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Complementarity between quantum coherence and mixedness: a majorization approach

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Abstract

Quantum coherence is a relevant resource for various quantum information processing tasks, but it is fragile since it is generally affected by environmental noise. This is reflected in the loss of purity of the system, which in turn limits the amount of quantum coherence of it. As a consequence, a complementarity relation between coherence and mixedness arises. Previous works characterize this complementarity through inequalities between the ℓ_1-norm of coherence and linear entropy, and between the relative entropy of coherence and von Neumann entropy. However, coherence–mixedness complementarity is expected to be a general feature of quantum systems, regardless of the measures used. Here, an alternative approach to coherence–mixedness complementarity, based on majorization theory, is proposed. Vectorial quantifiers of coherence and mixedness, namely the coherence vector and the spectrum, respectively, are used, instead of scalar measures. A majorization relation for the tensor product of both vectorial quantifiers is obtained, capturing general aspects of the trade-off between coherence and mixedness. The optimal bound for qubit systems and numerical bounds for qutrit systems are analyzed. Finally, coherence–mixedness complementarity relations are derived for a family of symmetric, concave and additive functions. These results provide a deeper insight into the relation between quantum coherence and mixedness.

Keywords: coherence, mixedness, complementarity, majorization

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum coherence is a fundamental property of quantum states related with the superposition principle. Within the paradigm of quantum resource theories, the coherence can also be interpreted as a resource and therefore it can be converted, used and quantified [1, 2]. Particularly, quantum coherence is a resource in several areas of quantum information processing, including quantum algorithms, quantum thermodynamics, and quantum metrology (see, for instance, [3] and references therein). In realistic scenarios, such as open quantum systems, coherence is often affected by noise from the environment, due to the phenomenon of decoherence [4].

The purity of a quantum state is related with how much it deviates from the maximally mixed state. In the framework of the resource theory of purity [5], it is interpreted as a quantum resource. Since the purity of a quantum state can be affected by noise, which increases the mixedness (or impurity) of the state, the degree of mixedness of a quantum state can be cast as an indicator of decoherence.

In this context, it becomes relevant to address the question: how is the coherence of a quantum system limited by its mixedness? [6] A standard way of dealing with this problem is to provide measures of coherence and mixedness, and then study complementarity relations between the given quantifiers [6–9]. Two trade-off relations between coherence and mixedness were obtained for the first time in [6], one given in terms of the ℓ_1-norm of coherence (C_{ℓ_1}) and the linear entropy (S_L), and the other one in terms of the relative entropy of coherence ($C_{\rm re}$) and von Neumann entropy ($S_{\rm vN}$).

In order to state those relations in a precise form, let us consider a quantum system with an associated d-dimensional Hilbert space $\mathcal{H}_d$, described by a density operator ρ, and let us take the computational basis $\mathcal{B} = \{|i\rangle\}_{i=0}^{d-1}$ as the incoherent basis. Then, the trade-off relations proved by Singh *et al* in [6] read as

$$\frac{C_{\ell_1}^{\;2}(\rho)}{(d-1)^2} + \frac{d\, S_L(\rho)}{d-1} \leqslant 1, \tag{1}$$

$$\frac{C_{\rm re}(\rho)}{\ln d} + \frac{S_{\rm vN}(\rho)}{\ln d} \leqslant 1, \tag{2}$$

where the coherence measures C_{ℓ_1} and $C_{\rm re}$ are given by

$$C_{\ell_1}(\rho) = \sum_{\substack{i,j=0\\ i\neq j}}^{d-1} |\langle i|\rho|j\rangle|, \tag{3}$$

$$C_{\rm re}(\rho) = S_{\rm vN}(\Delta(\rho)) - S_{\rm vN}(\rho), \tag{4}$$

with $\Delta(\rho) = \sum_{i=0}^{d-1}\langle i|\rho|i\rangle\,|i\rangle\langle i|$, and the mixedness measures S_L and $S_{\rm vN}$ are given by

$$S_L(\rho) = 1 - {\rm Tr}(\rho^2), \tag{5}$$

$$S_{\rm vN}(\rho) = -{\rm Tr}(\rho\ln\rho), \tag{6}$$

with ${\rm Tr}(\cdot)$ denoting the trace of an operator. Inequalities (1) and (2) were also obtained in [7] and [8], respectively. Other trade-off relations between these quantities were also discussed in [9].

Clearly, (1) and (2) capture the complementarity between coherence and mixedness, since each term on the l.h.s. is separately upper bounded by one, but the sum of both terms, when computed for the same density operator, cannot exceed unity. This compromise, however, is not exclusive of the coherence and mixedness measures employed above: indeed, it is a manifestation of the coherence sensitivity to noise, which is expressed by a loss of purity of the quantum system. Therefore, in order to fully characterize this quantum phenomenon, a more fundamental approach should be considered, by using more general quantifiers to account for coherence and mixedness.

Motivated by the above observation and by the relevance of finding hidden relations between quantum resources [10], we propose an alternative approach to characterize coherence–mixedness complementarity, based on majorization theory [11]. This approach is inspired by the development of uncertainty inequalities where majorization uncertainty relations [12–18] result more fundamental than those based on scalar quantities such as variance or entropy [19–23].

More precisely, we consider vectorial quantifiers of coherence and mixedness, instead of scalar ones, namely: the generalized coherence vector and the vector of eigenvalues (or spectrum), respectively [24]. We observe the existence of a compromise, in the form of a majorization relation, for the tensor product of both vectors. This captures the trade-off between coherence and mixedness, as desired. The optimal bound for the qubit system, as well as numerical bounds for the qutrit system, are provided.

Finally, we show how to obtain scalar coherence–mixedness complementarity inequalities from our majorization relation, appealing to symmetric, concave, and additive functions. In particular, we discuss the case of the family of Rényi entropies.

Our results provide a deeper insight into the relation between quantum coherence and mixedness from a resource-theoretic perspective [25].

In section 2 we introduce the concepts and properties that will be necessary throughout the work, in particular, the vectorial quantifiers to account for coherence and mixedness. Section 3 contains our proposal for characterizing the coherence–mixedness complementarity, and a series of results are discussed thoroughly for the qubit case as well as for qutrit systems. Finally, we present our concluding remarks in section 4, whereas detailed calculations and derivations are included in the appendices.

2. Preliminaries

We present here a review of some basic aspects of majorization lattice theory and the resource theories of coherence and purity, that will be necessary to derive our results.

2.1. Majorization lattice theory

We consider probability vectors which belong to a d-dimensional space ($d \geqslant 2$), constituting the $(d-1)$-probability simplex given by

$$\Delta_d = \left\{u = (u_1,\ldots,u_d) \in \mathbb{R}^d:\ u_i \geqslant 0, \sum_{i=1}^{d} u_i = 1\right\}. \tag{7}$$

In addition, we consider the set $\Delta_d^{\downarrow} \subset \Delta_d$ of d-dimensional probability vectors with their entries in decreasing order, that is, $u^{\downarrow} \in \Delta_d^{\downarrow}$ is such that $1 \geqslant u_1^{\downarrow} \geqslant u_2^{\downarrow} \geqslant \ldots \geqslant u_d^{\downarrow} \geqslant 0$.

A majorization relation between two probability vectors can be defined as follows (see, e.g. [11]). Given $u, v \in \Delta_d$, we say that u is majorized by v, denoted as $u \preceq v$, if

$$s_j(u^{\downarrow}) \leqslant s_j(v^{\downarrow}), \quad \forall j = 1,\ldots,d-1, \tag{8}$$

where $u^{\downarrow}$ and $v^{\downarrow}$ are vectors in $\Delta_d^{\downarrow}$ with the same entries, respectively, as u and v in Δ_d but sorted in decreasing order; and $s_j(u^{\downarrow}) = \sum_{i=1}^{j} u_i^{\downarrow}$ is the jth partial sum built with the first j entries of $u^{\downarrow}$. Since we are dealing with probability vectors, for $j = d$ we trivially have $s_d(u) = 1 = s_d(v)$. In addition, we say that u is strictly majorized by v, denoted as $u \prec v$, if $u \preceq v$ but $u^{\downarrow} \neq v^{\downarrow}$.

