 0, \end{aligned} \quad (4)$$

and

$$\begin{aligned} \mathbb{k}_2 &\equiv \ddot{y} + 2\epsilon \dot{y} + y(\alpha + (\beta - 3\gamma)x^2) \\ &+ (\beta + \gamma)y^3 - f_2(t) = 0, \end{aligned} \quad (5)$$

with

$$\begin{cases} x(0) = x_0, y(0) = y_0, \\ \dot{x}(0) = \dot{x}_0 \text{ and } \dot{y}(0) = \dot{y}_0. \end{cases} \quad (6)$$

Now, we try to analyze and solve the coupled system of equations (4)–(6) directly without decoupling using some

effective and accurate analytical approaches including both the H-HPM/KBM method and the MSM. The proposed methods are illustrated in detail below.

2.1. Hybrid homotopy perturbation method (H-HMP)

Let us construct the homotopy for the i.v.p. (3) as follows:

$$\begin{cases} \ddot{z} + \alpha z + p[2\varepsilon\dot{z} + \beta z|z|^2 + \gamma\bar{z}^3 - f_1(t) - if_2(t)] = 0, \\ z(0) = z_0 \text{ and } z'(0) = \dot{z}_0. \end{cases} \quad (7)$$

Based on the KBM method, the solution of the i.v.p. (3) is given by

$$z(t) = x(t) + iy(t), \quad (8)$$

with

$$\begin{aligned} x(t) &= a(t)\cos(\psi(t)) + \sum_{n=1}^N p^n U_n(a(t), b(t), \psi(t)), \\ y(t) &= b(t)\cos(\Psi(t)) + \sum_{n=1}^N p^n V_n(a(t), b(t), \Psi(t)), \\ a'(t) &\equiv \dot{a}(t) = \sum_{n=1}^N p^n A_n(a(t)), \\ b'(t) &\equiv \dot{b}(t) = \sum_{n=1}^N p^n B_n(b(t)), \\ \psi'(t) &\equiv \dot{\psi}(t) = \sqrt{\alpha} + \sum_{n=1}^N p^n \psi_n(a(t), b(t)), \\ \Psi'(t) &\equiv \dot{\Psi}(t) = \sqrt{\alpha} + \sum_{n=1}^N p^n \Psi_n(a(t), b(t)), \end{aligned} \quad (9)$$

where $a \equiv a(t)$, $b \equiv b(t)$, $\psi \equiv \psi(t)$, $A_n(a)$, $B_n(b)$, $\psi_n(a, b)$, $\Psi_n(a, b)$, $U_n(a, b, \psi)$, and $V_n(a, b, \Psi)$ are undetermined functions, whereas the parameter 'p' represents the perturbation parameter ($p \ll 1$).

We choose the solutions in order to avoid the presence of the so called secular terms For the first-order approximation ($N = 1$), we have

$$\begin{aligned} \ddot{z} + 2\varepsilon\dot{z} + \alpha z + \beta z|z|^2 + \gamma\bar{z}^3 \\ - f_1(t) - if_2(t) = \tilde{R} + i\tilde{S} + O(p^2), \end{aligned} \quad (10)$$

with

$$\begin{aligned} \tilde{R} = \frac{1}{4}p \left(4\psi_0^2 U^{(0,0,2)}(a, b, \psi) + 4\alpha U(a, b, \psi) + a^3\beta \cos(3\psi) \right. \\ \left. + a^3\gamma \cos(3\psi) + ab^2\beta \cos(\psi - 2\Psi) + ab^2\beta \cos(\psi + 2\Psi) \right. \\ \left. - 3ab^2\gamma \cos(\psi - 2\Psi) - 3ab^2\gamma \cos(\psi + 2\Psi) - 4f_1(t) \right) \\ + \frac{a}{4}p(-8\psi_0\psi_1(a, b) + 3a^2(\beta + \gamma) \\ + 2b^2(\beta - 3\gamma))\cos(\psi) \\ + a(\alpha - \psi_0^2)\cos(\psi) - 2p\psi_0(A_1(a) + a\varepsilon)\sin(\psi), \end{aligned}$$

and

$$\begin{aligned} \tilde{S} = \frac{1}{4}p \left(4\Psi_0^2 V^{(0,0,2)}(a, b, \Psi) + 4\alpha V(a, b, \Psi) + a^2\beta b \cos(2\psi - \Psi) \right. \\ \left. + a^2\beta b \cos(2\psi + \Psi) - 3a^2b\gamma \cos(2\psi - \Psi) \right. \\ \left. - 3a^2b\gamma \cos(2\psi + \Psi) + \beta b^3 \cos(3\Psi) + b^3\gamma \cos(3\Psi) - 4f_2(t) \right) \\ + \frac{b}{4}p(-8\Psi_0\Psi_1(a, b) + 2a^2(\beta - 3\gamma) + 3b^2(\beta + \gamma))\cos(\Psi) \\ - 2p\Psi_0(B_1(b) + b\varepsilon)\sin(\Psi) + b(\alpha - \Psi_0^2)\cos(\Psi). \end{aligned}$$

