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Analysis of the seventh-order Caputo fractional KdV equation: applications to the Sawada–Kotera–Ito and Lax equations

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Abstract

In this study, we investigate the seventh-order nonlinear Caputo time-fractional KdV equation. The suggested model's solutions, which have a series form, are obtained using the hybrid ZZ-transform under the aforementioned fractional operator. The proposed approach combines the homotopy perturbation method (HPM) and the ZZ-transform. We consider two specific examples with suitable initial conditions and find the series solution to test their applicability. To demonstrate the utility of the presented technique, we explore its applications to the fractional Sawada–Kotera–Ito problem and the Lax equation. We observe the impact of a few fractional orders on the wave solution evolution for the problems under consideration. We provide the efficiency and reliability of the ZZHPM by calculating the absolute error between the series solution and the exact solution of both the Sawada–Kotera–Ito and Lax equations. The convergence and uniqueness of the solution are portrayed via fixed-point theory.

Keywords: seventh-order KdV equation, soliton, homotopy perturbation method

(Some figures may appear in colour only in the online journal)

1. Introduction

Nonlinear evolution equations play a key role in many scientific and technical disciplines and are frequently employed as models to explain complex physical behavior. Specifically, the Korteweg–de Vries (KdV) equation has been widely used in fluid mechanics problems. There is a large family of KdV equations, which is extensively covered in the literature. Many extended and modified versions of the basic KdV equation have been formulated in the literature. The basic nonlinear KdV equation [1] is given by,

$$\frac{\partial}{\partial t}\Psi+\frac{\partial^3}{\partial t^3}\Psi+6\Psi\frac{\partial}{\partial t}\Psi=0, \tag{1}$$

where Ψ is the function of two variables x and t. There is an infinite number of constant-time motion integrals in the KdV equation. They can be clearly stated as,

$$\int_{-\infty}^{\infty}\mathcal{P}_{2j-1}\left(\Psi,\frac{\partial}{\partial x}\Psi,\frac{\partial^2}{\partial x^2}\Psi,\ldots\right)\mathrm{d}x,$$

where the polynomials are recursively expressed as,

$$\mathcal{P}_1=\Psi,$$
$$\mathcal{P}_j=-\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{P}_{j-1}+\sum_{k-1}^{j-2}\mathcal{P}_k\mathcal{P}_{j-1-k},\ \text{for}\ j\geqslant 2.$$

In the literature, researchers have defined modified and extended versions of the KdV equation with regard to the order. The expression for the most popular fifth-order KdV equation [1] is,

$$\frac{\partial}{\partial t}\Psi+a\Psi^2\frac{\partial}{\partial x}\Psi+b\frac{\partial}{\partial x}\Psi\frac{\partial^2}{\partial x^2}\Psi$$
$$+c\Psi\frac{\partial^3}{\partial x^3}+\frac{\partial^5}{\partial x^5}\Psi=0, \tag{2}$$

where a, b and c represent real constants. The readers are directed to access [1] for further information and the family of the KdV equation. The generalized seventh-order KdV equation [1] is expressed as,

$$\Psi_t = -(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x}). \tag{3}$$

The seventh-order KdV equation has been studied to find various wave solutions by using different techniques. For instance, Ganji and his co-author utilized the exp-function procedure to derive the exact solution of the seventh-order KdV equation [2]. Sayed and Kaya used Adomian decomposition technique to analyze the seventh-order Sawada–Kotera equation, which is a special case of the seventh-order KdV equation [3].

Lie groups describe symmetries of geometric objects or physical systems and have applications in mathematics and physics, including differential geometry, topology and quantum field theory. In this paper, Lie groups are relevant because they describe the symmetries of physical systems, such as the rotation of a rigid body in 3D space or the symmetries of a crystal lattice. Studying Lie groups can help researchers understand the behavior of physical systems and predict their behavior in various situations by examining continuous symmetries. For more details, the reader is referred to [4–6] .

At the intersection of applied mathematics and physics, fractional calculus (FC) is a fast-expanding topic of study [7, 8]. Numerous manuscripts have noted that modeling using the FC notion is particularly suitable and trustworthy for providing an exact explanation of memory and some scientific features of different materials and events, which are entirely absent in traditional or integer-order equations. Here, we describe a few applications of FC in various disciplines of science [9–11]. Saifullah *et al* investigated localized modes, shock waves and wave amplitude of a Klein–Gordon equation using fractional operators [12, 13]. Gulalai *et al* studied the soliton dynamics of a fractional-order modified KdV equation [14]. The third-order KdV equation has been studied via different fractional operators by Aljahdaly *et al* [15]. There are some more applications of FC in other fields of science [16–18].

Due to enormous nonlinear problems in FC, the researchers have given significant attention to the solutions of highly nonlinear models. In the literature, several analytical and numerical techniques have been devised to solve the problems of FC. For instance, modified double Laplace and natural transform methods have been implemented on a fractional-order Kawahara equation to extract series solutions [19]. A homotopy perturbation method (HPM) coupled with the Yang transform has been used to solve Caputo and Caputo–Fabrizio fractional partial differential equations (PDEs) [20, 21]. Ghandi *et al* used a Lie symmetry approach to investigate fractional KdV equations [22]. In [23], some new implication of integral transforms to the financial models with different fractional derivatives are presented. Furthermore, SARS CoV-2 with the Euler method is studied in [24]. The KdV equation is analyzed on critical flow over a hole with three fractional operators in [25]. For more applications of FC, please read [26–28]. Inspired by the above literature, we consider the seventh-order general KdV equation [1] in fractional Caputo sense as,

$${}_0^C\mathcal{D}_t^\rho\Psi = -(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x}), \tag{4}$$

with initial conditions,

$$\Psi(x, 0) = \mathcal{W}(x),$$

where Ψ is a function of x and t. $\mathcal{B}_{\mathtt{i}}$, $\mathtt{i} = 1, 2, 3 \ldots, 7$, denoted real parameters.

The novelty of this work lies in the proposed hybrid ZZ-transform approach for obtaining series solutions of the seventh-order nonlinear Caputo time-fractional KdV equation. The combination of the HPM and ZZ-transform is a novel approach to solving fractional differential equations. In addition, we explore the applicability of the presented technique to two specific problems, the fractional Sawad–Kotera–Ito (SKI) problem and the Lax equation, and investigate the impact of fractional orders on the wave solution evolution. The efficiency and reliability of the proposed approach are demonstrated through the absolute error analysis between the series solution and the exact solution of both problems. In addition, the convergence and uniqueness of the solution are established using fixed-point theory.

2. Preliminaries

Here, we offer some fundamental definitions that will be utilized across the rest of the article.

Definition 2.1. Suppose $\mathfrak{K}(t) \in \mathrm{H}^1(a, b)$, and $\rho \in \mathbb{R}^+$, then the Caputo operator is [29],

$${}_0^C\mathcal{D}_t^\rho\mathfrak{K}(t) = \frac{1}{\Gamma(k-\rho)}\int_0^t (t-\varphi)^{k-\rho-1}\mathfrak{K}'(\varphi)\mathrm{d}\varphi,$$
$$k - 1 < \rho < k,$$

where $k \in \mathbb{N}^+$, and $\frac{\partial^m\mathfrak{K}(t)}{\partial t^m}$ for $\rho = k$.