There are several equivalent definitions of majorization. A graphical one, and useful for our purposes, is given in terms of Lorenz curves [26]: given a probability vector $u \in \Delta_d$, its Lorenz curve is the plot of the piecewise linear function

L_u: $[0, d] \to [0, 1]$ where, for each $k \in \{0,\ldots,d-1\}$ one has $L_u(t) = s_k(u^\downarrow) + u^\downarrow_{k+1}(t-k)$ when $t \in [k, k+1]$, with the convention $s_0(u) \equiv 0$. Notice that for any u, L_u is an increasing, concave function, with $L_u(0) = 0$ and $L_u(d) = 1$; therefore all Lorenz curves are increasing and concave, besides they start at the origin and finish at the point $(d, 1)$. In this way, we have the equivalence: $u \preceq v \Longleftrightarrow L_u(t) \leqslant L_v(t)\ \forall t \in [0, d]$. The last inequality gives us a graphical way of accounting for majorization between probability vectors of a given dimension.

From the order-theoretic viewpoint, the set $\Delta_d^\downarrow$ equipped with the majorization relation $\preceq$ is a complete lattice [27, 28]. This means that $\Delta_d^\downarrow$ is a partially ordered set (poset)[4] such that, for any subset $\mathcal{U} \subseteq \Delta_d^\downarrow$, there exist the infimum and the supremum of $\mathcal{U}$. We denote them as $\bigwedge\mathcal{U}$ and $\bigvee\mathcal{U}$, respectively. In addition, the majorization lattice is lower bounded by the bottom element $\mathbf{0}_d = (1/d,\ldots,1/d)$, and upper bounded by the top element $\mathbf{1}_d = (1, 0, \ldots, 0)$, where both vectors have dimension d.

There exist algorithms to obtain the infimum as well as the supremum of any subset of the majorization lattice [28–30] (see also [31] for majorization infimum and supremum over linear constraints). The infimum of a set $\mathcal{U} \subseteq \Delta_d^\downarrow$ can be computed as follows. First, consider the set of points $\{(j, \underline{S}_j)\}_{j=0}^{d}$, where $\underline{S}_j = \inf\{s_j(u)\colon u \in \mathcal{U}\}$ for each $j = 1, \ldots, d$ and $\underline{S}_0 \equiv 0$, and then build the polygonal curve given by linear interpolation of those points. This piecewise linear path is the Lorenz curve of $\bigwedge\mathcal{U}$, denoted as $L_{\bigwedge\mathcal{U}}$. Finally, the infimum is the vector given by $\bigwedge\mathcal{U} = (L_{\bigwedge\mathcal{U}}(1), L_{\bigwedge\mathcal{U}}(2) - L_{\bigwedge\mathcal{U}}(1), \ldots, L_{\bigwedge\mathcal{U}}(d) - L_{\bigwedge\mathcal{U}}(d-1))$.

The supremum of a set $\mathcal{U} \subseteq \Delta_d^\downarrow$ is obtained in a similar way, but it requires a further step. First, we calculate the polygonal curve given by the linear interpolation of the set of points $\{(j, \bar{S}_j)\}_{j=0}^{d}$, where $\bar{S}_j = \sup\{s_j(u)\colon u \in \mathcal{U}\}$ and $\bar{S}_0 \equiv 0$. If this polygonal chain is concave, then it is precisely the Lorenz curve of $\bigvee\mathcal{U}$, namely $L_{\bigvee\mathcal{U}}$. Otherwise, $L_{\bigvee\mathcal{U}}$ is built up from the upper envelope of the obtained polygonal chain[5]. Finally, the desired vector is $\bigvee\mathcal{U} = (L_{\bigvee\mathcal{U}}(1), L_{\bigvee\mathcal{U}}(2) - L_{\bigvee\mathcal{U}}(1), \ldots, L_{\bigvee\mathcal{U}}(d) - L_{\bigvee\mathcal{U}}(d-1))$.

We remark that $\bigwedge\mathcal{U}$ and $\bigvee\mathcal{U}$ may or may not belong to the set $\mathcal{U}$. When $\bigwedge\mathcal{U} \in \mathcal{U}$ ($\bigvee\mathcal{U} \in \mathcal{U}$), one says that the infimum (supremum) is a minimum (maximum).

We recall here the notion of Schur-concavity, exhibited by real-valued functions that antipreserve the majorization relation. More precisely, a function $f\colon \Delta_d \to \mathbb{R}$ is said to be Schur-concave if $f(u) \geqslant f(v)$ for all $u, v \in \Delta_d$ such that $u \preceq v$. Another relevant notion is connected to additivity: let $u \otimes v$ denote the tensor product of u and v, then a function f from a probability vectors space (of any dimension) to $\mathbb{R}$ is is said to be additive under the tensor product if $f(u \otimes v) = f(u) + f(v)$ (see e.g. [33]). Examples of Schur-concave and additive functions are the Rényi entropies: $R_\alpha(u) = \frac{1}{1-\alpha}\ln\left(\sum_{i=1}^{d} u_i{}^\alpha\right)$, for any nonnegative real parameter α, with $R_1(u) \equiv \lim_{\alpha\to 1} R_\alpha(u) = -\sum_{i=1}^{d} u_i \ln u_i$, which is the Shannon entropy.

2.2. Resource theory of coherence: coherence vector as vectorial quantifier of coherence

Let us consider a quantum system with a d-dimensional Hilbert space $\mathcal{H}_d$. We denote the set of quantum states as $\mathcal{S}(\mathcal{H}_d)$ and the subset of pure states as $\mathcal{P}(\mathcal{H}_d)$. Once the incoherent basis $\mathcal{B} = \{|i\rangle\}_{i=0}^{d-1}$ is fixed, the incoherent states are those quantum states whose density matrix is diagonal in the incoherent basis. More precisely, ρ is an incoherent state if, and only if, $\rho = \sum_{i=0}^{d-1} \lambda_i |i\rangle\langle i|$, with $\lambda_i \geqslant 0$ for $i = 0, \ldots, d-1$, and $\sum_{i=0}^{d-1} \lambda_i = 1$. We denote the set of incoherent states as $\mathcal{I}(\mathcal{H}_d)$. Any quantum state $\rho \in \mathcal{S}(\mathcal{H}_d)$ that does not belong to $\mathcal{I}(\mathcal{H}_d)$, is a coherent state. In particular, maximally coherent states are pure states of the form $U_{\mathrm{IO}}|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|\,U_{\mathrm{IO}}^\dagger$, where $|\psi_d^{\mathrm{mcs}}\rangle = \sum_{i=0}^{d-1}\frac{1}{\sqrt{d}}|i\rangle$ and $U_{\mathrm{IO}} = \sum_{i=0}^{d-1} \mathrm{e}^{\mathrm{i}\theta_i}|\pi(i)\rangle\langle i|$, with π a permutation and phase factors $\{\theta_i\}$.

Let us recall the definition of incoherent operations [2], which are the free operations of this resource theory. These operations have the property that coherence cannot be created from an incoherent state, not even in a probabilistic way. More precisely, a completely positive trace-preserving map $\Lambda\colon \mathcal{S}(\mathcal{H}_d) \to \mathcal{S}(\mathcal{H}_d)$ is an incoherent operation (IO) if it admits a representation in terms of Kraus operators $\{K_n\}_{n=1}^{N}$ such that $K_n\rho K_n^\dagger/\mathrm{Tr}(K_n\rho K_n^\dagger) \in \mathcal{I}$ for all $n = 1, \ldots, N$ and $\rho \in \mathcal{I}$.

We introduce the notion of coherence vector for any quantum state [24]. Firstly, for a pure sate $|\psi\rangle\langle\psi| \in \mathcal{P}(\mathcal{H}_d)$ the coherence vector is defined as:

$$\mu(|\psi\rangle\langle\psi|) = (|\langle 0|\psi\rangle|^2,\ldots,|\langle d-1|\psi\rangle|^2). \tag{9}$$

Clearly, $\mu(|\psi\rangle\langle\psi|) \in \Delta_d$, and the ordered coherence vector $\mu^\downarrow(|\psi\rangle\langle\psi|) \in \Delta_d^\downarrow$.