Equating to zero the coefficients of p , $\cos(\psi(t))$, $\sin(\psi(t))$, $\cos(\Psi(t))$, and $\sin(\Psi(t))$, in the relations of (\tilde{R}, \tilde{S}) , the following system of algebraic-differential equations is obtained:

$$\begin{aligned} \alpha - \psi_0^2 &= 0, \\ \alpha - \Psi_0^2 &= 0, \\ A_1(a) + a\varepsilon &= 0, \\ B_1(b) + b\varepsilon &= 0, \\ -8\psi_0\psi_1(a, b) + 2b^2\beta + 3a^2\beta + 3a^2\gamma - 6b^2\gamma &= 0, \\ -8\Psi_0\Psi_1(a, b) + 2a^2\beta + 3b^2\beta - 6a^2\gamma + 3b^2\gamma &= 0, \end{aligned} \quad (11)$$

$$\begin{aligned} 4\psi_0^2 U_1^{(0,0,2)}(a, b, \psi) + 4\alpha U_1(a, b, \psi) + a^3\beta \cos(3\psi) \\ + a^3\gamma \cos(3\psi) + ab^2\beta \cos(\psi - 2\Psi) \\ + ab^2\beta \cos(\psi + 2\Psi) \\ - 3ab^2\gamma \cos(\psi - 2\Psi) - 3ab^2\gamma \cos(\psi + 2\Psi) \\ - 4f_1(t) = 0, \end{aligned} \quad (12)$$

and

$$\begin{aligned} 4\Psi_0^2 V_1^{(0,0,2)}(a, b, \Psi) + 4\alpha V_1(a, b, \Psi) \\ + a^2\beta b \cos(2\psi - \Psi) \\ + a^2\beta b \cos(2\psi + \Psi) - 3a^2b\gamma \cos(2\psi - \Psi) \\ - 3a^2b\gamma \cos(2\psi + \Psi) \\ + \beta b^3 \cos(3\Psi) + b^3\gamma \cos(3\Psi) - 4f_2(t) = 0. \end{aligned} \quad (13)$$

Solving system (11) yields

$$\begin{aligned} \psi_0 &= \Psi_0 = \sqrt{\alpha}, \\ A_1(a) &= -a\varepsilon, \\ B_1(b) &= -b\varepsilon, \\ \psi_1(a, b) &= \frac{1}{8\sqrt{\alpha}}(3a^2\beta + 3a^2\gamma + 2b^2\beta - 6b^2\gamma), \\ \Psi_1(a, b) &= \frac{1}{8\sqrt{\alpha}}(2a^2\beta - 6a^2\gamma + 3b^2\beta + 3b^2\gamma). \end{aligned} \quad (14)$$

Solving both equations (12) and (13) using the values of ψ_0 , Ψ_0 , $A_1(a)$, $B_1(a)$, $\psi_1(a, b)$, and $\Psi_1(a, b)$ given in equation (14), we have

$$\begin{aligned} U_1(a, b, \psi) &= \frac{1}{32\alpha} \\ &\times \left[32f_1(t) + a^3(\beta + \gamma)\cos(3\psi) \right. \\ &\quad \left. - 4ab^2(\beta - 3\gamma)\cos(2\Psi)(2\psi \sin(\psi) + \cos(\psi)) \right], \\ V_1(a, b, \Psi) &= \frac{1}{32\alpha} \\ &\times \left[32f_2(t) + b^3(\beta + \gamma)\cos(3\Psi) \right. \\ &\quad \left. - 4a^2b(\beta - 3\gamma)\cos(2\psi)(2\Psi \sin(\Psi) + \cos(\Psi)) \right]. \end{aligned} \quad (15)$$

Accordingly, the first-order approximation reads

$$z(t) = x(t) + iy(t), \quad (16)$$

with

$$\begin{cases} x(t) = a(t)\cos(\psi(t)) + U_1(a(t), b(t), \psi(t)), \\ y(t) = b(t)\cos(\Psi(t)) + V_1(a(t), b(t), \Psi(t)), \end{cases} \quad (17)$$

and

$$\begin{aligned} a(t) &= a_0 e^{-\varepsilon t}, \\ b(t) &= b_0 e^{-\varepsilon t}, \\ \psi(t) &= \frac{e^{-2\varepsilon t}}{16\sqrt{\alpha}\varepsilon} \left[2e^{2\varepsilon t}(8\sqrt{\alpha}a_1\varepsilon + b_0^2(\beta - 3\gamma) + 8\alpha\varepsilon t) \right. \\ &\quad \left. + 3a_0^2(\beta + \gamma)(e^{2\varepsilon t} - 1) - 2b_0^2(\beta - 3\gamma) \right], \\ \Psi(t) &= \frac{e^{-2\varepsilon t}}{16\sqrt{\alpha}\varepsilon} \left[\left(2a_0^2(\beta - 3\gamma)(e^{2\varepsilon t} - 1) \right) \right. \\ &\quad \left. + 16\sqrt{\alpha}\varepsilon e^{2\varepsilon t}(b_1 + \sqrt{\alpha}t) \right. \\ &\quad \left. + 3b_0^2(\beta + \gamma)(e^{2\varepsilon t} - 1) \right], \end{aligned}$$

where the constants (a_0, a_1, b_0, b_1) are obtained from the ICs (6).

2.2. Multiple scales method (MSM)

To apply the MSM [31, 32] for analyzing the general forced damped complex Duffing i.v.p. (3), we first construct the p -problem to the system of equations (4) and (5) for $\alpha > 0$ as follows:

$$\begin{cases} \ddot{x} + \alpha x + p[2\varepsilon\dot{x} + x(\beta - 3\gamma)y^2 + (\beta + \gamma)x^3 - f_1(t)] = 0, \\ \ddot{y} + \alpha y + p[2\varepsilon\dot{y} + y(\beta - 3\gamma)x^2 + (\beta + \gamma)y^3 - f_2(t)] = 0, \end{cases} \quad (18)$$

where the parameter ‘ p ’ represents the perturbation parameter ($p \ll 1$).