Definition 2.2. Consider $\mathfrak{K}(t)$ is a function, which is defined as $\forall\, t \geqslant 0$. Then, the $\mathcal{ZZ}$-transform of $\mathfrak{K}(t)$ is $P(\nu, \varsigma)$ and is given by [30],

$$\mathcal{Z}[\mathfrak{K}(t)] = P(\nu, \varsigma) = \varsigma\int_0^\infty \mathfrak{K}(\nu t)\exp(-\varsigma t)\mathrm{d}t.$$

Definition 2.3. The $\mathcal{ZZ}$-transform of the m^{th} integer-order derivative of $\mathfrak{K}(t)$ is,

$$\mathcal{Z}(\mathfrak{K}^k(t)) = \left(\frac{\varsigma}{\nu}\right)^m \mathcal{Z}(\mathfrak{K}(t)) - \sum_{r=0}^{k}\frac{\varsigma^{m-r}}{\nu^{k-r}}\mathfrak{K}^r(0).$$

Definition 2.4. The $\mathcal{ZZ}$-transform of the Caputo operator can be defined as [30],

$$\mathcal{Z}[{}_0^C\mathcal{D}_t^\rho \mathfrak{K}(t);(\nu,\varsigma)] = \left(\frac{\varsigma}{\nu}\right)^\rho \mathcal{Z}[\mathfrak{K}(t)] - \sum_{r=0}^{k-1} \frac{\varsigma^{\rho-r}}{\nu^{\rho-r}} \mathfrak{K}^r(0),$$
$$k-1<\rho\leqslant k, \quad k=1,2,3,\ldots,$$

where ς and ν are the variables of the $\mathcal{ZZ}$-transform.

3. General solution using ZZHPM

Here, we briefly discuss the suggested ZZHPM method. This technique is the combination of the $\mathcal{ZZ}$-transform and HPM. It is a very effective method to analyze the analytical solution of various real-world problems. The ZZ-transform has several advantages over numerical methods for solving differential equations and analyzing signals. For instance, it can provide exact solutions to differential equations, which is not always possible with numerical methods. Numerical methods rely on approximations and iterative calculations, which can introduce errors and inaccuracies into the solution. ZZ-transform provides analytical insight into the behavior of systems, which can help engineers and scientists gain a deeper understanding of the underlying physics and dynamics. Furthermore, it is generally more robust than numerical methods since it can handle a wide range of problems and system configurations.

Here first we discuss the ZZHPM approach and then apply it to two different models arising from equation (4). Let us suppose the general non-linear fractional problem as,

$${}_0^C\mathcal{D}_t^\rho \Psi(x,t) + \mathscr{U}[\Psi(x,t)] + \mathscr{V}[\Psi(x,t)] = \vartheta(x,t), \tag{5}$$

where $0<\rho\leqslant 1$, $t>0$, $x\in\mathbb{R}^+$. ${}_0^C\mathcal{D}_t^\rho\Psi(x,t)$ denotes the fractional operator of $\Psi(x,t)$ in Caputo sense. $\mathscr{U}$ is the differential operator containing the linear terms, $\mathscr{V}$ is the differential operator containing the nonlinear terms and $\vartheta(x,t)$ is an external function. The initial condition for equation (5) is as follows:

$$\Psi(x,0)=\vartheta(x). \tag{6}$$

Using definition 2.4, we obtain the following:

$$\frac{\varsigma^\rho}{\nu^\rho}\mathcal{Z}[\Psi(x,t)] - \frac{\varsigma^\rho}{\nu^\rho}\Psi(x,0)$$
$$=-\mathcal{Z}[\mathscr{U}[\Psi(x,t)] + \mathscr{V}[\Psi(x,t)] + \vartheta(x,t)], \tag{7}$$

and after simplification, we obtain,

$$\mathcal{Z}[\Psi(x,t)] = \Psi(x,0) - \frac{\nu^\rho}{\varsigma^\rho}(\mathcal{Z}[\mathscr{U}[\Psi(x,t)]$$
$$+\mathscr{V}[\Psi(x,t)] + \vartheta(x,t)]). \tag{8}$$

Using inverse $\mathcal{ZZ}$-transform and initial condition equation (6), we obtain:

$$\Psi(x,t) = F(x,t) - \mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}(\mathcal{Z}[\mathscr{U}[\Psi(x,t)]\right.$$
$$\left.+\mathscr{V}[\Psi(x,t)] + \vartheta(x,t)])\right], \tag{9}$$

where $F(x,t)$ is a term resulting from the source terms and initial condition. Furthermore, we use the perturbation method and consider that the required solution might be written in the terms of power series of p as,

$$\Psi(x,t) = \sum_{v=0}^{\infty} p^v \Psi_v(x,t), \tag{10}$$

where $p\in[0,1]$. The decomposition of the nonlinear term is,

$$\mathscr{V}[\Psi(x,t)] = \sum_{v=0}^{\infty} H^v(\Psi_v), \tag{11}$$

where H_v is He's polynomial [31] of Ψ_0, Ψ_1, Ψ_2, Ψ_3, …, which can be obtained using the following formula:

$$H_v(\Psi_0,\Psi_1,\Psi_2,\ldots,\Psi_v) = \frac{1}{n!}\frac{\partial^v}{\partial p^v}\left[\mathscr{V}\left(\sum_{r=0}^{v} p^r\Psi_r\right)\right]_{p=0},$$
$$v=0,1,2,3,\ldots.. \tag{12}$$

Using equations (10) and (11), we can write equation (9) as,

$$\sum_{v=0}^{\infty} p^v\Psi_v(x,t) = F(x,t)$$
$$-\sum_{v=0}^{\infty} p\left(\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\left(\mathcal{Z}\left[\mathscr{U}\left[\sum_{v=0}^{\infty} p^v\Psi_v(x,t)\right]\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\sum_{v=0}^{\infty} p^v H_v(\Psi(x,t)) + \vartheta(x,t)\right]\right)\right]\right). \tag{13}$$

Comparing the terms on both sides of equation (13), we achieve the required series solution as,

$$p^0:\ \Psi_0(x,t) = \mathcal{F}(x,t),$$
$$p^1:\ \Psi_1(x,t) = -\mathcal{Z}^{-1}\left(\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathscr{U}\Psi_0(x,t) + H_0(\Psi(x,t))]\right),$$
$$p^2:\ \Psi_2(x,t) = -\mathcal{Z}^{-1}\left(\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathscr{U}\Psi_1(x,t) + H_1(\Psi(x,t))]\right),$$
$$p^3:\ \Psi_3(x,t) = -\mathcal{Z}^{-1}\left(\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathscr{U}\Psi_2(x,t) + H_2(\Psi(x,t))]\right),$$
$$\vdots$$
$$p^v:\ \Psi_v(x,t) = -\mathcal{Z}^{-1}\left(\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathscr{U}\Psi_{v-1}(x,t) + H_{v-1}(\Psi(x,t))]\right), \tag{14}$$

where $v\in\mathbb{N}^+$. One can continue in the same manner to obtain the final general solution. Finally, the approximate solution can be expressed as,

$$\Psi(x,t) = \lim_{v\to\infty}\sum_{v=0}^{\infty}\Psi_v(x,t). \tag{15}$$

For the convergence of the above series equation (15) we refer to [31].