In general, for any given state ρ, let

$$\mathcal{D}(\rho) = \left\{\{q_k, |\psi_k\rangle\}_{k=1}^{M}\colon\ \rho = \sum_{k=1}^{M} q_k|\psi_k\rangle\langle\psi_k|\right\}, \tag{10}$$

be the set of all the pure-state decompositions of ρ. Also, let $\mathcal{U}^{\mathrm{psd}}(\rho) \subseteq \Delta_d^\downarrow$ be the set of probability vectors given by

$$\mathcal{U}^{\mathrm{psd}}(\rho) = \left\{\sum_{k=1}^{M} q_k\, \mu^\downarrow(|\psi_k\rangle\langle\psi_k|)\colon\ \{q_k, |\psi_k\rangle\}_{k=1}^{M} \in \mathcal{D}(\rho)\right\}. \tag{11}$$

Then, the coherence vector of the state ρ, denoted as $\nu(\rho)$, is defined as the supremum (with respect to the majorization relation) of the set $\mathcal{U}^{\mathrm{psd}}(\rho)$. More precisely, $\nu\colon \mathcal{S}(\mathcal{H}_d) \to \Delta_d^\downarrow$ is given by

$$\nu(\rho) = \bigvee\mathcal{U}^{\mathrm{psd}}(\rho). \tag{12}$$

[4] A partial order relation $\preceq$ over a set X is a binary relation over X and itself, such that, it is transitive and antisymmetric. A set with a partial order is said to be a partially ordered set (poset).

[5] We recall that the upper envelope of a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is defined as $\inf\{g\colon f \leqslant g$ and g is continuous and concave$\}$ (see e.g. [32], Def.4.1.5).

Clearly, for a pure state $|\psi\rangle\langle\psi|$ we have $\nu(|\psi\rangle\langle\psi|)=\mu^{\downarrow}(|\psi\rangle\langle\psi|)$. In addition, the coherence vector satisfies the following properties [24]:

(C_1) Bottom on maximally coherent states: $\rho = U_{\mathrm{IO}}|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|\,U_{\mathrm{IO}}^{\dagger}\Longleftrightarrow\nu(\rho)=\mathbf{0}_d$.

(C_2) Top on incoherent states: $\rho\in\mathcal{I}(\mathcal{H}_d)\Longleftrightarrow\nu(\rho)=\mathbf{1}_d$.

(C_3) Monotonicity under arbitrary incoherent operations: $\nu(\rho)\preceq\nu(\Lambda(\rho))$ for any incoherent operation Λ and any state ρ.

(C_4) Monotonicity under selective incoherent operations: $\nu(\rho)\preceq\sum_{n=1}^{N}p_n\,\nu(\sigma_n)$ for any state ρ and for any incoherent operation Λ with incoherent Kraus operators $\{K_n\}_{n=1}^{N}$, where $p_n=\mathrm{Tr}(\mathrm{K}_n\rho\mathrm{K}_n^{\dagger})$ and $\sigma_n=K_n\rho K_n^{\dagger}/p_n$.

(C_5) Convexity for pure-state decompositions: $\nu(\sum_{k=1}^{M}q_k|\psi_k\rangle\langle\psi_k|)\preceq\sum_{k=1}^{M}q_k\,\nu(|\psi_k\rangle\langle\psi_k|)$.

These properties provide the physical interpretation of the coherence vector ν as vectorial quantifier of coherence. Specifically, the coherence vector completely characterizes both maximally coherent and incoherent states. Besides, monotonicity under incoherent operations captures the intuition that coherence cannot be created for free.

In general, $\nu(\rho)$ can be difficult to calculate, since it involves an optimization problem over the set of pure-state decompositions of ρ. However, for qubit systems we have an analytical expression as follows. Let us consider a qubit system in a state $\rho=\frac{1}{2}(1+\vec{r}\cdot\vec{\sigma})$, with $\vec{r}=(r_x, r_y, r_z)$ the Bloch vector ($\|\vec{r}\|\leqslant 1$) and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ the vector formed by the Pauli matrices. Then, it can be shown that the coherence vector is $\nu(\rho)=\left(\frac{1+\sqrt{1-r_x^2-r_y^2}}{2},\frac{1-\sqrt{1-r_x^2-r_y^2}}{2}\right)$.

2.3. Resource theory of purity: spectrum as vectorial quantifier of mixedness

We consider the resource theory of purity introduced in [5], where the free operations of the theory are unital operations that preserve the dimension of the Hilbert space. More precisely, a completely positive trace-preserving map Λ: $\mathcal{S}(\mathcal{H}_d)\to\mathcal{S}(\mathcal{H}_d)$ is a unital operation (UO) if $\Lambda\left(\frac{1}{d}\mathbb{1}_d\right)=\frac{1}{d}\mathbb{1}_d$, where 1_d is the d-dimensional identity matrix. An example of an UO is a random unitary matrix operation, $\Lambda(\rho)=\sum_{n=1}^{N}p_nU_n\rho U_n^{\dagger}$, where U_n are $d\times d$ unitary matrices and $p=(p_1,\ldots,p_N)\in\Delta_N$. In this way, the only free state is the maximally mixed state $\frac{1}{d}\mathbb{1}_d$, which has maximal mixedness (or null purity). Any other quantum state has some degree of purity.

Let λ: $\mathcal{S}(\mathcal{H}_d)\to\Delta_d^{\downarrow}$ be the function that assigns to each state ρ the vector formed by its eigenvalues sorted in decreasing order, that is, $\lambda(\rho)$ is the spectrum of ρ. It can be shown that λ satisfies the following properties, which makes it suitable as a vectorial quantifier of mixedness:

(M_1) Bottom on maximally mixed state: $\rho=\frac{1}{d}\mathbb{1}_d\Longleftrightarrow\lambda(\rho)=\mathbf{0}_d$.

(M_2) Top on pure states: $\rho\in\mathcal{P}(\mathcal{H}_d)\Longleftrightarrow\lambda(\rho)=\mathbf{1}_d$.

(M_3) Monotonicity under arbitrary unital operations: $\lambda(\Lambda(\rho))\preceq\lambda(\rho)$ for any unital operation Λ and any state ρ.

(M_4) Monotonicity under selective unital operations: $\sum_{n=1}^{N}p_n\,\lambda(\sigma_n)\preceq\lambda(\rho)$ for any state ρ and for any unital operation Λ with Kraus operators $\{K_n\}_{n=1}^{N}$, where $p_n=\mathrm{Tr}(\mathrm{K}_n\rho\mathrm{K}_n^{\dagger})$ and $\sigma_n=K_n\rho K_n^{\dagger}/p_n$.

(M_5) Convexity: $\lambda(\sum_{k=1}^{M}q_k\rho_k)\preceq\sum_{k=1}^{M}q_k\,\lambda(\rho_k)$.

The spectrum λ is a vectorial quantifier of mixedness in the resource theory of purity. The spectrum univocally characterizes both pure states and the maximally mixed state. Moreover, monotonicity under unital operations captures the intuition that purity cannot freely be created.

3. Results

Our aim is to quantify how mixedness imposes limits to coherence in a quantum system, taking advantage of the coherence vector ν and the spectrum λ. To address this problem, we analyze different majorization constraints that arise when both vectors are considered simultaneously for a given quantum state.

3.1. The bottom for coherence and mixedness cannot be reached simultaneously by any state

Let $\rho\in\mathcal{S}(\mathcal{H}_d)$. Since $\nu(\rho),\lambda(\rho)\in\Delta_d^{\downarrow}$, the following majorization relations are satisfied:

$$\mathbf{0}_d\preceq\nu(\rho)\preceq\mathbf{1}_d, \tag{13}$$

and

$$\mathbf{0}_d\preceq\lambda(\rho)\preceq\mathbf{1}_d, \tag{14}$$

where the bounds are attainable. Moreover, we observe that the top vector is simultaneously attained, since for any incoherent pure state $|i\rangle\langle i|$, we have $\nu(|i\rangle\langle i|)=\mathbf{1}_d=\lambda(|i\rangle\langle i|)$. A natural question arises: is there a quantum state ρ such that $\nu(\rho)=\mathbf{0}_d=\lambda(\rho)$? Our first observation is that such a state does not exist.