Based on the MSM, the solutions of system (18) are defined in the following form:

$$\begin{cases} x = u_0(T_0, T_1) + pu_1(T_0, T_1) + O(p^2), \\ y = v_0(T_0, T_1) + pv_1(T_0, T_1) + O(p^2), \end{cases} \quad (19)$$

with

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial T_0} + p\frac{\partial}{\partial T_1} + \dots, \\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial T_0^2} + 2p\frac{\partial}{\partial T_0}\frac{\partial}{\partial T_1} + \dots, \end{aligned} \quad (20)$$

where $u_0 \equiv u_0(T_0, T_1)$, $u_1 \equiv u_1(T_0, T_1)$, $v_0 \equiv v_0(T_0, T_1)$, $v_1 \equiv v_1(T_0, T_1)$, and the independent time-scales $T_j = p^j t$ where $j = 0, 1, \dots$.

For simplicity, we assume that

$$\begin{cases} R_p(t) \equiv \ddot{x} + \alpha x + p[2\varepsilon\dot{x} + x(\beta - 3\gamma)y^2 + (\beta + \gamma)x^3 - f_1(t)], \\ S_p(t) \equiv \ddot{y} + \alpha y + p[2\varepsilon\dot{y} + y(\beta - 3\gamma)x^2 + (\beta + \gamma)y^3 - f_2(t)]. \end{cases} \quad (21)$$

Putting equation (19) with the differential operators (20) into system (21) and collecting the terms of the same power of p ,

we get

$$\begin{aligned} R_p(t) &= u_0^{(2,0)} + \alpha u_0 \\ &\quad + p(2\varepsilon u_0^{(1,0)} + 2u_0^{(1,1)} + u_1^{(2,0)} + \alpha u_1 \\ &\quad + u_0^3(\beta + \gamma) + u_0 v_0^2(\beta - 3\gamma) - f_1(t)), \end{aligned} \quad (22)$$

and

$$\begin{aligned} S_p(t) &= v_0^{(2,0)} + \alpha v_0 \\ &\quad + p(2\varepsilon v_0^{(1,0)} + 2v_0^{(1,1)} + v_1^{(2,0)} + \alpha v_1 \\ &\quad + u_0^2 v_0(\beta - 3\gamma) + v_0^3(\beta + \gamma) - f_2(t)). \end{aligned} \quad (23)$$

Equating to zero the coefficients of p^j ($j = 0, 1$), the following system of partial DEs is obtained

$$\begin{cases} u_0^{(2,0)} + \alpha u_0 = 0, \\ v_0^{(2,0)} + \alpha v_0 = 0, \end{cases} \quad (24)$$

and

$$\begin{cases} 2\varepsilon u_0^{(1,0)} + 2u_0^{(1,1)} + u_1^{(2,0)} + \alpha u_1 + (\beta + \gamma)u_0^3 + (\beta - 3\gamma)u_0 v_0^2 - f_1(t) = 0, \\ 2\varepsilon v_0^{(1,0)} + 2v_0^{(1,1)} + v_1^{(2,0)} + \alpha v_1 + (\beta - 3\gamma)u_0^2 v_0 + (\beta + \gamma)v_0^3 - f_2(t) = 0, \end{cases} \quad (25)$$

with

$$\begin{aligned} u_j^{(q,r)} &= \frac{\partial^{q+r}}{\partial T_0^q \partial T_1^r} u_j(T_0, T_1), \\ v_j^{(q,r)} &= \frac{\partial^{q+r}}{\partial T_0^q \partial T_1^r} v_j(T_0, T_1) \text{ for any } (j, q, r). \end{aligned}$$

Solving system (24) yields the values of (u_0, v_0) as follows:

$$\begin{cases} u_0 = a(T_1) \cos(\sqrt{\alpha} T_0 + \varphi(T_1)), \\ v_0 = b(T_1) \cos(\sqrt{\alpha} T_0 + \phi(T_1)), \end{cases} \quad (26)$$

where $a(T_1)$, $b(T_1)$, $\varphi(T_1)$, and $\phi(T_1)$ are undetermined time-dependent functions.

Solving system (25) yields the values of (u_1, v_1) as follows:

$$\begin{cases} u_1 = S_0 + S_1 \sin(\sqrt{\alpha} t + \varphi(t)) + S_2 \cos(\sqrt{\alpha} t + \varphi(t)), \\ v_1 = W_0 + W_1 \sin(\sqrt{\alpha} t + \phi(t)) + W_2 \cos(\sqrt{\alpha} t + \phi(t)), \end{cases} \quad (27)$$

with

$$\begin{aligned} S_0 &= \frac{\gamma_1 \cos(t\omega_1)}{\alpha - \omega_1^2} + \frac{a(t)^3}{32\alpha}(\beta + \gamma)\cos(3(\sqrt{\alpha}t + \varphi(t))) \\ &\quad + \frac{a(t)}{32\alpha}(\beta - 3\gamma)b(t)^2(-4\sqrt{\alpha}t \sin(\Theta_1) \\ &\quad - 2\cos(\Theta_1) + \cos(\bar{\Theta}_1)), \\ S_1 &= \frac{1}{8\sqrt{\alpha}}[4a'(t) + a(t)(-2t(\beta - 3\gamma)b(t)^2 \\ &\quad + 4\varepsilon + 8\sqrt{\alpha}t\varphi'(t) - 3ta(t)^3(\beta + \gamma)), \\ S_2 &= -\frac{1}{16\alpha}[16\alpha ta'(t) + 2a(t)(\beta - 3\gamma)b(t)^2 \\ &\quad + 8\alpha\varepsilon t - 4\sqrt{\alpha}\varphi'(t) + 3a(t)^3(\beta + \gamma)], \end{aligned}$$