4. General series solution of the considered equation with ZZHPM

In this section, we calculate the general series solution of the seventh-order KdV with a fractional Caputo operator. Thus, we consider the seventh-order KdV in Caputo sense as,

$$ {}_0^C\mathcal{D}_t^\rho\Psi = -(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x}), \tag{16} $$

with initial condition,

$$ \Psi(x, 0) = \mathcal{W}(x). \tag{17} $$

Applying $\mathcal{ZZ}$- transform to equation (16), we obtain,

$$ \mathcal{Z}[{}_0^C\mathcal{D}_t^\rho\Psi] = -\mathcal{Z}(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x}). \tag{18} $$

Using definition 2.4 in equation (18) and doing some simplification, we obtain,

$$ \mathcal{Z}[\Psi] = \mathcal{W}(x) - \mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x})\right]. \tag{19} $$

Then, taking the inverse $\mathcal{ZZ}$- transform of equation (19), we obtain,

$$ \Psi = \mathcal{W}(x) - \mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1\Psi^3\Psi_x + \mathcal{B}_2\Psi_x^3 + \mathcal{B}_3\Psi\Psi_x\Psi_{2x} + \mathcal{B}_4\Psi^2\Psi_{3x} + \mathcal{B}_5\Psi_{2x}\Psi_{3x} + \mathcal{B}_6\Psi_x\Psi_{4x} + \mathcal{B}_7\Psi\Psi_{5x} + \Psi_{7x})\right]. \tag{20} $$

Next, we consider the solution in the following form:

$$ \Psi = \sum_{v=0}^{\infty} p^v\Psi_n. \tag{21} $$

Then, we decompose the nonlinear terms using the He's polynomials as follows:

$$ \begin{aligned} &\Psi^3\Psi_x = \sum_{v=0}^{\infty} H_v,\ \ \Psi_x^3 = \sum_{v=0}^{\infty} I_v,\\ &\Psi\Psi_x\Psi_{2x} = \sum_{v=0}^{\infty} J_v,\ \ \Psi^2\Psi_{3x} = \sum_{v=0}^{\infty} K_v,\\ &\Psi_{2x}\Psi_{3x} = \sum_{v=0}^{\infty} L_v,\\ &\Psi_x\Psi_{4x} = \sum_{v=0}^{\infty} M_v,\ \ \Psi\Psi_{5x} = \sum_{v=0}^{\infty} N_v. \end{aligned} \tag{22} $$

Using equations (21) and (22) in equation (20), we obtain,

$$ \sum_{v=0}^{\infty} p^v\Psi_v = \mathcal{W}(x) - \sum_{v=0}^{\infty} p^v\left\{\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}\left(\mathcal{B}_1\sum_{v=0}^{\infty} H_v + \mathcal{B}_2\sum_{v=0}^{\infty} I_v + \mathcal{B}_3\sum_{v=0}^{\infty} J_v + \mathcal{B}_4\sum_{v=0}^{\infty} K_v + \mathcal{B}_5\sum_{v=0}^{\infty} L_v + \mathcal{B}_6\sum_{v=0}^{\infty} M_v + \mathcal{B}_7\sum_{v=0}^{\infty} N_v + \Psi_{7x}\right)\right]\right\}. \tag{23} $$

Comparing the like powers of p, we obtain,

$$ \begin{aligned} p^0:&\ \Psi_0 = \mathcal{W}(x),\\ p^1:&\ \Psi_1 = -\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1H_0 + \mathcal{B}_2I_0 + \mathcal{B}_3J_0 + \mathcal{B}_4K_0 + \mathcal{B}_5L_0 + \mathcal{B}_6M_0 + \mathcal{B}_7N_0 + \Psi_{7x})\right],\\ p^2:&\ \Psi_2 = -\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1H_1 + \mathcal{B}_2I_1 + \mathcal{B}_3J_1 + \mathcal{B}_4K_1 + \mathcal{B}_5L_1 + \mathcal{B}_6M_1 + \mathcal{B}_7N_1 + \Psi_{7x})\right],\\ p^3:&\ \Psi_3 = -\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1H_2 + \mathcal{B}_2I_2 + \mathcal{B}_3J_2 + \mathcal{B}_4K_2 + \mathcal{B}_5L_2 + \mathcal{B}_6M_2 + \mathcal{B}_7N_2 + \Psi_{7x})\right],\\ &\vdots\\ p^v:&\ \Psi_v = -\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}(\mathcal{B}_1H_{v-1} + \mathcal{B}_2I_{v-1} + \mathcal{B}_3J_{v-1} + \mathcal{B}_4K_{v-1} + \mathcal{B}_5L_{v-1} + \mathcal{B}_6M_{v-1} + \mathcal{B}_7N_{v-1} + \Psi_{7x})\right]. \end{aligned} \tag{24} $$

We calculate first two terms for the He's polynomials of each of the nonlinear terms which are as follows:

$$ \begin{aligned} &H_0 = \Psi_0^3\Psi_{0x},\ I_0 = \Psi_0^3,\ J_0 = \Psi_{0x}\Psi_{0xx}\Psi_0,\\ &K_0 = \Psi_0^2\Psi_{0xxx},\ L_0 = \Psi_{0xx}\Psi_{0xxx},\ M_0 = \Psi_{0x}\Psi_{0xxxx},\ N_0 = \Psi_0\Psi_{0xxxxx},\\ &H_1 = 3\Psi_0^2\Psi_1\Psi_{0x} + \Psi_0^3\Psi_{1x},\ I_1 = \Psi_0^3\Psi_1,\\ &J_1 = 2\Psi_1\Psi_{0x}\Psi_{0xx} + \Psi_0\Psi_{1x}\Psi_{0xx} + \Psi_0\Psi_{0x}\Psi_{1xx},\\ &K_1 = 2\Psi_0\Psi_{0xxx}\Psi_1 + \Psi_0^2\Psi_{1xxx},\ L_1 = \Psi_{1xx}\Psi_{0xxx} + \Psi_{0xx}\Psi_{1xxx},\\ &M_1 = \Psi_{0x}\Psi_{1xxxx} + \Psi_{1x}\Psi_{0xxxx},\\ &N_1 = \Psi_{0x}\Psi_{1xxxxx} + \Psi_{0xxxxx}\Psi_{1x}. \end{aligned} \tag{25} $$

After inserting equations (25) into (24), one may achieve the required series solution of the seventh-order KdV equation (16).

5. Convergence and uniqueness of the solution

This section demonstrates the results concerned with the uniqueness and convergence of the solution of the considered equation, which is obtained through ZZHPM.

Theorem 5.1. *The series solution obtained by ZZHPM is unique if,*

$$ \varpi = 1 - \frac{T^\rho}{\rho\Gamma(\rho)}(\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4 + \mathcal{B}_5 + \mathcal{B}_6 + \mathcal{B}_7 + \kappa^7) < 1. $$

Proof. The series solution of the considered equation is as follows:

$$ \Psi(x, t) = \sum_{v=0}^{\infty} \Psi_v(x, t), \tag{26} $$

where

$$\Psi_v(x,t)=\Psi_{v-1}(x,t)-\mathcal{Z}^{-1}\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathcal{B}_1H(\Psi_{v-1})\right.$$
$$+\mathcal{B}_2I(\Psi_{v-1})+\mathcal{B}_3J(\Psi_{v-1})$$
$$+\mathcal{B}_4K(\Psi_{v-1})+\mathcal{B}_5L(\Psi_{v-1})$$
$$\left.+\mathcal{B}_6M(\Psi_{v-1})+\mathcal{B}_7N(\Psi_{v-1})+(\Psi_{v-1})_{7x}]\right]. \tag{27}$$