Lemma 1. *There is no $\rho\in\mathcal{S}(\mathcal{H}_d)$ such that $\nu(\rho)=\lambda(\rho)=\mathbf{0}_d$, for any $d\geqslant 2$.*

Proof. Assume that there is a state $\rho\in\mathcal{S}(\mathcal{H}_d)$ such that $\nu(\rho)=\mathbf{0}_d=\lambda(\rho)$. By property ($C_1$), ρ is a maximally coherent state, that is, $\rho=U_{\mathrm{IO}}|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|\,U_{\mathrm{IO}}^{\dagger}$, whereas by property M_1, ρ is the maximally mixed state $\rho=\frac{1}{d}\mathbb{1}_d$. But this is not possible, since $U_{\mathrm{IO}}|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|\,U_{\mathrm{IO}}^{\dagger}\neq\frac{1}{d}\mathbb{1}_d$. Therefore, there is no $\rho\in\mathcal{S}(\mathcal{H}_d)$ such that $\nu(\rho)=\mathbf{0}_d=\lambda(\rho)$. □

An important remark follows: one might think that the result of lemma 1 does not guarantee a coherence–mixedness compromise, because it could be the case that there exists a sequence of quantum states for which the coherence vector and spectrum converge both to the bottom vector simultaneously. In lemma 2 below, we do show that this is not the case. Then, as a consequence of both lemmas, a vectorial majorization-based complementarity relation will be proposed.

3.2. Coherence–mixedness complementarity from a tensor-product majorization relation

As mentioned, we look for a trade-off between coherence and mixedness in a vectorial form. There are several ways to combine two probability vectors in order to obtain a new one. We consider the tensor product of both probability vectors for a given ρ acting on d-dimensional Hilbert space, namely:

$$\nu(\rho)\otimes\lambda(\rho)=(\nu_1(\rho)\lambda_1(\rho),\ldots,\nu_1(\rho)\lambda_d(\rho),\ldots,\nu_d(\rho)\lambda_1(\rho),\ldots,\nu_d(\rho)\lambda_d(\rho)). \tag{15}$$

The choice of the tensor product is inspired in the majorization uncertainty relations (MURs) [12–14, 17, 18]. When dealing with MURs, it is important to determine a nontrivial upper bound for the probability vectors associated with the measurements. In our case, as previously noted, a trivial upper bound does exist. However, it is unclear whether a nontrivial lower bound exists. We investigate here bounding the tensor-product (15) from below.

Let us introduce the following set of probability vectors

$$\mathcal{U}_d^{\otimes}=\{(\nu(\rho)\otimes\lambda(\rho))^{\downarrow}:\ \rho\in\mathcal{S}(\mathcal{H}_d)\}. \tag{16}$$

Clearly, $\mathcal{U}_d^{\otimes}\subseteq\Delta_{d^2}^{\downarrow}$. Therefore, there exist the infimum and the supremum of that set. In particular, the supremum is a maximum, since it is reached by any incoherent pure state $|i\rangle\langle i|$: $\bigvee\mathcal{U}_d^{\otimes}=\mathbf{1}_{d^2}$.

We show here that the infimum of that set, $\bigwedge\mathcal{U}_d^{\otimes}$, is different from the trivial lower bound, $\mathbf{0}_{d^2}$. As a consequence, a tensor-product majorization relation between the coherence vector and the spectrum is obtained. This relation captures the trade-off between coherence and mixedness.

Let us first prove the following lemma that, together with lemma 1, guarantees a coherence–mixedness complementarity relation:

Lemma 2. *There is no sequence of quantum states* $\{\rho_n\in\mathcal{S}(\mathcal{H}_d)\}_{n\in\mathbb{N}}$ *such that* $\nu_1(\rho_n)\lambda_1(\rho_n)\longrightarrow 1/d^2$ *as* $n\longrightarrow\infty$.

Proof. Let us assume that there is a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}g(\rho_n)=1/d^2$, with g: $\mathcal{S}(\mathcal{H}_d)\to\mathbb{R}$ given by $g(\rho)=\nu_1(\rho)\lambda_1(\rho)$. Since $\nu_1(\rho)$ and $\lambda_1(\rho)$ are continuous functions, g is continuous.

The set of density matrices $\mathcal{S}(\mathcal{H}_d)$ is compact and g is continuous, then its image Img is a compact subset of $\mathbb{R}$, in particular, it is closed.

The sequence $\{g(\rho_n)\}_{n\in\mathbb{N}}\subseteq$ Img is convergent to $1/d^2$ in a closed set, then $1/d^2\in$ Img. Therefore, there is a ρ^* such that $g(\rho^*)=1/d^2$.

If $g(\rho^*)=\nu_1(\rho^*)\lambda_1(\rho^*)=1/d^2$, then all the components of the tensor-product vector $\nu(\rho^*)\otimes\lambda(\rho^*)$ are equal to $1/d^2$, and thus $\nu(\rho^*)=\mathbf{0}_d=\lambda(\rho^*)$. But this is in contradiction with lemma 1. Therefore, we conclude that there is no sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\nu_1(\rho)\lambda_1(\rho)=1/d^2$. □

Now we are in a position to state and prove one of our main results with reference to the tensor product of the vectors that acquaint for coherence and mixedness:

Proposition 3. *The bottom element* $\mathbf{0}_{d^2}$ *is strictly majorized by the infimum* $\bigwedge\mathcal{U}_d^{\otimes}$, *that is,* $\mathbf{0}_{d^2}\prec\bigwedge\mathcal{U}_d^{\otimes}$.

Proof. Since $\mathbf{0}_{d^2}$ is the bottom element of $\Delta_{d^2}^{\downarrow}$, then $\mathbf{0}_{d^2}\preceq\bigwedge\mathcal{U}_d^{\otimes}$.

Let us assume that $\mathbf{0}_{d^2}=\bigwedge\mathcal{U}_d^{\otimes}$. Then, the first component of $\bigwedge\mathcal{U}_d^{\otimes}$ is equal to $1/d^2$.

From the definition of the infimum of the set $\mathcal{U}_d^{\otimes}$, we have that $(\bigwedge\mathcal{U}_d^{\otimes})_1=\inf\{\nu_1(\rho)\lambda_1(\rho):\ \rho\in\mathcal{S}(\mathcal{H}_d)\}$. Therefore $\inf\{\nu_1(\rho)\lambda_1(\rho):\ \rho\in\mathcal{S}(\mathcal{H}_d)\}=1/d^2$, which implies that there is a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\nu_1(\rho_n)\lambda_1(\rho_n)=1/d^2$. But, due to lemma 2, this is not possible. Therefore, we conclude that $\bigwedge\mathcal{U}_d^{\otimes}\neq\mathbf{0}_{d^2}$, and thus $\mathbf{0}_{d^2}\prec\bigwedge\mathcal{U}_d^{\otimes}$. □

As a consequence of proposition 3, we obtain the following important coherence–mixedness complementarity relation, in a vectorial form:

$$\mathbf{0}_{d^2}\prec\bigwedge\mathcal{U}_d^{\otimes}\preceq\nu(\rho)\otimes\lambda(\rho). \tag{17}$$

This result implies that for both vectorial quantifiers, the coherence vector and the spectrum, there is no sequence of quantum states that converges to the bottom simultaneously.

3.3. Optimal lower bound for qubit systems: analytical result

In general, obtaining an analytical expression for the infimum of $\mathcal{U}_d^{\otimes}$ is a hard task. However, for qubit systems ($d=2$), we have been able to solve the optimization problem.

Proposition 4. *For a qubit system, the infimum of the tensor product for coherence and spectrum vectors is*

$$\bigwedge\mathcal{U}_2^{\otimes}=\left(\frac{1}{2},\frac{\sqrt{2}}{4},\frac{1}{8},\frac{3}{8}-\frac{\sqrt{2}}{4}\right). \tag{18}$$

Proof. See appendix A. □

We remark that the vector 18 is an infimum, although not a minimum of the tensor-product set in the qubit case. In other words, there is no quantum state such that the tensor product between the coherence vector and the spectrum for qubits is equal to the infimum $\bigwedge\mathcal{U}_2^{\otimes}$. However, this should not be considered as a drawback of the majorization approach. Indeed, a similar situation usually arises in the case of majorization uncertainty relations, where states corresponding to minimal uncertainty in general do not exist [12].