and

$$\begin{aligned} W_0 &= \frac{\gamma_2 \cos(t\omega_2)}{\alpha - \omega_2^2} + \frac{(\beta + \gamma)}{32\alpha} b(t)^3 \cos(3(\sqrt{\alpha}t + \phi(t))) \\ &\quad + \frac{a(t)^2}{32\alpha} (\beta - 3\gamma) b(t) (-4\sqrt{\alpha}t \sin(\Theta_2) \\ &\quad - 2\cos(\Theta_2) + \cos(\bar{\Theta}_2)), \\ W_1 &= \frac{1}{8\sqrt{\alpha}} [b(t)(-2ta(t)^2(\beta - 3\gamma) + 4\epsilon \\ &\quad + 8\sqrt{\alpha}t\phi'(t) + 4b'(t) - 3tb(t)^3(\beta + \gamma))], \\ W_2 &= -\frac{1}{16\alpha} [2b(t)(a(t)^2(\beta - 3\gamma) + 8\alpha\epsilon t \\ &\quad - 4\sqrt{\alpha}\phi'(t) + 16\alpha tb'(t) + 3b(t)^3(\beta + \gamma))], \end{aligned}$$

where $\Theta_1 = \sqrt{\alpha}t - \varphi(t) + 2\phi(t)$, $\bar{\Theta}_1 = 3\sqrt{\alpha}t + \varphi(t) + 2\phi(t)$, $\Theta_2 = \sqrt{\alpha}t + 2\varphi(t) - \phi(t)$, and $\bar{\Theta}_2 = 3\sqrt{\alpha}t + 2\varphi(t) + \phi(t)$.

Now, to avoid the secularity and to determine the values of $a(T_1)$, $b(T_1)$, $\phi(t)$, and $\varphi(t)$, we should solve the ode system $S_{1,2} = 0$ and $W_{1,2} = 0$, which leads to

$$\begin{cases} a(T_1) = c_0 e^{-\epsilon T_1}, b(T_1) = d_0 e^{-\epsilon T_1}, \\ \varphi(T_1) = d_1 - \frac{(3\beta c_0^2 + 3\gamma c_0^2 + 2\beta d_0^2 - 6\gamma d_0^2)}{16\sqrt{\alpha}\epsilon} e^{-2\epsilon T_1}, \\ \phi(T_1) = c_1 - \frac{(2\beta c_0^2 - 6\gamma c_0^2 + 3\beta d_0^2 + 3\gamma d_0^2)}{16\sqrt{\alpha}\epsilon} e^{-2\epsilon T_1}, \end{cases} \quad (28)$$

while for $\epsilon = 0$, we get

$$\begin{cases} a(T_1) = c_0, b(T_1) = d_0, \\ \phi(T_1) = \frac{T_1(2c_0^2(\beta - 3\gamma) + 3d_0^2(\beta + \gamma))}{8\sqrt{\alpha}} + c_1, \\ \varphi(T_1) = \frac{T_1(3c_0^2(\beta + \gamma) + 2d_0^2(\beta - 3\gamma))}{8\sqrt{\alpha}} + d_1. \end{cases} \quad (29)$$

Accordingly, system (27) reduces to $u_1 = S_0$ and $v_1 = W_0$. Inserting equations (26)–(28) into system (19), and for $p \rightarrow 1$, we finally get the first-order approximations to (x, y, z) as follows:

$$z = x(t) + iy(t), \quad (30)$$

with

$$\begin{aligned} x(t) &= \frac{\gamma_1 \cos(t\omega_1)}{\alpha - \omega_1^2} + a(t) \cos(\sqrt{\alpha}t + \varphi(t)) \\ &\quad + \frac{a(t)^3}{32\alpha} (\beta + \gamma) \cos(3(\sqrt{\alpha}t + \varphi(t))) \\ &\quad + \frac{a(t)b(t)^2}{32\alpha} (\beta - 3\gamma) (-4\sqrt{\alpha}t \sin(\Theta_1) \\ &\quad - 2\cos(\Theta_1) + \cos(\bar{\Theta}_1)) \end{aligned} \quad (31)$$

and

$$\begin{aligned} y(t) &= \frac{\gamma_2 \cos(\omega_2 t)}{\alpha - \omega_2^2} + b(t) \cos(\sqrt{\alpha}t + \phi(t)) \\ &\quad + \frac{b(t)^3}{32\alpha} (\beta + \gamma) \cos(3(\sqrt{\alpha}t + \phi(t))) \\ &\quad + \frac{a(t)^2 b(t)}{32\alpha} (\beta - 3\gamma) (-4\sqrt{\alpha}t \sin(\Theta_2) \\ &\quad - 2\cos(\Theta_2) + \cos(\bar{\Theta}_2)), \end{aligned} \quad (32)$$

where the constants c_0 , d_0 , c_1 and d_1 are found from the initial conditions.

It is shown that the approximations (30)–(32) do not recover only the solution of the forced damped CDO (2), but can also recover the solution of the unforced damped CDO, i.e., equation (2) for $F(t) = 0$ or $\gamma_1 = \gamma_2 = 0$, as follows:

$$\begin{aligned} x(t) &= a(t) \cos(\sqrt{\alpha}t + \varphi(t)) \\ &\quad + \frac{a(t)^3}{32\alpha} (\beta + \gamma) \cos(3(\sqrt{\alpha}t + \varphi(t))) \\ &\quad + \frac{a(t)b(t)^2}{32\alpha} (\beta - 3\gamma) (-4\sqrt{\alpha}t \sin(\Theta_1) \\ &\quad - 2\cos(\Theta_1) + \cos(\bar{\Theta}_1)) \end{aligned} \quad (33)$$

and

$$\begin{aligned} y(t) &= b(t) \cos(\sqrt{\alpha}t + \phi(t)) \\ &\quad + \frac{b(t)^3}{32\alpha} (\beta + \gamma) \cos(3(\sqrt{\alpha}t + \phi(t))) \\ &\quad + \frac{a(t)^2 b(t)}{32\alpha} (\beta - 3\gamma) (-4\sqrt{\alpha}t \sin(\Theta_2) \\ &\quad - 2\cos(\Theta_2) + \cos(\bar{\Theta}_2)). \end{aligned} \quad (34)$$