In contrast, to prove that the obtained series solution is unique, we take two solutions, such as Ψ and Ψ^*, of the suggested equation. Now, consider,

$$|\Psi-\Psi^*|=|(\Psi-\Psi^*)-\mathcal{Z}^{-1}$$
$$\times\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathcal{B}_1H(\Psi-\Psi^*)+\mathcal{B}_2I(\Psi-\Psi^*)\right.$$
$$+\mathcal{B}_3J(\Psi-\Psi^*)+\mathcal{B}_4K(\Psi-\Psi^*)+\mathcal{B}_5L(\Psi-\Psi^*)$$
$$\left.+\mathcal{B}_6M(\Psi-\Psi^*)+\mathcal{B}_7N(\Psi-\Psi^*)+(\Psi-\Psi^*)_{7x}]\right]|. \tag{28}$$

Using the convolution property of the ZZ-transform, we obtain,

$$|\Psi-\Psi^*|=\left|(\Psi-\Psi^*)-\frac{1}{\Gamma(\rho)}\int_0^t(t-\varphi)^{-\rho}\right.$$
$$\times[\mathcal{B}_1H(\Psi-\Psi^*)+\mathcal{B}_2I(\Psi-\Psi^*)$$
$$+\mathcal{B}_3J(\Psi-\Psi^*)+\mathcal{B}_4K(\Psi-\Psi^*)+\mathcal{B}_5L(\Psi-\Psi^*)$$
$$+\mathcal{B}_6M(\Psi-\Psi^*)+\mathcal{B}_7N(\Psi-\Psi^*)+(\Psi-\Psi^*)_{7x}]\mathrm{d}\varphi|.$$

Let $\kappa^n=\frac{\partial^n}{\partial x^n}$ and applying the integral mean value result, we have,

$$|\Psi-\Psi^*|=|\Psi-\Psi^*|-\frac{T^\rho}{\rho\Gamma(\rho)}$$
$$\times[(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{B}_5$$
$$+\mathcal{B}_6+\mathcal{B}_7+\kappa^7)|\Psi-\Psi^*|]$$
$$\leqslant\left[1-\frac{T^\rho}{\rho\Gamma(\rho)}(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{B}_5\right.$$
$$\left.+\mathcal{B}_6+\mathcal{B}_7+\kappa^7)\right]|\Psi-\Psi^*|$$
$$\leqslant\varpi|\Psi-\Psi^*|.$$

It follows that,

$$(1-\varpi)|\Psi-\Psi^*|\leqslant 0, \tag{29}$$

since $\varpi\in(0,1)$. Hence, equation (29) gives $\Psi=\Psi^*$. Thus, the solution obtained through ZZHPM is unique. □

Theorem 5.2. *Let Ξ be a Banach space and Π: $\Xi\to\Xi$ be a mapping so that,*

$$\|\Pi(\Psi)-\Pi(\Psi^*)\|\leqslant\psi\|\Psi-\Psi^*\|,\quad\forall\ \Psi,\Psi^*\in\Xi.$$

With the help of the Banach fixed-point result, it is established that Π has a fixed point. Then, the series solution obtained by ZZHPM converges to the fixed point of Π.

Proof. Let $(\mathscr{C}(0,T),\|.\|)$ denote the space of continuous functions associated with the norm $\|\Psi\|=\max_{t\in[0,T]}|\Psi(t)|$. Take a sequence$\{\Psi_r\}$in Ξ and consider,

$$\|\Psi_r-\Psi_f\|=\max_{t\in[0,T]}\|(\Psi_{r-1}-\Psi_{f-1})-\mathcal{Z}^{-1}$$
$$\times\left[\frac{\nu^\rho}{\varsigma^\rho}\mathcal{Z}[\mathcal{B}_1H(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_2I(\Psi_{r-1}-\Psi_{f-1})\right.$$
$$+\mathcal{B}_3J(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_4K(\Psi_{r-1}-\Psi_{f-1})$$
$$+\mathcal{B}_5L(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_6M(\Psi_{r-1}-\Psi_{f-1})$$
$$\left.+\mathcal{B}_7N(\Psi_{r-1}-\Psi_{f-1})+(\Psi_{r-1}-\Psi_{f-1})_{7x}]\right]\|. \tag{30}$$

Using the convolution property of the ZZ-transform, we have,

$$\|\Psi_r-\Psi_f\|=\max_{t\in[0,T]}\left\|(\Psi_{r-1}-\Psi_{f-1})-\frac{1}{\Gamma(\rho)}\int_0^t(t-\varphi)^{-\rho}\right.$$
$$\times[\mathcal{B}_1H(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_2I(\Psi_{r-1}-\Psi_{f-1})$$
$$+\mathcal{B}_3J(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_4K(\Psi_{r-1}-\Psi_{f-1})$$
$$+\mathcal{B}_5L(\Psi_{r-1}-\Psi_{f-1})+\mathcal{B}_6M(\Psi_{r-1}-\Psi_{f-1})$$
$$+\mathcal{B}_7N(\Psi_{r-1}-\Psi_{f-1})+(\Psi_{r-1}-\Psi_{f-1})_{7x}]\mathrm{d}\varphi\|.$$

Let $\kappa^n=\frac{\partial^n}{\partial x^n}$ and applying the integral mean value result, we have,

$$\|\Psi_r-\Psi_f\|=\max_{t\in[0,T]}|\Psi_{r-1}-\Psi_{f-1}|-\frac{T^\rho}{\rho\Gamma(\rho)}$$
$$\times[\mathcal{B}_1|\Psi_{r-1}-\Psi_{f-1}|+\mathcal{B}_2|\Psi_{r-1}-\Psi_{f-1}|$$
$$+\mathcal{B}_3|\Psi_{r-1}-\Psi_{f-1}|+\mathcal{B}_4|\Psi_{r-1}-\Psi_{f-1}|+\mathcal{B}_5|\Psi_{r-1}-\Psi_{f-1}|$$
$$+\mathcal{B}_6|\Psi_{r-1}-\Psi_{f-1}|+\mathcal{B}_7|\Psi_{r-1}-\Psi_{f-1}|+\kappa^7|\Psi_{r-1}-\Psi_{f-1}|]$$
$$\leqslant\left[1-\frac{T^\rho}{\rho\Gamma(\rho)}(\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3+\mathcal{B}_4+\mathcal{B}_5+\mathcal{B}_6+\mathcal{B}_7+\kappa^7)\right]$$
$$\times\|\Psi_{r-1}-\Psi_{f-1}\|\ \varpi\|\Psi_{r-1}-\Psi_{f-1}\|.$$

Thus, $\|\Psi_r-\Psi_f\|\leqslant\varpi\|\Psi_{r-1}-\Psi_{f-1}\|$. By putting $r=d+1$, we obtain,

$$\|\Psi_r-\Psi_f\|\leqslant\varpi\|\Psi_f-\Psi_{f-1}\|$$
$$\leqslant\varpi^2\|\Psi_{f-1}-\Psi_{f-2}\|$$
$$\leqslant\varpi^3\|\Psi_{f-2}-\Psi_{f-3}\|\leqslant\cdots\leqslant\varpi^n\|\Psi_1-\Psi_0\|.$$

We can then write the following:

$$\|\Psi_r-\Psi_f\|\leqslant\|\Psi_{f+1}-\Psi_f\|\leqslant\|\Psi_{f+2}-\Psi_{f+1}\|$$
$$\leqslant\cdots\leqslant\|\Psi_r-\Psi_{m-1}\|$$
$$\leqslant[\varpi^f+\varpi^{f+1}+\cdots+\varpi^{r-1}]\|\Psi_1-\Psi_0\|$$
$$\leqslant\varpi^f[1+\varpi^f+\cdots+\varpi^{r-f-1}]\|\Psi_1-\Psi_0\|$$
$$\leqslant\varpi^f\left[\frac{1-\varpi^{r-f-1}}{1-\varpi}\right]\|\Psi_1-\Psi_0\|.$$

Since $0<\varpi<1$, $\|\Psi_r-\Psi_f\|$ gives a finite value when $f\to\infty$. Hence, the considered sequence $\{\Psi_r\}$ is a Cauchy sequence in Ξ. It follows that the sequence $\{\Psi_r\}$ is convergent.