For the sake of illustration of the tensor-product majorization relation, we take as an example the maximally coherent state, $|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|$, going through a depolarizing

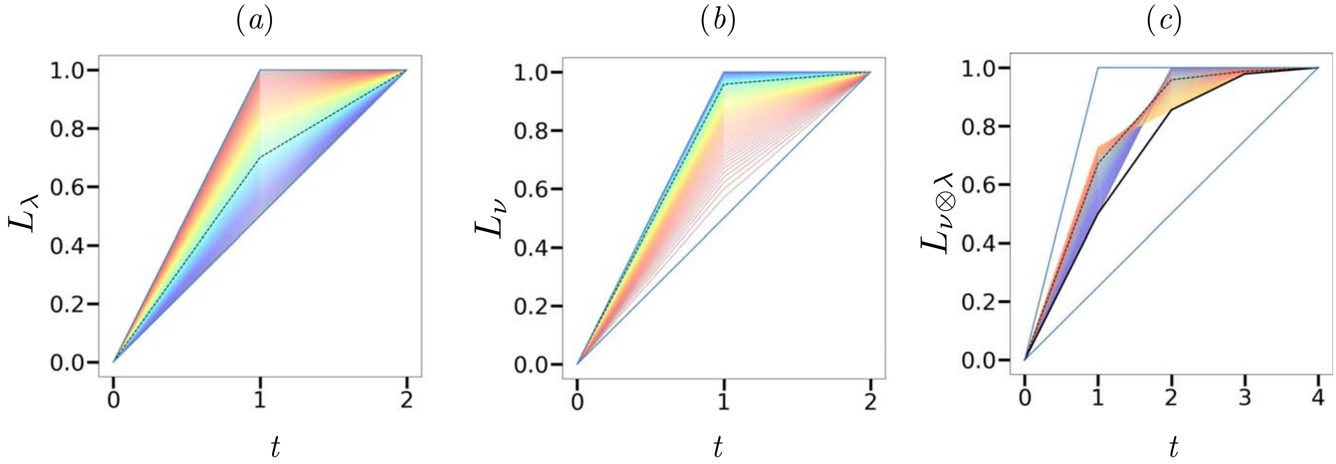

Figure 1. The Lorenz curves of (a) $\lambda(\rho_p)$, (b) $\nu(\rho_p)$, and (c) $\nu(\rho_p)\otimes\lambda(\rho_p)$) for 100 qubit maximally coherent mixed states are plotted. The values of p are taken equally spaced from the interval [0, 1] (the color map indicates p increasing for blue to red colors). For the sake of clarity, these Lorenz curves are highlighted (dashed lines) for a particular state. The Lorenz curves of the bottom and top elements are also depicted (light blue lines). In addition, the Lorenz curve of the infimum $\bigwedge\mathcal{U}_2^{\otimes} = (1/2,\ \sqrt{2}/4,\ 1/8,\ 3/8 - \sqrt{2}/4)$, which coincides with the one of $\bigwedge\mathcal{U}_3^{\otimes\ \mathrm{mcms}}$, is depicted ((c) black line).

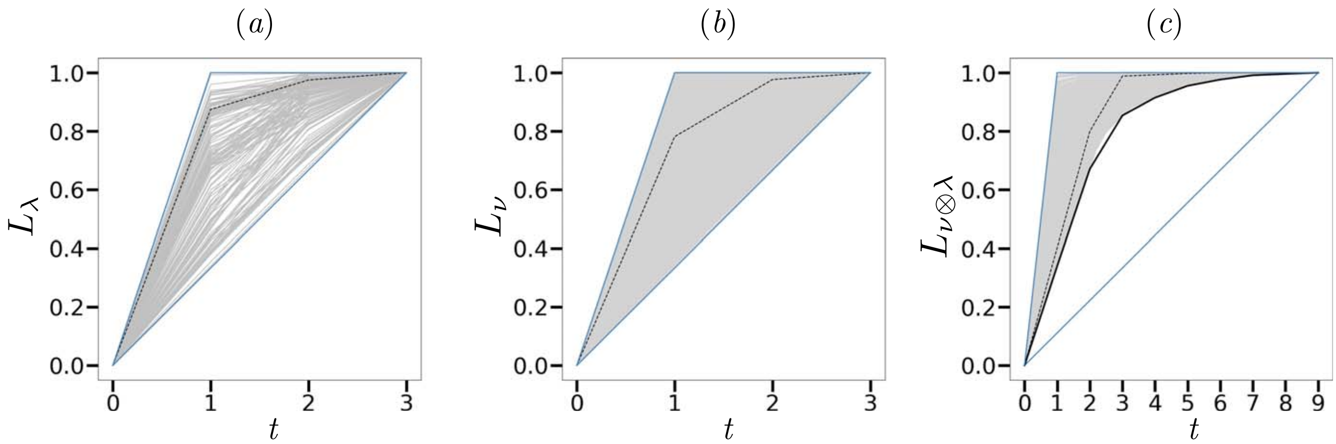

Figure 2. The Lorenz curves of (a) $\lambda(\rho)$, (b) $\nu(\rho)$, and (c) $\nu(\rho)\otimes\lambda(\rho)$ for $N\,M$ qutrit states (randomly sorted as is indicated in the main text) are plotted (gray lines). For the sake of clarity, these Lorenz curves are highlighted (dashed lines) for a particular state. The Lorenz curves of the bottom and top elements are also depicted (light blue lines). The coherence–mixedness complementarity relation for qutrit systems can be seen from (c) where the Lorenz curve of the infimum $\bigwedge\mathcal{U}_3^{\otimes\ \mathrm{rdm}} \approx (0.3396, 0.3307, 0.1833, 0.0610, 0.0405, 0.0215, 0.0147, 0.0046, 0.0041)$ (black line) is above to the Lorenz curve of the bottom $\mathbf{0}_9$.

channel Λ_p with depolarizing probability $p\in[0, 1]$. The state at the output of the channel is given by

$$\rho_p = \Lambda_p(|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|) = p\frac{\mathbb{1}_d}{d} + (1-p)|\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|. \tag{19}$$

These states are called maximally coherent mixed states, and they saturate inequality (1) (see [6]).

In figure 1, we depict the Lorenz curves of $\lambda(\rho_p)$, $\nu(\rho_p)$ and $\nu(\rho_p)\otimes\lambda(\rho_p)$, for different qubit maximally coherent mixed states, varying the parameter p. In this case, for the spectrum as well as for the coherence vector separately, we can observe that the lowest Lorenz curves coincide with the Lorenz curve of the bottom element $\mathbf{0}_2$. However, for the tensor product between the spectrum and coherence vector, all Lorenz curves lie above the one corresponding to the infimum $\bigwedge\mathcal{U}_2^{\otimes} = \left(\frac{1}{2}, \frac{\sqrt{2}}{4}, \frac{1}{8}, \frac{3}{8} - \frac{\sqrt{2}}{4}\right)$, which, in turn, is strictly above the Lorenz curve of the bottom $\mathbf{0}_4$.

Interestingly enough, for a qubit system, it can be shown that the infimum of the set $\mathcal{U}_2^{\otimes}$ is equal to the infimum of $\mathcal{U}_2^{\otimes\ \mathrm{mcms}} = \{(\nu(\rho_p)\otimes\lambda(\rho_p))^{\downarrow}\colon p\in[0, 1]\}$.

3.4. Bounds for qutrit systems: numerical results

For a qutrit system we provide a numerical analysis of the majorization-based coherence–mixedness complementarity relation. In particular, we numerically obtain the infimum $\bigwedge\mathcal{U}_3^{\otimes}$.

The procedure employed is as follows. We start by sorting N spectra $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ uniformly distributed over the 2-dimensional simplex. For each spectrum λ we generate