3. Results and discussion

3.1. Example (1)

First, let us check the behavior of the obtained analytical approximations to the i.v.p. (3) for the unforced case, i.e., for $\gamma_1 = \gamma_2 = 0$, by using the following numerical example

$$\begin{cases} \ddot{x} + 0.2\dot{x} + x(1 + 0.4y^2) + 1.2x^3 = 0, \\ \ddot{y} + 0.2\dot{y} + y(1 + 0.4x^2) + 1.2y^3 = 0, \\ x(0) = 0.1, y(0) = 0.1, \\ \dot{x}(0) = 0 \text{ and } \dot{y}(0) = 0. \end{cases} \quad (35)$$

The approximations to real and imaginary parts (x, y) using both the KBM method (approximations (16) and (17)) and the MSM (approximations (33) and (34)) are compared with the RK4 approximations as illustrated in figure 1. Moreover, the absolute approximations $|z| = |x + iy|$ using both the KBM method and MSM are compared with the absolute approximation using the RK4 approach as shown in figure 2. In addition, the maximum residual distance error (MRDE) L_{MRDE} to the real, imaginary, and absolute approximations,

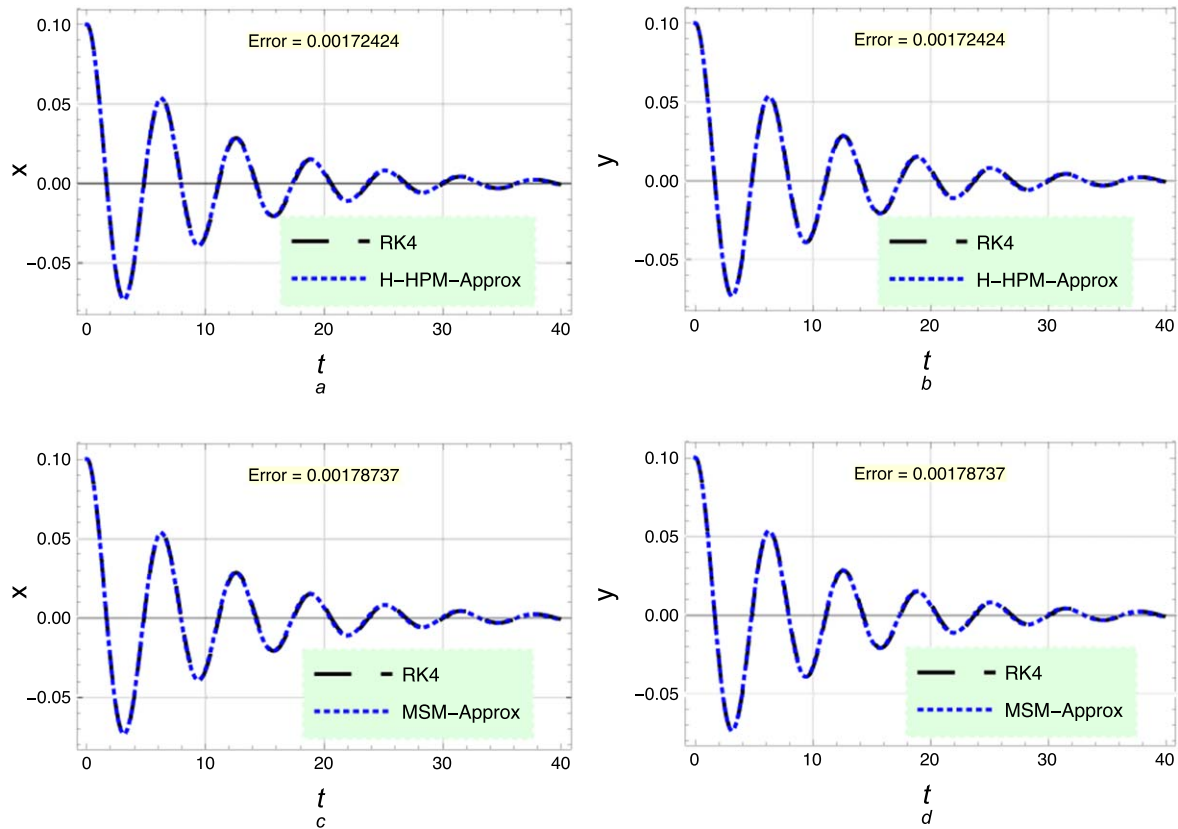


Figure 1. The analytical approximations (17), (31), and (32) using H-HPM and MSM as well as the RK4 numerical approximations to the i.v.p. (3) for (a), (c) the real part (4) and (b), (d) the imaginary part (5) are considered using data from example (1).

