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A No-Go Theorem for Nonlinear Canonical Quantization*

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Abstract *We want to point out the following strengthening of the classical theorem of Groenewold and van Hove: There exists no mapping* $\mathtt{Op}$ *from polynomial observables* $f(p,q)$ *on the phase space* $\boldsymbol{R}^{2n}$ *into linear operators on* $L^2(\boldsymbol{R}^n)$ *which would map Poisson brackets into commutators, the position and momentum observables* p *and* q *into the usual (Schrödinger) position and momentum operators, and would obey the von Neumann rule* $\mathtt{Op}(cf^k) = c\,\mathtt{Op}(f)^k$ *for* $k = 1, 2, 3$ *and* $c \in \boldsymbol{R}$*. The point is that neither linearity, nor continuity etc. of* $\mathtt{Op}$ *are assumed.*

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The original concept of quantization, going back to Weyl, von Neumann and Dirac,[1−3] consists in assigning (trying to assign) to the observables — real-valued functions $f(p,q)$ of $(p,q) \in \boldsymbol{R}^n \times \boldsymbol{R}^n$ — self-adjoint operators $\mathtt{Op}(f)$ on the Hilbert space $L^2(\boldsymbol{R}^n)$ in such a way that

(i) the correspondence $f \mapsto \mathtt{Op}(f)$ is linear;

(ii) for any function $\phi : \boldsymbol{R} \to \boldsymbol{R}$ for which this makes sense, $\mathtt{Op}(\phi \circ f) = \phi(\mathtt{Op}(f))$ (the von Neumann rule);

(iii) the operators $\mathtt{Op}(p_j)$ and $\mathtt{Op}(q_j)$ corresponding to the coordinate functions p_j, q_j $(j = 1, \ldots, n)$ are given by

$$\mathtt{Op}(q_j)f = q_j f, \qquad \mathtt{Op}(p_j)f = \frac{\mathrm{i}h}{2\pi}\frac{\partial f}{\partial q_j}\,,$$
$$\forall f = f(q) \in L^2(\boldsymbol{R}^n)\,;$$

(iv) $[\mathtt{Op}(f), \mathtt{Op}(g)] = \frac{\mathrm{i}h}{2\pi}\mathtt{Op}(\{f,g\})$, where

$$\{f,g\} = \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\Big)$$

is the Poisson bracket of f and g.

Note that it is a special case $\phi = \mathbf{1}$ of axiom (i), as well as $f = p_1$, $g = q_1$ of axiom (ii), that

$$\mathtt{Op}(\mathbf{1}) = I\,,$$

where $\mathbf{1}$ is the function constant one and I the identity operator. The domain of definition of the mapping $\mathtt{Op}$ — the space of quantizable observables — must contain the constants and the coordinate functions p_j and q_j $(j = 1, \ldots, n)$, and one would of course like this set to be as large as possible — ideally, it should include at least $C^\infty(\boldsymbol{R}^n)$, or some other nice function space.

Unfortunately, it is well known that the axioms (i) ∼ (iv) are not quite consistent. First of all, using axioms (i) ∼ (iii) it is possible to express $\mathtt{Op}(f)$ for $f(p,q) = p_1^2 q_1^2 = (p_1 q_1)^2$ in two ways with two different results; thus axiom (ii) cannot be satisfied if axioms (i) and (iii) hold and p_1^2, q_1^2, $p_1 q_1$ and $p_1^2 q_1^2 \in \mathrm{dom}(\mathtt{Op})$. (The von Neumann rule (ii) is even used only for the squaring function $\phi(t) = t^2$; see Folland[4] and Arens and Babbitt[5]). Secondly, it is a result of Groenewold,[6] later elaborated further by van Hove,[7] that axiom (iv) fails whenever axioms (i) and (iii) are satisfied and the space of quantizable observables contains all polynomials in p, q of degree not exceeding four. (see e.g. Gotay[8] for a recent survey of these matters.)

There are two traditional approaches to handle this disappointing situation. The first is to keep the three axioms (i), (iii) and (iv) (possibly giving up only the von Neumann rule (ii)) but cut down the space of quantizable observables (for instance, to functions at most linear in p). This is one of the underlying ideas of geometric quantization. The second approach is to keep axioms (i) and (iii), but require axiom (iv) to hold only asymptotically as the Planck constant h tends to zero. This leads to deformation quantization.

There is, however, a third possibility — namely, to insist on the axioms (ii), (iii) and (iv), but discard (i) (the linearity). (Note that by axiom (ii) with $\phi(t) = ct$, we will still have at least homogeneity, i.e., $\mathtt{Op}(cf) = c\mathtt{Op}(f)$ for any constant c.) This idea goes back to Tuynman, who writes in §5.1 in Ref. [9] that he does not "$\cdots$ know of a physical motivation for this linearity condition, but it is certainly desirable from the computational point of view."