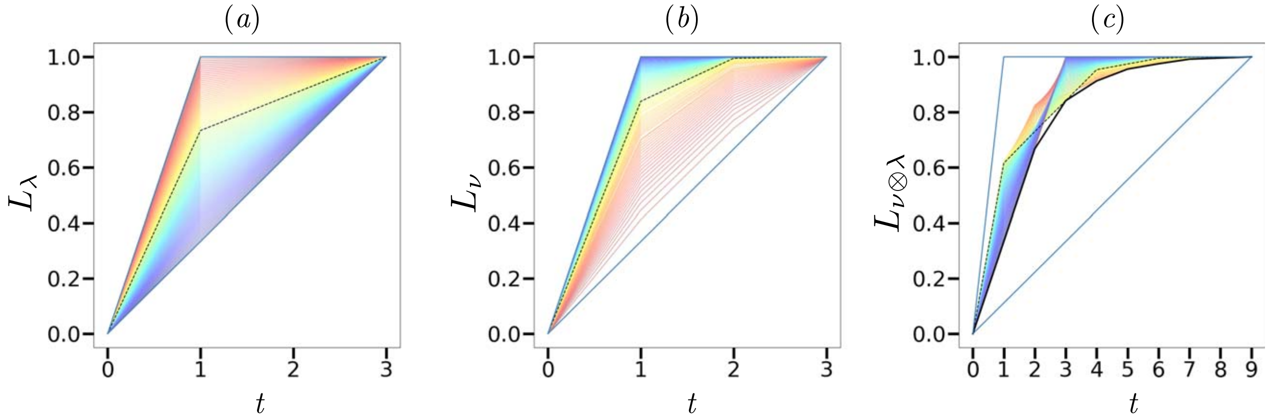

Figure 3. The Lorenz curves of (a) $\lambda(\rho_p)$, (b) $\nu(\rho_p)$, and (c) $\nu(\rho_p) \otimes \lambda(\rho_p)$ for 100 qutrit maximally coherent mixed states are plotted. The values of p are taken equally spaced from the range [0, 1] (the color map indicates p increasing for blue to red colors). For the sake of clarity, these Lorenz curves are highlighted (dashed lines) for a particular state. The Lorenz curves of the bottom and top elements are also depicted (light blue lines). In addition, the Lorenz curve of the infimum $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}} \approx (0.3333, 0.3333, 0.1747, 0.0714, 0.0428, 0.0194, 0.0166, 0.0042, 0.0042)$((c) black line) is depicted. For qutrit systems, it seems to be sufficient to consider the set of maximally coherent mixed states in order to estimate the infimum of $\bigwedge \mathcal{U}_3^{\otimes}$.

M quantum states $\rho = U \mathrm{diag}(\lambda) U^\dagger$, where U is a unitary matrix randomly generated according to the Haar measure, and diag(λ) stands for the 3×3 matrix with the spectrum elements in the diagonal and 0 elsewhere. In this way, we obtain a set of NM qutrit random states, which we denote as $\mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$. Then, for each quantum state $\rho \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$, we compute numerically its coherence vector $\nu(\rho)$ (see appendix B for details). With these ingredients, we calculate the tensor product $\nu(\rho) \otimes \lambda(\rho)$ for all $\rho \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$, obtaining the set $\mathcal{U}_3^{\otimes\,\mathrm{rdm}} = \{(\nu(\rho) \otimes \lambda(\rho))^{\downarrow}: \rho \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)\}$. The infimum of this set, $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}}$, is an upper bound, and a suitable approximation, of $\bigwedge \mathcal{U}_3^{\otimes}$; that is, $\bigwedge \mathcal{U}_3^{\otimes} \lessapprox \bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}}$.

In figure 2, we show the Lorenz curves of $\lambda(\rho)$, $\nu(\rho)$ and $\nu(\rho) \otimes \lambda(\rho)$ for each $\rho \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$. The procedure to obtain the Lorenz curve corresponding to the infimum $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}}$, consists of connecting with line segments the set of points $\{(k, \min_{\rho \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)} s_k(\nu(\rho) \otimes \lambda(\rho)))\}_{k=0}^{9}$; in other words, we pick the lowest partial sum s_k for each k to construct this special broken line. We observe in figures 2(a) and (b) that the Lorenz curves of $\lambda(\rho)$ and $\nu(\rho)$ are trivially bounded from below as given in relations (13) and (14), whereas the Lorenz curves of the tensor product $\nu(\rho) \otimes \lambda(\rho)$ shown in figure 2(c) are not trivially lower-bounded. In particular, we obtain that $\mathbf{0}_9 \prec \bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}}$, which is indeed a manifestation of the coherence–mixedness complementary relation from a majorization approach.

Another approach to the problem is the following. We numerically calculate $\lambda(\rho_p)$ and $\nu(\rho_p)$ for different qutrit maximally coherent mixed states ρ_p, that is, states of the form given in (19) with $d = 3$. For these vectors we obtain the set $\mathcal{U}_3^{\otimes\,\mathrm{mcms}} = \{(\nu(\rho_p) \otimes \lambda(\rho_p))^{\downarrow}: p \in [0, 1]\}$ and its infimum $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}}$. In figures 3(a)–(c), we plot the Lorenz curves of $\lambda(\rho_p)$, $\nu(\rho_p)$ and $\nu(\rho_p) \otimes \lambda(\rho_p)$, for 100 equally spaced values of p in the interval [0, 1]. The Lorenz curve of $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}}$ is obtained as the linear interpolation of the set of points $\{(k, \min_{\rho_p} s_k(\nu(\rho_p) \otimes \lambda(\rho_p)))\}_{k=0}^{9}$, for which we obtain $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}}$. As before, the Lorenz curves of $\lambda(\rho_p)$ and $\nu(\rho_p)$ have trivial bounds from below, but the Lorenz curves of $\nu(\rho_p) \otimes \lambda(\rho_p)$ are not trivially lower bounded.

We can observe that $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}}$ and $\bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}}$ are similar probability vectors, indeed $\| \bigwedge \mathcal{U}_3^{\otimes\,\mathrm{rdm}} - \bigwedge \mathcal{U}_3^{\otimes\,\mathrm{mcms}} \|_2 \approx 0.01$. This is in agreement with what was obtained in the qubit case: it is sufficient to consider the set of qutrit maximally coherent mixed states in order to estimate the infimum of $\bigwedge \mathcal{U}_3^{\otimes}$.

Finally, we state a conjecture for the greatest component of $\nu(\rho) \otimes \lambda(\rho)$ that allows us to obtain a nonoptimal but nontrivial bound for the tensor-product majorization relation, which depends only on the dimension of the Hilbert space.

Conjecture 5. Let $\rho \in \mathcal{S}(\mathcal{H}_d)$.

Then

$$\nu_1(\rho)\, \lambda_1(\rho) \geqslant \frac{1}{d}. \tag{20}$$

If this conjecture is true, it follows that

$$\mathbf{0}_d \prec \boldsymbol{b}_d \preceq \bigwedge \mathcal{U}_d^{\otimes} \preceq \nu(\rho) \otimes \lambda(\rho), \tag{21}$$

with $\boldsymbol{b}_d = \left(\frac{1}{d}, \frac{d-1}{d(d^2-1)}, \ldots, \frac{d-1}{d(d^2-1)}\right)$.

For $d = 2$ the conjecture is clearly valid, whereas for $d = 3$ we have numerical evidence of its validity. In figure 4, we plot $\nu_1(\rho_m)\lambda_1(\rho_m)$ for all the qutrit states randomly generated, that is, $\rho_m \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$; we did not find any counterexample that violates inequality (20).

3.5. From vectorial to scalar coherence–mixedness complementarity relations

Finally, we discuss how to obtain from the previous results scalar coherence–mixedness complementarity relations,

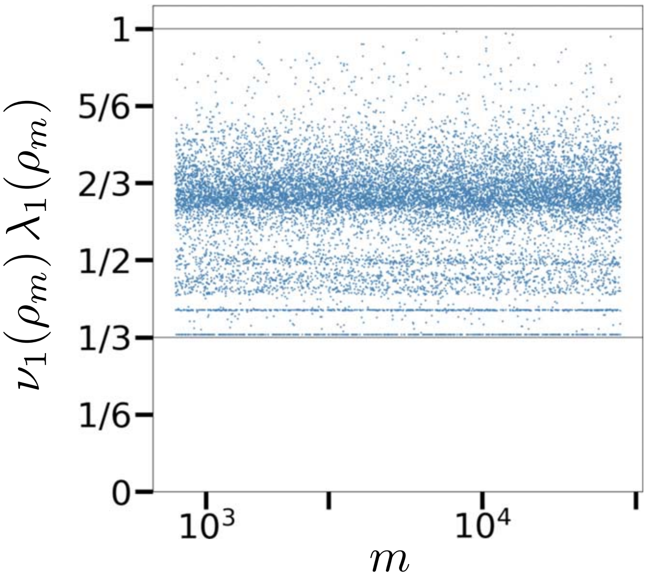

Figure 4. Plot of the points $(m,\ \nu_1(\rho_m)\lambda_1(\rho_m))$ for all the qutrit states randomly generated, that is, $\rho_m \in \mathcal{S}^{\mathrm{rdm}}(\mathcal{H}_3)$. There are no points below the straight line $\nu_1(\rho)\lambda_1(\rho) = 1/3$.

similar to inequalities (1) and (2). More precisely, for concave and additive functions we have the following result.