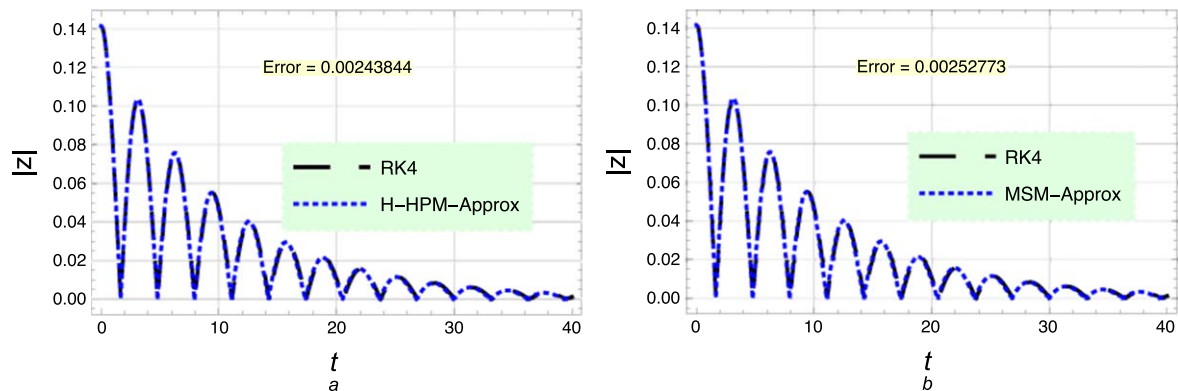


Figure 2. The analytical approximations (a) (16) using H-HPM and (b) (30) using MSM are compared with the RK4 numerical approximations to the i.v.p. (3) for the absolute value $|z|$ using data from example (1).

according to the following relations

$$\begin{cases} L_{MRDE}|_x = |x - \text{Re}(\text{RK4})|, \\ L_{MRDE}|_y = |y - \text{Im}(\text{RK4})|, \\ L_{MRDE}|_z = ||z| - |\text{RK4}||, \end{cases}$$

is estimated on the whole study domain as demonstrated in table 1.

3.2. Example (2)

Now, we can investigate the characteristics of the obtained analytical approximations of the i.v.p. (3) in the existence of excited force, i.e., $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ by using the following

numerical example:

$$\begin{cases} \ddot{x} + 0.2\dot{x} + x(1 + 0.4y^2) + 1.2x^3 = 0.1 \cos(0.1t), \\ \ddot{y} + 0.2\dot{y} + y(1 + 0.4x^2) + 1.2y^3 = -0.02 \cos(0.17t), \\ x(0) = 0, y(0) = 0, \\ \dot{x}(0) = 0 \text{ and } \dot{y}(0) = 0. \end{cases} \quad (36)$$

Both real and imaginary parts approximations (x, y) using the H-HPM (approximations (16) and (17)) and the MSM (approximations (30)–(32)) are compared with the numerical approximations using the RK4 approach as illustrated in figure 3. In addition, the absolute approximations $|z|$ using the proposed methods (H-HPM and MSM) are compared with the

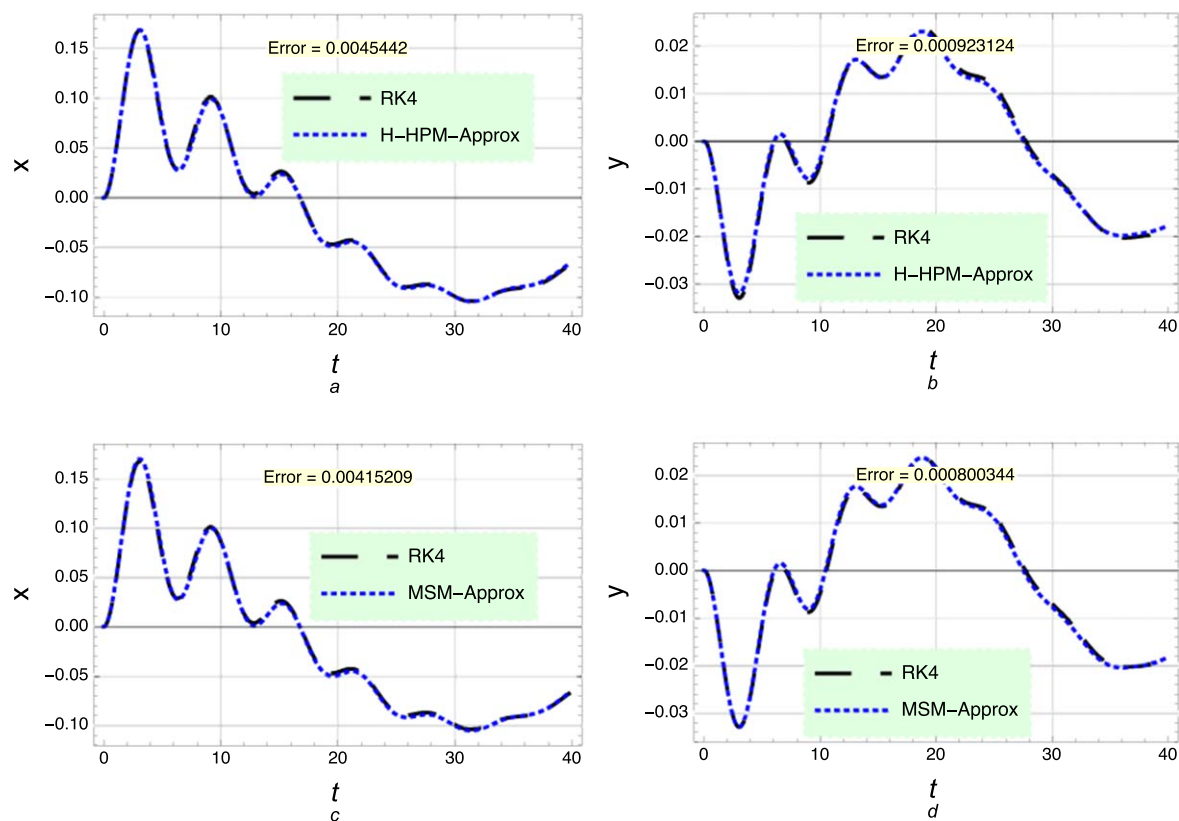


Figure 3. The analytical approximations (17), (31), and (32) using H-HPM and MSM as well as the RK4 numerical approximations to the i.v. p. (3) for (a), (c) the real part (4) and (b), (d) the imaginary part (5) are considered using data from example (2).