Theorem 5.3. *Let* $\sum_{v=0}^{\infty}\Psi_v(x,t)<\infty$ *and* $\sum_{v=0}^{g}\Psi_v(x,t)$ *be a* gth-*order series solution of* $\Psi(x,t)$. *Let* $\lambda>0$ *so that* $\|\Psi_{v+1}(x,t)\|\leqslant\lambda\|\Psi_v(x,t)\|$, *then the maximum absolute error holds the inequality:*

$$\left\|\Psi(x,t)-\sum_{v=0}^{g}\Psi_v(x,t)\right\|<\frac{\lambda^{g+1}}{1-\lambda}\|\Psi_0(x,t)\|.$$

Proof. Since $\sum_{v=0}^{\infty}\Psi_v(x,t)<\infty$, it follows that $\sum_{v=0}^{g}\Psi_v(x,t)$ is finite. Consider,

$$\begin{aligned}\left\|\Psi(x,t)-\sum_{v=0}^{g}\Psi_v(x,t)\right\|&=\left\|\sum_{v=g+1}^{\infty}\Psi_v(x,t)\right\|\\&\leqslant\sum_{v=g+1}^{\infty}\|\Psi_v(x,t)\|\\&\leqslant\sum_{v=g+1}^{\infty}\lambda^g\|\Psi_0(x,t)\|\\&\leqslant\lambda^{g+1}(1+\lambda+\lambda^2+\cdots)\|\Psi_0(x,t)\|\\&\leqslant\frac{\lambda^{g+1}}{1-\lambda}\|\Psi_0(x,t)\|.\end{aligned}$$

The proof is completed.

6. Application of HPZZM

Here, we present the application of the HPZZM. To do so, we consider the seventh-order SKI equation in the first example, then the seventh-order Lax equation in the second example. We calculate the series solution of the considered models and present graphical visualization and discussion on the results and findings.

CASE I: When we substitute the values of parameters $\mathcal{B}_i$ as $\mathcal{B}_1=252,\ \mathcal{B}_2=63,\ \mathcal{B}_3=378,\ \mathcal{B}_4=126,\ \mathcal{B}_5=63,\ \mathcal{B}_6=42,$ and $\mathcal{B}_7=21,$ in equation (4), we obtain the seventh-order SKI equation in Caputo sense as,

$$\begin{aligned}{}_0^C\mathcal{D}_t^{\rho}\Psi=&-(252\Psi^3\Psi_x+63\Psi_x^3+378\Psi\Psi_x\Psi_{2x}\\&+126\Psi^2\Psi_{3x}+63\Psi_{2x}\Psi_{3x}+42\Psi_x\Psi_{4x}\\&+21\Psi\Psi_{5x}+\Psi_{7x}),\end{aligned}\tag{31}$$

with initial condition,

$$\Psi(x,0)=\frac{2{\kappa_1}^2\mathrm{e}^{\kappa_1x}}{(1+\mathrm{e}^{\kappa_1x})^2}.\tag{32}$$

The exact solution of the seventh-order SKI equation (31) can be obtained [1]:

$$\Psi(x,t)=\frac{2{\kappa_1}^2\mathrm{e}^{\kappa_1x-\kappa_1^7t}}{(1+\mathrm{e}^{\kappa_1x-\kappa_1^7t})^2}.\tag{33}$$

An approximate solution of equation (31) with initial condition equation (32) using the method discussed in section 6 is obtained:

$$\begin{aligned}\Psi_0=&\frac{2{\kappa_1}^2\mathrm{e}^{\kappa_1x}}{(1+\mathrm{e}^{\kappa_1x})^2},\\\Psi_1=&\frac{t^{\rho}}{\Gamma(\rho+1)}\frac{1}{128}\kappa_1^8\operatorname{sech}^9\left(\frac{\kappa_1x}{2}\right)\\&\left(\kappa_1\left(-5147\sinh\left(\frac{\kappa_1x}{2}\right)+1521\sinh\left(\frac{3\kappa_1x}{2}\right)\right.\right.\\&\left.-51\sinh\left(\frac{5\kappa_1x}{2}\right)+\sinh\left(\frac{7\kappa_1x}{2}\right)\right)\\&\left.-2240\cosh\left(\frac{\kappa_1x}{2}\right)+1400\cosh\left(\frac{3\kappa_1x}{2}\right)-56\cosh\left(\frac{5\kappa_1x}{2}\right)\right),\\\Psi_2=&\frac{t^{2\rho}}{\Gamma(2\rho+1)}\left[\frac{4\kappa_1^{14}\mathrm{e}^{8\kappa_1x}}{(\mathrm{e}^{\kappa_1x}+1)^{16}}((99757403915\kappa_1^2-98062720)\cosh(\kappa_1x)\right.\\&-8(4075792667\kappa_1^2+55632640)\cosh(2\kappa_1x)\\&+32(-2248096037\kappa_1^2+4210570\cosh(3\kappa_1x)\\&-252840\cosh(4\kappa_1x)+1666\cosh(5\kappa_1x)+15308776)\\&+\kappa_1(112(195757004\sinh(\kappa_1x)-125580469\sinh(2\kappa_1x)\\&+26403974\sinh(3\kappa_1x)-1882484\sinh(4\kappa_1x)\\&+35322\sinh(5\kappa_1x)-65\sinh(6\kappa_1x))\\&+\kappa_1(4706851313\cosh(3\kappa_1x)-261485088\cosh(4\kappa_1x)\\&\left.+4153091\cosh(5\kappa_1x)-7272\cosh(6\kappa_1x)+\cosh(7\kappa_1x)))\right],\end{aligned}$$