Corollary 6. *Let* $\rho \in \mathcal{S}(\mathcal{H}_d)$ *and let* $f\colon \mathbb{R}^d \to \mathbb{R}$ *be a symmetric, strictly concave and additive function such that* $\mathrm{argmax}_{u\in\Delta_d} f(u) = \mathbf{0}_d$ *and* $f(\mathbf{1}_d) = 0$. *Then*

$$\frac{f(\nu(\rho))}{f(\mathbf{0}_d)} + \frac{f(\lambda(\rho))}{f(\mathbf{0}_d)} \leqslant \frac{f(\bigwedge\mathcal{U}_d^{\otimes})}{f(\mathbf{0}_d)} < 2, \tag{22}$$

where $f(\nu(\rho))$ *is a coherence monotone and* $f(\lambda(\rho))$ *is a mixedness measure.*

Proof. Since f is symmetric and strictly concave, f is strictly Schur-concave and continuous [11]. Then, $f(\nu(\rho))$ is a coherence monotone and $f(\lambda(\rho))$ is a mixedness measure [5, 24].

Given a state $\rho \in \mathcal{S}(\mathcal{H}_d)$, from the majorization relation (17), the strictly Schur-concavity and the additive of the function f, we have

$$f(\bigwedge\mathcal{U}_d^{\otimes}) \geqslant f(\nu(\rho)\otimes\lambda(\rho)) \tag{23}$$

$$= f(\nu(\rho)) + f(\lambda(\rho)). \tag{24}$$

Dividing by $f(\mathbf{0}_d)$ we obtain the first inequality of (22). On the other hand, from proposition 3 and the strictly Schur-concavity of f, we have $f(\bigwedge\mathcal{U}_d^{\otimes}) < f(\mathbf{0}_{d^2})$. Finally, from the additivity of f and taking into account that $\mathbf{0}_{d^2} = \mathbf{0}_d \otimes \mathbf{0}_d$, we obtain $\frac{f(\bigwedge\mathcal{U}_d^{\otimes})}{f(\mathbf{0}_d)} < 2$. □

Corollary 7. *There is no sequence* $\{\rho_n \in \mathcal{S}(\mathcal{H}_d)\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty}\frac{f(\nu(\rho_n))}{f(\mathbf{0}_d)} + \frac{f(\lambda(\rho_n))}{f(\mathbf{0}_d)} = 2$.

Proof. Let us assume that there is a sequence $\{\rho_n \in \mathcal{S}(\mathcal{H}_d)\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\frac{f(\nu(\rho_n))}{f(\mathbf{0}_d)} + \frac{f(\lambda(\rho_n))}{f(\mathbf{0}_d)} = 2$. Then, $\lim_{n\to\infty} f(\nu(\rho_n)\otimes\lambda(\rho_n)) = 2f(\mathbf{0}_d) = f(\mathbf{0}_d^2)$. Since $\{\nu(\rho_n)\otimes\lambda(\rho_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^{d^2}$ is a bounded sequence, then, due to Bolzano–Weierstrass theorem, there exists a convergent subsequence $\{\nu(\rho_{n_j})\otimes\lambda(\rho_{n_j})\}_{j\in\mathbb{N}}$, that is, $\lim_{j\to\infty}\nu(\rho_{n_j})\otimes\lambda(\rho_{n_j}) = u \in \Delta_{d^2}$. Since f is continuous, $f(\lim_{j\to\infty}\nu(\rho_{n_j})\otimes\lambda(\rho_{n_j})) = \lim_{j\to\infty} f(\nu(\rho_{n_j})\otimes\lambda(\rho_{n_j}))$. Then, $f(u) = f(\mathbf{0}_d^2)$. Since f is strictly Schur-concave, $u = \mathbf{0}_d^2$. Thus, $\lim_{j\to\infty}\nu(\rho_{n_j})\otimes\lambda(\rho_{n_j}) = \mathbf{0}_d^2 = (1/d^2,\ldots,1/d^2)$. But this is in contradiction with lemma 2. Therefore, we conclude that there is no sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\frac{f(\nu(\rho_n))}{f(\mathbf{0}_d)} + \frac{f(\lambda(\rho_n))}{f(\mathbf{0}_d)} = 2$. □

Clearly, both terms of the l.h.s. of (22) are lower bounded by 1, but the sum cannot exceed $\frac{f(\bigwedge\mathcal{U}_d^{\otimes})}{f(\mathbf{0}_d)}$, which is strictly lower than 2. The fact that there is no sequence of quantum states such that the l.h.s. of (22) converges to 2 guarantees the existence of a scalar coherence–mixedness complementarity relation for each function f satisfying the hypotheses of corollary 6.

With our method we provide a family of scalar coherence–mixedness complementarity relations that are different from the previous ones given in the literature [6–9], as inequalities (1) and (2) and others, which have been obtained one by one.

As an example, we consider the family of Rényi entropies, R_α, with entropic parameter $\alpha \in [0,\ 1]$, which satisfies the hypotheses of corollary 6. In particular, for qubit system, we obtain that

$$\frac{R_\alpha(\nu(\rho))}{\ln 2} + \frac{R_\alpha(\lambda(\rho))}{\ln 2} \leqslant \frac{R_\alpha\left(\frac{1}{2}, \frac{\sqrt{2}}{4}, \frac{1}{8}, \frac{3}{8} - \frac{\sqrt{2}}{4}\right)}{\ln 2} < 2, \tag{25}$$

which is a scalar coherence–mixedness complementarity relation.

4. Concluding remarks

We have tackled the problem of quantifying the complementarity relation between quantum coherence and mixedness in quantum systems, by appealing to majorization theory.

Our main result is the existence of a nontrivial majorization relation involving the tensor product between the coherence vector and the spectrum of a quantum system. This is a vectorial coherence–mixedness complementarity relation, alternative to those based on scalar quantifiers. In the case of the qubit system, we have obtained analytically the optimal bound for the tensor product of both vectors. In addition, we have provided numerical results for qutrit systems taking advantage of the notion of Lorenz curves.

Moreover, we have shown how to obtain scalar coherence–mixedness complementarity inequalities from the majorization relation and appealing to symmetric, concave, and additive functions. In particular, the case of measures in terms of Rényi entropies have been analyzed.

Our results provide a deeper insight into the relation between quantum coherence and mixedness, and take a step forward in the direction of finding hidden trade-off relations between quantum resources.

Acknowledgments

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Appendix A. Proof of proposition 4

Let us consider a qubit system in a state $\rho = \frac{1}{2}(1 + \vec{r}\cdot\vec{\sigma})$, with $\vec{r} = (r_x, r_y, r_z)$ the Bloch vector ($\|\vec{r}\| = \sqrt{r_x^2 + r_y^2 + r_z^2} \leqslant 1$), and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the vector formed by the Pauli matrices.

The coherence vector of ρ is equal to $\nu(\rho) = \left(\frac{1+b}{2}, \frac{1-b}{2}\right)$, with $b = \sqrt{1-u^2}$ and $u = \sqrt{r_x^2 + r_y^2}$, and the spectrum is given by $\lambda(\rho) = \left(\frac{1+\|\vec{r}\|}{2}, \frac{1-\|\vec{r}\|}{2}\right)$.

Accordingly, $\nu(\rho)\otimes\lambda(\rho)$ is a four-dimensional probability vector given by

$$\nu(\rho)\otimes\lambda(\rho) = \left(\frac{1+b}{2}\frac{1+\|\vec{r}\|}{2}, \frac{1+b}{2}\frac{1-\|\vec{r}\|}{2}, \frac{1-b}{2}\frac{1+\|\vec{r}\|}{2}, \frac{1-b}{2}\frac{1-\|\vec{r}\|}{2}\right). \tag{A.1}$$

The entries of $\nu(\rho)\otimes\lambda(\rho)$ are not necessarily decreasingly ordered. However, we have $(\nu(\rho)\otimes\lambda(\rho))_1^{\downarrow} = \nu_1(\rho)\lambda_1(\rho)$ and $(\nu(\rho)\otimes\lambda(\rho))_4^{\downarrow} = \nu_2(\rho)\lambda_2(\rho)$.

In order to obtain the infimum $\bigwedge\mathcal{U}_2^{\otimes}$, we have to minimize the partial sums of $(\nu(\rho)\otimes\lambda(\rho))^{\downarrow}$, that is, $s_k = \sum_{i=1}^{k}(\nu(\rho)\otimes\lambda(\rho))_i^{\downarrow}$. Therefore, we have to solve minimization problems of the form

$$\underset{u,r_z}{\text{minimize}} \quad s_k(u, r_z) \tag{A.2a}$$

$$\text{subject to} \quad \sqrt{u^2 + r_z^2} \leqslant 1, \tag{A.2b}$$

for $k = 1, 2, 3$.