In fact, such nonlinear assignments do actually arise already in some existing approaches to geometric quantization, namely when one tries to extend the space of quantizable observables by using the Blattner–Kostant–Sternberg kernel method: for instance, for observables of the form $p^2 f(q)$ on $\boldsymbol{R}^1$ it can be computed (see Bao and

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Zhu[10]) that the corresponding operator assigned by the BKS method is given by

$$\mathtt{Op}(p^2 f(q))\psi = \Big(\frac{\mathrm{i}h}{2\pi}\Big)^2 \Big[f\psi'' + f'\psi' + \Big(\frac{f''}{4} - \frac{f'^2}{16f}\Big)\psi\Big],$$

which is manifestly nonlinear in f. Note that, remarkably, since the operator corresponding to $pf(q)$ is given by (loc. cit.) $\mathtt{Op}(pf(q))\psi = -(\mathrm{i}h/2\pi)(f\psi' + \frac{1}{2}f'\psi)$, this correspondence further satisfies the von Neumann rule (ii) for the squaring function, i.e. $\mathtt{Op}(pf(q))^2 = \mathtt{Op}(p^2 f(q)^2)$.

These considerations lead to the following question: Does there exist a map $\mathtt{Op}$ from, say, the space of polynomials in p, q into formally self-adjoint linear operators on $L^2(\boldsymbol{R}^n)$ which would obey the rules (ii) (von Neumann), (iii) (Schrödinger) and (iv) (brackets), but not necessarily (i) (linearity)?

We want to point out that, unfortunately, there is a simple no-go theorem also in this case.

Theorem Let $\mathcal{P}$ be the vector space of all real polynomials in p, q of degree ≤ 3. Then there is no mapping $\mathtt{Op}$ from $\mathcal{P}$ into operators which would satisfy (ii) $\sim$ (iv).

Proof By Eq. (1), for any $k, m = 1, 2, \ldots$,

$$\{p_1^k, q_1^m\} = km\, p_1^{k-1} q_1^{m-1}.$$

Thus by axiom (iv),

$$\mathtt{Op}(p_1^{k-1} q_1^{m-1}) = \frac{1}{ckm}[P^k, Q^m],$$

where for brevity we write $\mathtt{Op}(p_1) = P$, $\mathtt{Op}(q_1) = Q$ and $c = \mathrm{i}h/2\pi$. Now by axiom (iii) and the Leibniz rule,

$$\begin{aligned} P^k Q^m f &= \sum_{j=0}^{k} \binom{k}{j} (P^{k-j} q_1^m) P^j f \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{m!\, c^{k-j}}{(m-k+j)!} Q^{m-k+j} P^j f, \end{aligned}$$

so

$$\begin{aligned} [P^k, Q^m] &= \sum_{j=0}^{k-1} \binom{k}{j} \frac{m!\, c^{k-j}}{(m-k+j)!} Q^{m-k+j} P^j \\ &= \sum_{l=1}^{k} \binom{k}{l} \frac{m!\, c^l}{(m-l)!} Q^{m-l} P^{k-l}. \end{aligned}$$

In particular, taking $k = m = 2$ gives

$$\mathtt{Op}(p_1 q_1) = \frac{2c^2 + 4cQP}{4c} = \frac{PQ + QP}{2},$$

while for $k = m = 3$ we get

$$\begin{aligned} \mathtt{Op}(p_1^2 q_1^2) &= \frac{6c^3 + 18c^2 QP + 9cQ^2P^2}{9c} \\ &= \frac{2}{3}c^2 + 2cQP + Q^2P^2. \end{aligned}$$

As

$$\Big(\frac{PQ + QP}{2}\Big)^2 = \Big(\frac{c}{2} + QP\Big)^2 = \frac{c^2}{4} + 2cQP + Q^2P^2,$$

we thus see that $\mathtt{Op}(p_1^2 q_1^2) \neq \mathtt{Op}(p_1 q_1)^2$, contradicting axiom (ii). Thus $\mathtt{Op}$ cannot exist.

Note that not only the correspondence $\mathtt{Op} : f \mapsto \mathtt{Op}(f)$, but even the operators $\mathtt{Op}(f)$ themselves need not be assumed to be linear. Also, axiom (ii) was used only for $\phi(t) = t^2$, $\phi(t) = t^3$ and $\phi(t) = at$, $a \in \boldsymbol{R}$.

We remark that from a purely mathematical viewpoint, it can, in fact, be shown that already axioms (ii) and (iii) by themselves lead to a contradiction, unless one puts some restriction on the functions ϕ in (ii) or on the domain of the mapping $\mathtt{Op}$ (the space of quantizable observables). Namely, recall that there exists a continuous function f (Peáno curve) which maps $\boldsymbol{R}$ continuously and surjectively onto $\boldsymbol{R}^{2n}$. Let g be a right inverse for f, so that $g : \boldsymbol{R}^{2n} \to \boldsymbol{R}$ and $f \circ g = \mathrm{id}$; such g exists owing to the surjectivity of f, and can be chosen to be measurable and locally bounded. Set $T = \mathtt{Op}(g)$ and consider the functions $\phi = p_1 \circ f$, $\psi = q_1 \circ f$. Then by axiom (ii),

$$\phi(T) = \mathtt{Op}(p_1 \circ f \circ g) = P, \quad \psi(T) = \mathtt{Op}(q_1 \circ f \circ g) = Q,$$

and

$$\begin{aligned} 0 = (\phi\psi - \psi\phi)(T) &= \phi(T)\psi(T) - \psi(T)\phi(T) \\ &= [P, Q] = \frac{\mathrm{i}h}{2\pi} I, \end{aligned}$$

a contradiction. Thus axiom (ii) cannot hold for all $\phi \in C(\boldsymbol{R})$ if axiom (iii) holds and g belongs to the domain of $\mathtt{Op}$ (i.e. to the quantizable observables). In the physical realm, however, one usually deals only with smooth observables, which rules out such pathologies.

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