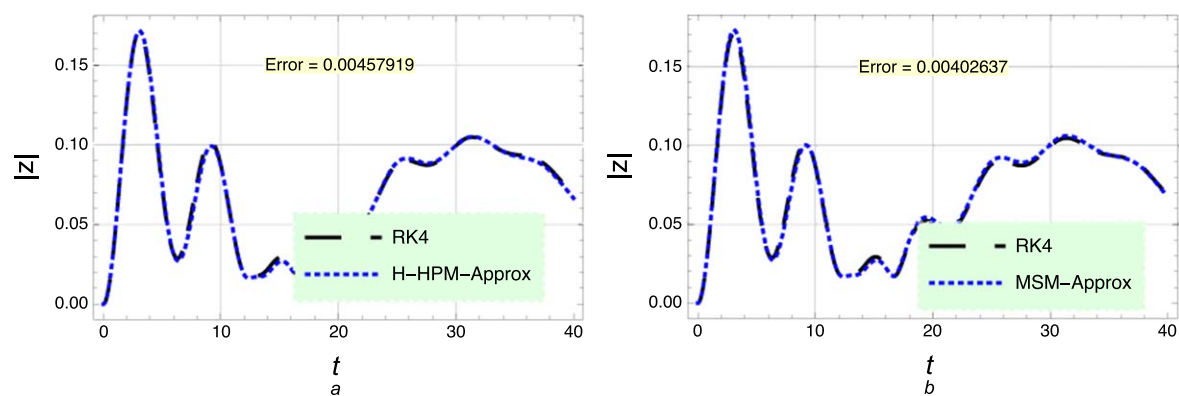


Figure 4. The analytical approximations (a) (16) using H-HPM and (b) (30) using MSM are compared with the RK4 numerical approximations to the i.v.p. (3) for the absolute value $|z|$ using data from example (2).

Table 1. The MRDE L_{MRDE} for all proposed methods as compared to the RK4 approach using data from example (1).

Method	$L_{MRDE} _x$	$L_{MRDE} _y$	$L_{MRDE} _{ z }$
H-HPM	0.001 724 24	0.001 724 24	0.002 438 44
MSM	0.001 803	0.001 803	0.002 549 82

Table 2. The MRDE L_{MRDE} for all proposed methods as compared to the RK4 approach using data from example (2).

Method	$L_{MRDE} _x$	$L_{MRDE} _y$	$L_{MRDE} _{ z }$
H-HPM	0.004 544 2	0.000 923 124	0.004 579 19
MSM	0.004 152 09	0.000 800 344	0.004 026 37

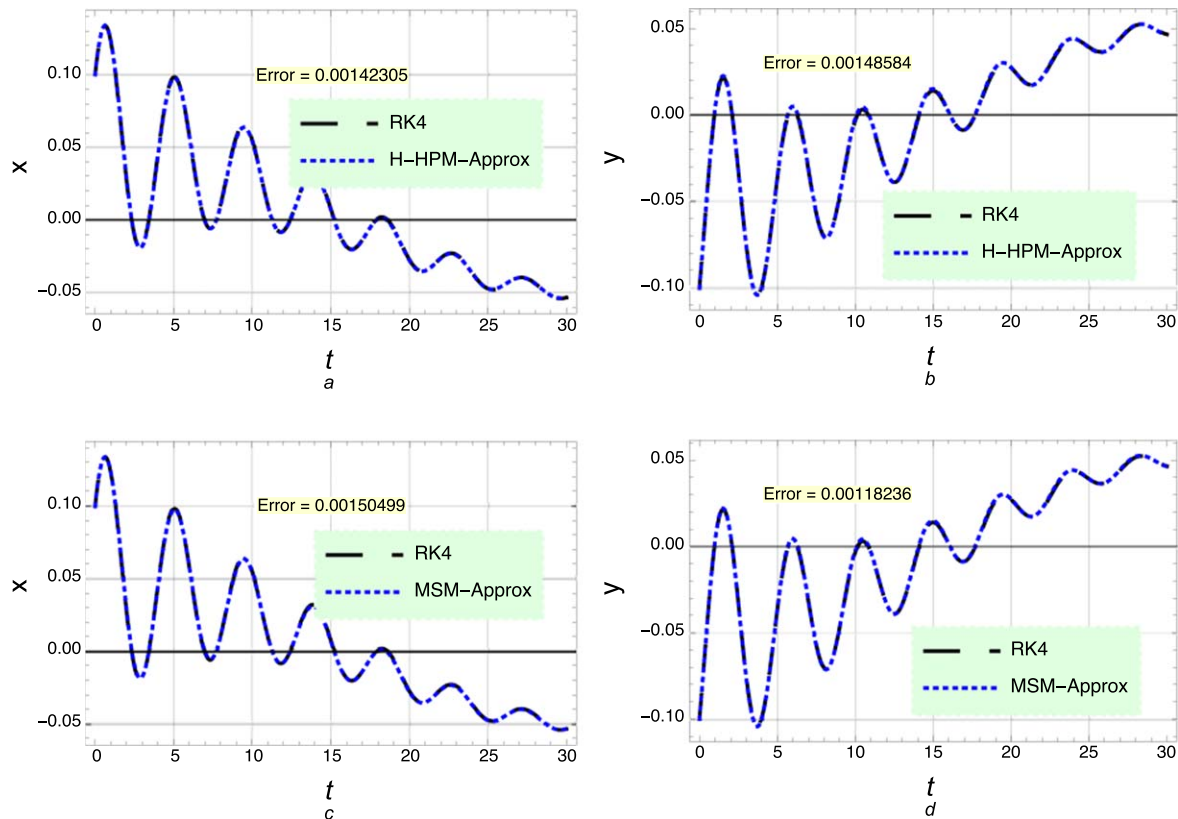


Figure 5. The analytical approximations (17), (31), and (32) using H-HPM and MSM as well as the RK4 numerical approximations to the i.v. p. (3) for (a), (c) the real part (4) and (b), (d) the imaginary part (5) are considered using data from example (3).