$$\Psi_3=\frac{t^3\rho}{\Gamma(3\rho+1)}\left[\frac{4\kappa_1^{15}\mathrm{e}^{12\kappa_1 x}}{(\mathrm{e}^{\kappa_1 x}+1)^{24}}(28(5\kappa_1^8-256)\sinh(10\kappa_1 x)\right.$$
$$+112(6115661797204083\kappa_1^8-708065009581720\kappa_1^6+2632269946)\sinh(\kappa_1 x)$$
$$+56(-10834920885580987\kappa_1^8+443100117195008\kappa_1^6+3820429728)\sinh(2\kappa_1 x)$$
$$+448(522674185659433\kappa_1^8+48369304270784\kappa_1^6-20324091)\sinh(3\kappa_1 x)$$
$$-224(207947261714893\kappa_1^8+57529983937888\kappa_1^6+291567048)\sinh(4\kappa_1 x)$$
$$+448(11008824532820\kappa_1^8+5106039771560\kappa_1^6-38433057)\sinh(5\kappa_1 x)$$
$$-28(9309498016225\kappa_1^8+5437896171008\kappa_1^6-153323328)\sinh(6\kappa_1 x)$$
$$+56(100131269125\kappa_1^8+61330008376\kappa_1^6+24148026)\sinh(7\kappa_1 x)$$
$$-112(282949699\kappa_1^8+163593248\kappa_1^6+1664888)\sinh(8\kappa_1 x)$$
$$+56(242119\kappa_1^8+124712\kappa_1^6+71286)\sinh(9\kappa_1 x)$$
$$-12\kappa_1(35\kappa_1^8+14\kappa_1^6+638)\cosh(10\kappa_1 x)$$
$$+\kappa_1(560(36732\kappa_1^2+11)\kappa_1^6+4244631)\cosh(9\kappa_1 x)$$
$$+2\kappa_1(112\kappa_1^4(33985056751133990\mathrm{k}1^4-476296709445287\kappa_1^2+2492386176)-114768408117)\cosh(\kappa_1 x)$$
$$-8\kappa_1(7\kappa_1^4(64600732031888135\kappa_1^4+2371048269479222\kappa_1^2-30550428491520)-38302625590)\cosh(2\kappa_1 x)+$$
$$2\kappa_1(224(2304437774357495\kappa_1^4+234055410353843\kappa_1^2-305954181120)\kappa_1^4+116252521965)\cosh(3\kappa_1 x)$$
$$+\kappa_1(448\kappa_1^4(34687267432845\kappa_1^4+6117318503119\kappa_1^2+206328033408)-45540422859)\cosh(5\kappa_1 x)$$
$$-4\kappa_1(7(24384334865635\kappa_1^4+3821604892558\kappa_1^2$$
$$+241861541376)\kappa_1^4+445814514)\cosh(6\kappa_1 x)$$
$$+\kappa_1(112(105375281720\kappa_1^4+10835785487\kappa_1^2$$
$$+1106568960)\kappa_1^4+2897305795)\cosh(7\kappa_1 x)$$
$$-40\kappa_1(7(191506793\kappa_1^4+5734642\kappa_1^2$$
$$+884352)\kappa_1^4+6118578)\cosh(8\kappa_1 x)$$
$$-8\kappa_1(7(87004384081718295\kappa_1^4$$
$$-2865020151372530\kappa_1^2+25058532030336)\kappa_1^4$$
$$+40828169166)-32\kappa_1(5358393277701345\kappa_1^8$$
$$+832727675724322\kappa_1^6+10772628593664\kappa_1^4$$
$$+544946478)\cosh(4\kappa_1 x)$$
$$+\kappa_1\cosh(11\kappa_1 x)$$
$$\vdots$$

The final series solution is,

$$\Psi(x,t)=\sum_{v=0}^{\infty}\Psi_v(x,t). \tag{34}$$

CASE II: When we substitute the values of parameters $\mathcal{B}_i$ as $\mathcal{B}_1=140, \mathcal{B}_2=70, \mathcal{B}_3=280, \mathcal{B}_4=70, \mathcal{B}_5=70, \mathcal{B}_6=42, \mathcal{B}_7=14$, in equation (4), we obtain the seventh-order Lax equation in fractional-order Caputo sense as,

$$\begin{aligned}&{}_0^C\mathcal{D}_t^\rho\Psi=-(140\Psi^3\Psi_x+70\Psi_x^3+280\Psi\Psi_x\Psi_{2x}\\&+70\Psi^2\Psi_{3x}+70\Psi_{2x}\Psi_{3x}+42\Psi_x\Psi_{4x}+14\Psi\Psi_{5x}+\Psi_{7x}),\end{aligned} \tag{35}$$

with initial condition,

$$\Psi(x,0)=\beta-2\varrho^2\tanh^2(\varrho x). \tag{36}$$

The exact solution for $\rho=1$ of the above equation is [1]:

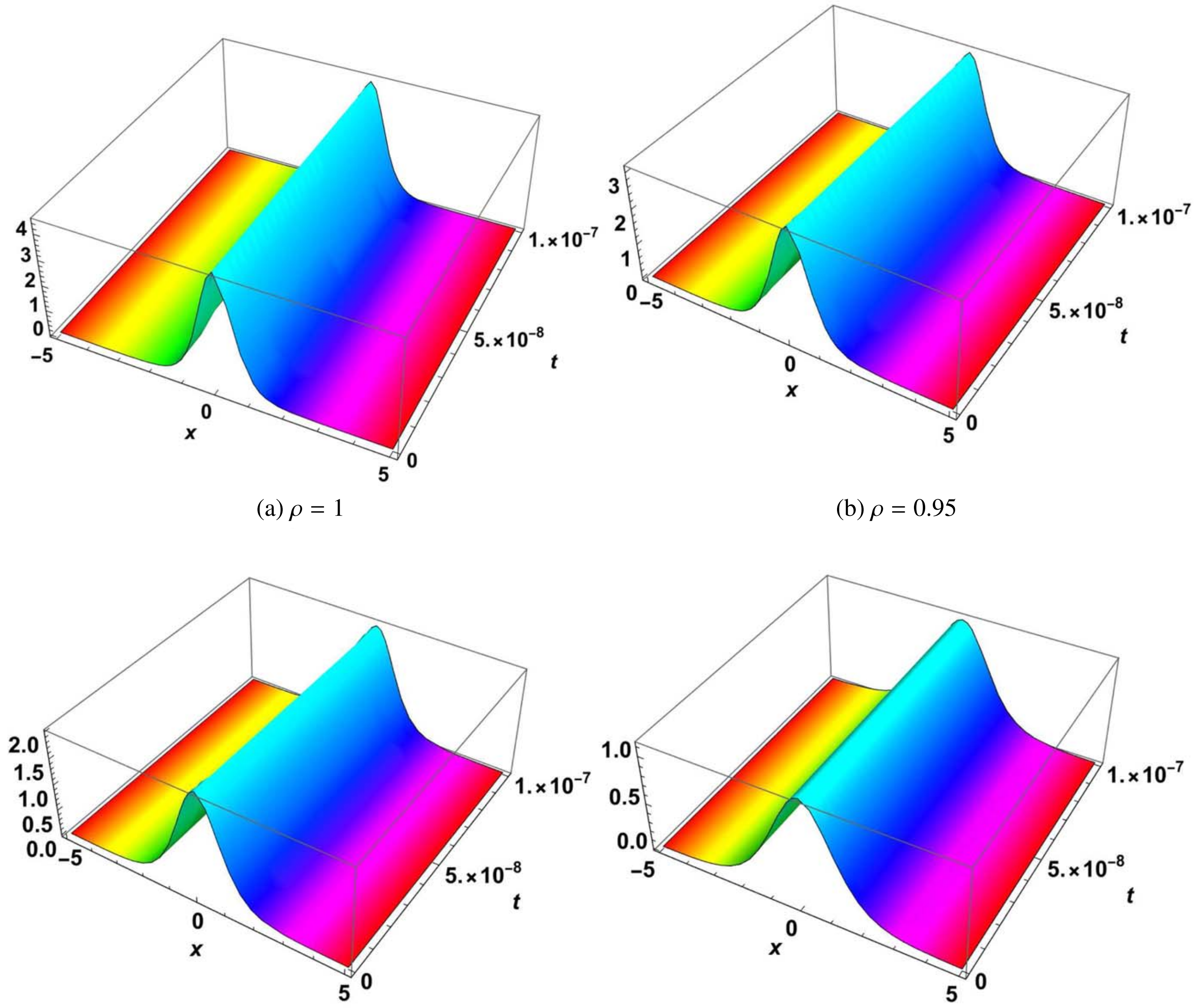
$$\begin{aligned}\Psi(x,0)=\beta-2\varrho^2\tanh^2(\varrho x-(-384\varrho^6\\+784\beta\varrho^4-560\beta^2\varrho^2+140\varrho^3)t).\end{aligned} \tag{37}$$