A.1. Minimum of s_1

In this case, the objective function to minimize is $s_1(u, r_z) = \nu_1(\rho)\lambda_1(\rho) = \frac{1+\sqrt{1-u^2}}{2}\frac{1+\sqrt{u^2+r_z^2}}{2}$.

For all u and r_z satisfying the constraint, we have

$$\begin{aligned} s_1(u, r_z) &\geqslant s_1(u, 0) \\ &= \frac{1+\sqrt{1-u^2}}{2}\frac{1+u}{2} \\ &\geqslant s_1(0, 0) = s_1(1, 0) = \frac{1}{2}. \end{aligned} \tag{A.3}$$

This value is reached at Bloch vectors $\vec{r} = (0, 0, 0)$ (maximally mixed state) and $\vec{r} = (r_x, r_y, 0)$, with $r_x^2 + r_y^2 = 1$ (maximally coherent states).

A.2. Minimum of s_2

In this case, the objective function to minimize is $s_2(u, r_z) = \nu_1(\rho)\lambda_1(\rho) + \max\{\nu_1(\rho)\lambda_2(\rho), \nu_2(\rho)\lambda_1(\rho)\} = \frac{1+\sqrt{\max\{1-u^2, u^2+r_z^2\}}}{2}$.

For all u and r_z satisfying the constraints, we have

$$\begin{aligned} s_2(u, r_z) &\geqslant s_2(u, 0) \\ &= \frac{1+\sqrt{\max\{1-u^2, u^2\}}}{2} \\ &\geqslant s_2\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{1}{2}\left(1 + \frac{\sqrt{2}}{2}\right). \end{aligned} \tag{A.4}$$

This value is reached at Bloch vectors $\vec{r} = (r_x, r_y, 0)$, with $r_x^2 + r_y^2 = 1/2$.

A.3. Minimum of s_3

In this case, the objective function to minimize is $s_3(u, r_z) = \frac{1+\sqrt{1-u^2}}{2} + \frac{1-\sqrt{1-u^2}}{2}\frac{1+\sqrt{u^2+r_z^2}}{2}$.

For all u and r_z satisfying the constraints, we have

$$\begin{aligned} s_3(u, r_z) &\geqslant s_3(u, 0) \\ &= \frac{1+\sqrt{1-u^2}}{2} + \frac{1-\sqrt{1-u^2}}{2}\frac{1+u}{2} \\ &\geqslant s_3\left(\frac{1}{\sqrt{2}}, 0\right) = \frac{5+2\sqrt{2}}{8}. \end{aligned} \tag{A.5}$$

This value is reached at Bloch vectors $\vec{r} = (r_x, r_y, 0)$, with $r_x^2 + r_y^2 = 1/2$.

Appendix B. Numerical calculation of the coherence vector

In this appendix, we review how to numerically obtain the coherence vector $\nu(\rho) = (\nu_1(\rho),\ldots,\nu_d(\rho))$ of a quantum sate ρ of a d-dimensional Hilbert space.

First, we recall the Schrödinger mixture theorem [34]: any ensemble $\{p_k, |\phi_k\rangle\}_{k=1}^{M}$ is a pure-state decomposition of ρ if, and only if, there exists an $M\times M$ unitary matrix V such that

$$|\phi_k\rangle = \frac{1}{\sqrt{p_k}}\sum_{j=1}^{d}\sqrt{\lambda_j}V_{k,j}|e_j\rangle, \tag{B.1}$$

where λ_j and $|e_j\rangle$ are the eigenvalues and eigenvectors of ρ, respectively. There may be more elements in the ensemble than eigenvectors, that is, $M > d$. If this is the case, only the first d columns of V appear in equation (B.1). The remaining $M-d$ columns are just added so that V is a unitary matrix.

The coherence vector of a state ρ is defined as $\nu(\rho) = \bigvee \mathcal{U}^{\rm psd}(\rho)$, with $\mathcal{U}^{\rm psd}(\rho)$ given by

$$\mathcal{U}^{\rm psd}(\rho) = \left\{ \sum_{k=1}^{M} q_k\ \mu^{\downarrow}(|\psi_k\rangle\langle\psi_k|):\ \ \{q_k, |\psi_k\rangle\}_{k=1}^{M} \in \mathcal{D}(\rho) \right\}. \tag{B.2}$$

According to the Schrödinger mixture theorem, we can generate different pure-state decompositions of a given ρ by choosing a different unitary matrix V. In particular, for each pure-state decomposition $\{q_k, |\psi_k\rangle\}_{k=1}^{M}$ of ρ, generated from a unitary matrix V, we define the d-dimensional probability vector $\tilde{\nu}(V;\rho) = (\tilde{\nu}_1(V;\rho),\ldots,\tilde{\nu}_d(V;\rho))$, as follows

$$\tilde{\nu}_{i+1}(V;\rho) = \sum_{k=1}^{M} \left| \sum_{j=1}^{d} \sqrt{\lambda_j}\, V_{k,j} \langle i|e_j\rangle \right|^2, \tag{B.3}$$

with $i = 0, \ldots, d-1$. In general, $\tilde{\nu}(V;\rho)$ is not ordered. Rearranging it in decreasing order denoted as $\tilde{\nu}^{\downarrow}(V;\rho)$, we can rewrite the set $\mathcal{U}^{\rm psd}(\rho)$ as $\bigcup_{M\in\mathbb{N}} \mathcal{U}_M^{\rm psd}(\rho)$, where

$$\mathcal{U}_M^{\rm psd}(\rho) = \{\tilde{\nu}^{\downarrow}(V;\rho):\ \ V \in U(M)\}, \tag{B.4}$$

with $U(M)$ the set of $M \times M$ unitary matrices.

According to the algorithm of the supremum (see section 2 or [28]), we have to solve $d-1$ maximization problems: $\bar{S}_k = \max\{s_k(u):\ u \in \mathcal{U}^{\rm psd}(\rho)\}$, with $k = 1, \ldots, d-1$, where $s_k(u) = \sum_{j=1}^{k} u_j$. For numerical considerations, we consider only pure-state decompositions with $M = d$ elements. We will numerically obtain $\tilde{S}_k = \max\{s_k(u):\ u \in \mathcal{U}_d^{\rm psd}(\rho)\}$, which is a lower bound of $\bar{S}_k$.

By fixing a parametrization of the unitary matrix V in terms of $K = d^2$ independent real parameters $\alpha = (\alpha_1,\ldots,\alpha_K) \in \mathcal{A} \subset \mathbb{R}^K$, with $\mathcal{A}$ the domain where the parameters belong, we can numerically solve the following $d-1$ maximization problems:

$$\tilde{S}_1 = \max_{\alpha\in\mathcal{A}} \tilde{\nu}_1^{\downarrow}(V(\alpha);\rho), \tag{B.5a}$$

$$\tilde{S}_2 = \max_{\alpha\in\mathcal{A}} [\tilde{\nu}_1^{\downarrow}(V(\alpha);\rho) + \tilde{\nu}_2^{\downarrow}(V(\alpha);\rho)], \tag{B.5b}$$

$$\vdots$$

$$\tilde{S}_{d-1} = \max_{\alpha\in\mathcal{A}} [\tilde{\nu}_1^{\downarrow}(V(\alpha);\rho) + \cdots + \tilde{\nu}_{d-1}^{\downarrow}(V(\alpha);\rho)]. \tag{B.5c}$$

Finally, as it was explained in section 2, the coherence vector is obtained from the upper envelope of the linear interpolation of the points $\{(j, \bar{S}_j)\}_{j=0}^{d}$ (with $\bar{S}_0 = 0$ and $\bar{S}_d = 1$). Using $\tilde{S}_k$ instead of $\bar{S}_k$, we obtain a suitable lower bound of the coherence vector, which is enough for our purposes.

To obtain the coherence vectors of figures 2(b) and 3(b), which corresponds to the case of the qutrit ($d = 3$), we have used the polarization parametrization, with $K = 9$ parameters, given in [35]. For higher dimensions, one can use the parametrization given in [36]. The corresponding maximization problems were solved numerically using the differential evolution method for global optimization [37].

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