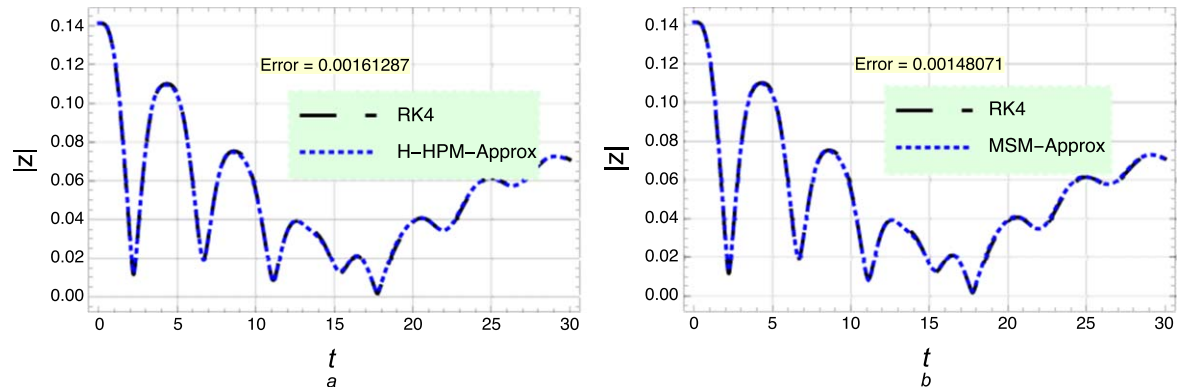


Figure 6. The analytical approximations (a) (16) using H-HPM and (b) (30) using MSM are compared with the RK4 numerical approximations to the i.v.p. (3) for the absolute value $|z|$ using data from example (3).

Table 3. The MRDE L_{MRDE} for all proposed methods as compared to RK4 approach using data of example (3).

Method	$L_{MRDE} _x$	$L_{MRDE} _y$	$L_{MRDE} _{ z }$
H-HPM	0.001 423 05	0.001 485 84	0.001 612 87
MSM	0.001 504 99	0.001 182 36	0.001 480 71

absolute numerical approximations using the RK4 approach as shown in figure 4. Moreover, the MRDE L_{MRDE} for the real, imaginary, and absolute approximations is estimated as shown in table 2.

3.3. Example (3)

Here, we consider the following new values to the relevant parameters with non-zero ICs:

$$\begin{cases} \ddot{x} + 0.2\dot{x} + x(2 - 2y^2) + 2x^3 = 0.1 \cos(0.1t), \\ \ddot{y} + 0.2\dot{y} + y(2 - 2x^2) + 2y^3 = -0.1 \cos(0.1t), \\ x(0) = 0.1, y(0) = -0.1, \\ \dot{x}(0) = 0.1 \text{ and } \dot{y}(0) = 0.1. \end{cases} \quad (37)$$

Figure 5 represents the real and imaginary parts approximations (x , y) using the KBM method (approximations (16) and

(17)) and the MSM (approximations (30)–(32)) as well as the RK4 numerical approximations. Also, the absolute approximations $|z|$ using all mentioned approaches are presented in figure 6. Furthermore, the MRDE L_{MRDE} for the real, imaginary, and absolute approximations to all proposed methods is estimated as illustrated in table 3. It is clear from tables 1–3 that the accuracies of all obtained analytical approximations using both the KBM method and MSM are highly compatible with each other. Moreover, all obtained approximations are characterized by high accuracy and are more stable for a long time, as illustrated in figures 1–6. Furthermore, one can see that the MSM accuracy is sometimes better than those of H-HPM. Despite this, there is great harmony and agreement between both the analytical and numerical approximations, which enhances the high accuracy of all obtained analytic approximations.

4. Conclusion

The nonlinear complex Duffing oscillators including both unforced and forced damped complex Duffing oscillators (CDO), have been analyzed using some different approaches. Two different approaches have been applied for deriving some effective, accurate, and stable approximations. In the first approach, the hybrid homotopy perturbation method (H-HPM)/Krylov-Bogoliubov-Mitropolsky (KBM) method has been carried out for deriving an approximation in terms of trigonometric functions. In the second approach, the proposed problem has been analyzed via the multiple scales method (MSM). In both the H-HPM/KBM method and MSM, only the first-order approximations have been derived because the first-order approximations are sufficient to obtain high accuracy and more stable solutions.

It was clear from the numerical results as well as the graphical analysis that all proposed methods give good results compared to the numerical approximations using the 4th-order Runge Kutta (RK4) method. Moreover, it was noted that all of the obtained approximations are characterized by high accuracy and are more stable for a long time. Note that in this investigation, we applied all mentioned approaches for solving and analyzing the nonlinear complex Duffing oscillators problem without decoupling.

Author contributions

All authors contributed equally and approved the final manuscript.

Conflicts of interest

The authors declare that they have no conflicts of interest.

Data Availability

All data generated or analyzed during this study are included in this published article (More details can be requested from El-Tantawy).

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