An approximate solution of equation (35) with initial condition equation (36) using the method discussed in section 6 is obtained:

$$\Psi_0=\beta-2\varrho^2\tanh^2(\varrho x),$$
$$\Psi_1=\frac{t^\rho}{\Gamma(\rho+1)}2\varrho^3(\varrho^6(\mathrm{Tanh}^2)^{(7)}(\varrho x)+280\tanh(\varrho x)\mathrm{sech}^2(\varrho x)$$
$$\times(\beta-2\varrho^2\tanh^2(\varrho x))^3+35\varrho^2(\sinh(3\varrho x)$$
$$-11\sinh(\varrho x))\mathrm{sech}^9(\varrho x)(\beta+2\varrho^2+(\beta-2\varrho^2)\cosh(2\varrho x))^2$$
$$+9800\varrho^4(\cosh(2\varrho x)-2)$$
$$\times\tanh(\varrho x)\mathrm{sech}^{14}(\varrho x)(\beta+2\varrho^2+(\beta-2\varrho^2)\cosh(2\varrho x))^4$$
$$-224\varrho^7(-56\cosh(2\varrho x)+\cosh(4\varrho x)+123)$$
$$\times\tanh^2(\varrho x)\mathrm{sech}^8(\varrho x)+280\varrho^6(56\sinh(\varrho x)$$
$$-15\sinh(3\varrho x)+\sinh(5\varrho x))\mathrm{sech}^9(\varrho x)$$
$$+168\varrho^6(92\sinh(\varrho x)-27\sinh(3\varrho x)$$
$$+\sinh(5\varrho x))\mathrm{sech}^9(\varrho x),$$
$$\vdots$$

The final series solution is,

$$\Psi(x,t)=\sum_{v=0}^{\infty}\Psi_v(x,t). \tag{38}$$



(a) ρ = 1 (b) ρ = 0.95

(c) ρ = 0.90 (d) ρ = 0.85

Figure 1. Evolution of wave solution equation (34) with different fractional orders and $\kappa_1 = 0.5$.

7. Discussion

In this section, the dynamics of the obtained results using the HPZZM method are discussed. The approximate solution of the seventh-order SKI equation (31) is presented in equation (34), and its graphical visualization is shown in figures 1 and 2. In figure 1, it can be seen that the fractional order ρ affects the amplitude of the wave solution, which varies directly with ρ. Figure 1(a) shows the evolution of one soliton with fractional order $\rho = 1$. Similarly, figures 1(b)–(d) depict the behavior of the one soliton with fractional orders $\rho = 0.95,\ 0.90$ and 0.85, respectively.

To simulate these results, Mathematica 13.0 software was used, and the results are quite reliable and efficient in solving higher-order nonlinear problems. The numerical error data for Case I are given in table 1, and for Case II, they are provided in table 2. The absolute error in the numeric data form is provided

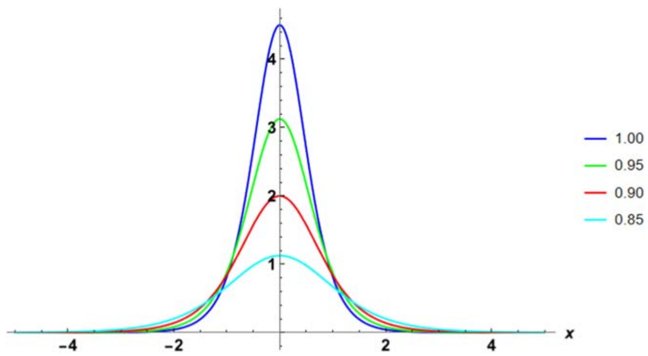


Figure 2. Visualization of 2D behavior of evolution of wave solution equation (34) with different fractional orders and $\kappa_1 = 0.5$.

to explore the efficiency of the proposed approach. The numerical error data for both tables are displayed in 3D graphs, i.e. figures 3 and 6.

Table 1. Error analysis for Example 1.

(x,t)	Approximate solutio	Exact solution	Absolute error
(−1,0.0001)	0.9560	0.9550	0.0010
(−0.9,0.0001)	1.0930	1.0915	0.0015
(−0.8,0.0001)	1.2429	1.2409	0.0020
(−0.7,0.0001)	1.4032	1.4010	0.0021
(−0.6,0.0001)	1.5698	1.5681	0.0017
(−0.5,0.0001)	1.7363	1.7360	0.0003
(−0.4,0.0001)	1.8939	1.8962	0.0023
(−0.3,0.0001)	2.0325	2.0385	0.0060
(−0.2,0.0001)	2.1410	2.1513	0.0104
(−0.1,0.0001)	2.2097	2.2244	0.0147
(0,0.0001)	2.2321	2.2500	0.0179
(0.1,0.0001)	2.2062	2.2255	0.0193
(0.2,0.0001)	2.1351	2.1535	0.0184
(0.3,0.0001)	2.0263	2.0414	0.0151
(0.4,0.0001)	1.8895	1.8997	0.0102
(0.5,0.0001)	1.7351	1.7398	0.0046
(0.6,0.0001)	1.5726	1.5720	0.0007
(0.7,0.0001)	1.4098	1.4048	0.0050
(0.8,0.0001)	1.2522	1.2444	0.0078
(0.9,0.0001)	1.1039	1.0948	0.0092
(1,0.0001)	0.9672	0.9579	0.0093

Table 2. Error analysis for Example 2.

(x,t)	Approximate solutio	Exact solution	Absolute error
(−10,0.01)	0.6804	0.6805	0.0001
(−9,0.01)	0.6809	0.6811	0.0002
(−8,0.01)	0.6820	0.6824	0.0004
(−7,0.01)	0.6844	0.6853	0.0009
(−6,0.01)	0.6896	0.6917	0.0021
(−5,0.01)	0.7004	0.7055	0.0051
(−4,0.01)	0.7211	0.7340	0.0307
(−3,0.01)	0.7574	0.7881	0.0399
(−2,0.01)	0.8340	0.8739	0.0499
(−1,0.01)	1.0160	0.9661	0.0012
(0,0.01)	1.0000	0.9988	0.0491
(1,0.01)	0.8912	0.9403	0.0395
(2,0.01)	0.8839	0.8444	0.0300
(3,0.01)	0.7979	0.7679	0.0123
(4,0.01)	0.7352	0.7229	0.0048
(5,0.01)	0.7048	0.7001	0.0020
(6,0.01)	0.6911	0.6892	0.0008
(7,0.01)	0.6850	0.6842	0.0004
(8,0.01)	0.6822	0.6819	0.0002
(9,0.01)	0.6810	0.6808	0.0001
(10,0.01)	0.6805	0.6804	0.00001

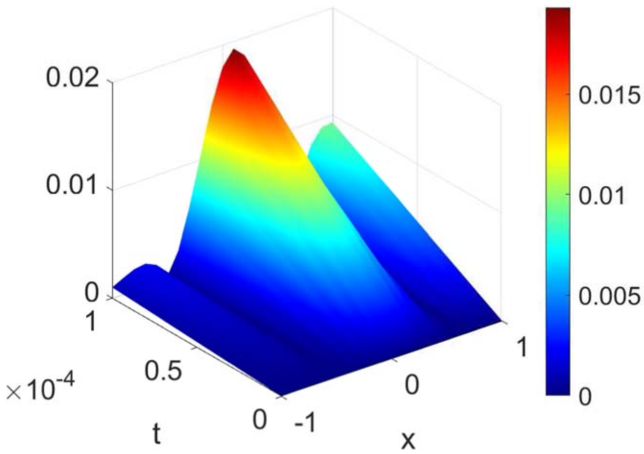


Figure 3. Simulation of error estimate for Case 1.

Moreover, figures 4 and 5 demonstrate the graphical visualization of the approximate solution of the seventh-order Lax equation (16), which is presented in equation (38). The parameters used are $\varrho = 0.5$ and $\beta = 1$. In figure 2, the time t is considered as 20. Figure 1(a) shows the evolution of one soliton with $\rho = 1$. Likewise, figure 1(b)–(d) depicts the behavior of the wave solution with fractional orders $\rho = 0.9$, 0.8 and 0.7, respectively.

From the tables and their corresponding plots, it is observed that at a very small time, the approximate solution is very close to the exact solution, and their associated absolute error is small enough. This indicates that the proposed approach is reliable and efficient in solving higher-order nonlinear problems.

8. Conclusion

Our focus is on investigating the KdV equation and its modified versions due to their applications in fluid mechanics and wave phenomena. Scientists of the twenty-first century

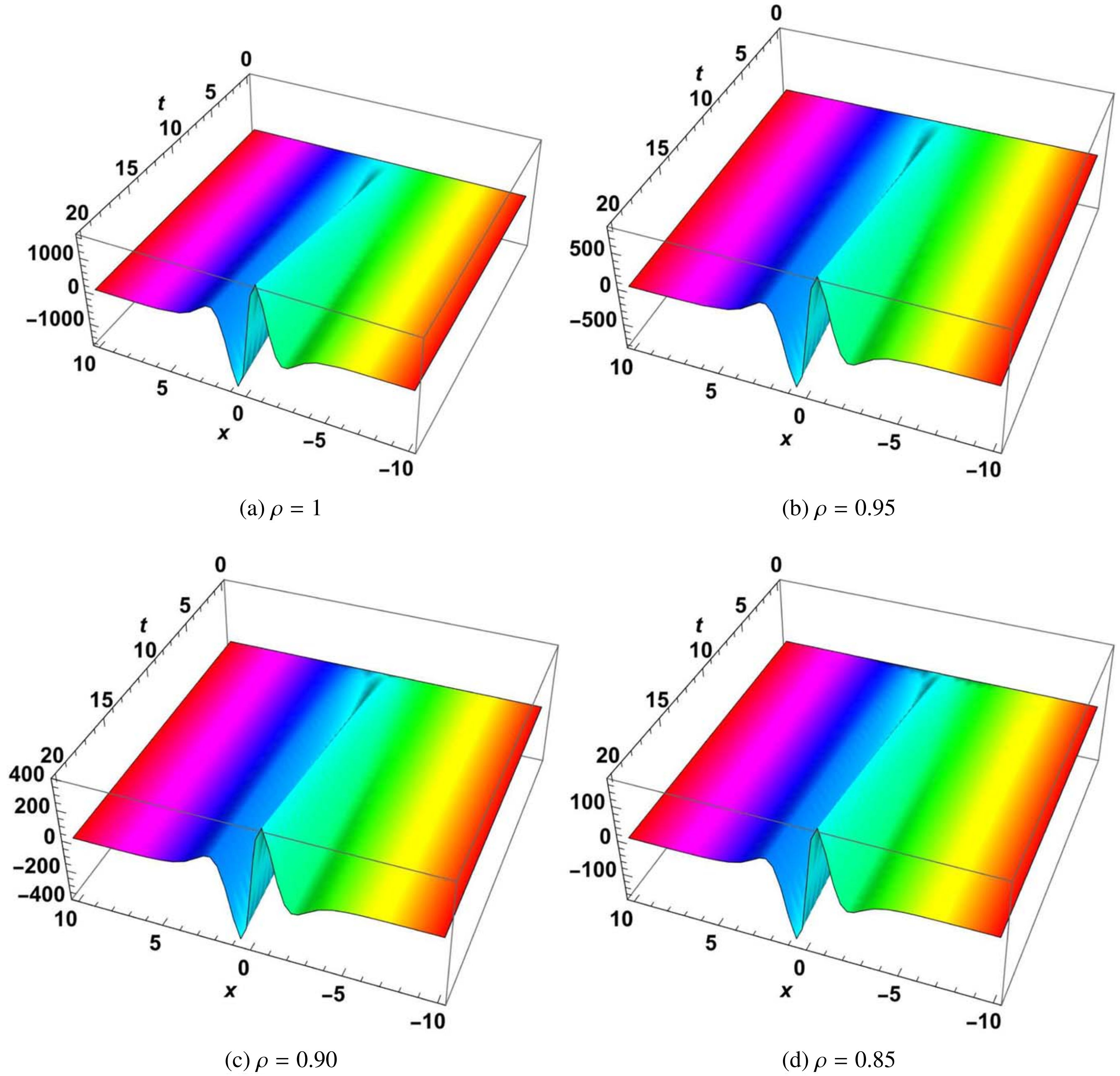


Figure 4. Evolution of wave solution equation (34) with different fractional orders.

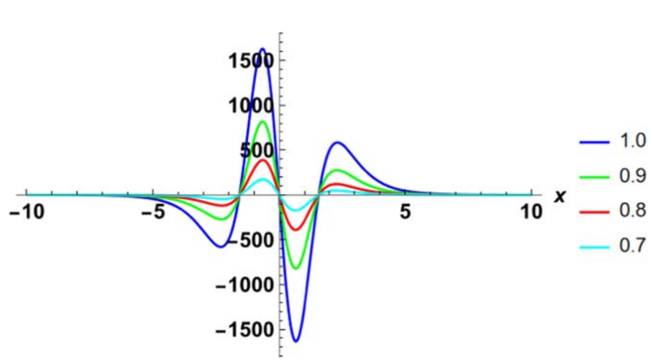


Figure 5. Visualization of 2D behavior of evolution of wave solution equation (34) with different fractional orders.

have a significant challenge in studying higher-order nonlinear KdV equations. In this research, the seventh-order time-fractional nonlinear KdV equation has been studied using FC application. In the analysis, the Caputo differential operator is used. The vital issue is for us to find the solution to higher nonlinear PDEs. Several analytical and numerical methods have been used to analyze the approximate and exact solutions to these problems. Here, the series solution to the problem under consideration was derived using the ZZ-transform in conjunction with HPM. To ensure the applicability of the presented method, we have explored two particular examples of the presented problem. All results have been simulated with the help of Mathematica software. The reliability and

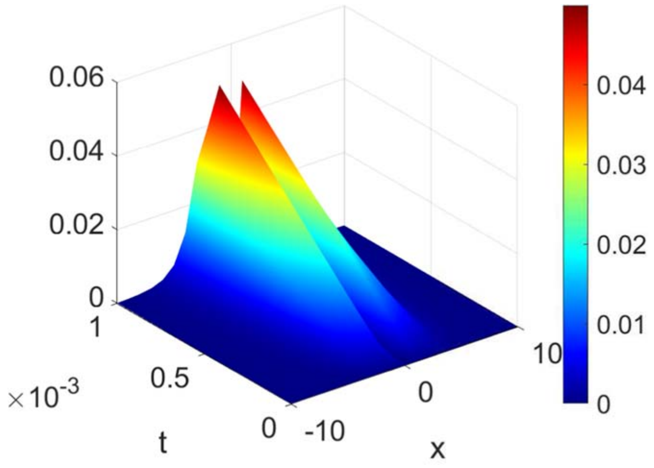


Figure 6. Simulation of error estimate for Case II.

efficiency of the considered HPM is provided in the form of tables and graphs. From the table and graphs, we concluded that the proposed ZZHPM is simple and accurate for solving higher-order fractional PDEs.

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