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General Wigner Rotations in D Dimensions*

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Abstract *We construct general Wigner rotations for both massive and massless particles in D-dimensional spacetime. We work out the explicit expressions of these Wigner rotations for arbitrary Lorentz transformations. We study the relation between the electromagnetic gauge invariance and the non-uniqueness of Wigner rotation.*

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1 Introduction and Summary

In quantum field theory, one-particle states are classified according to the representations of little groups of the Lorentz group.[1] For a systematic introduction of little groups or Wigner rotations for both massive and massless particles in four-dimensional spacetime, see Ref. [2].

In this paper, we wish to study the Wigner rotations for both massive and massless particles in an arbitrary D-dimensional spacetime. We begin by introducing the little groups in D-dimensional spacetime. For a chosen "standard" D-momentum k^ν,‡ the little group or Wigner rotation is defined as $W^\mu{}_\nu k^\nu = k^\mu$, $\mu,\nu = 0,1,\ldots,D-1$. For an arbitrary Lorentz transformation Λ and a given momentum p^μ, the little group can be constructed as follows,[2]

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p)\,. \tag{1}$$

Here $L(p)$ is some standard Lorentz transformation, bringing k^μ to p^μ, i.e. $p^\mu = L^\mu{}_\nu(p)k^\nu$.

In this paper, we work out the explicit expressions of little group elements (1) for both massive and massless particles in D-dimensional spacetime.

Our main idea is to use spinor algebra to construct the little groups or Wigner rotations. Generally speaking, the spinor algebra in D dimensions is slightly easier than the tensor algebra. Nevertheless, the spinors can still furnish faithful representations of the little groups; So they can be used to work out (1). The technical details will be introduced in the next section. For the massive particle case, we use two distinct methods to derive the explicit expression for the Wigner rotations; In the special case of $4D$, we provide a third way to work out the explicit expression for the Wigner rotation.

The spinor representation of little group for massless particles is particularly interesting. For instance, in the case of $4D$, the little group is ISO(2), with the rotation generator J^3 and two translation generators T^1 and T^2. If the physical state is a superposition of the eigenvectors of T^1 and T^2, and if the eigenvalues of T^1 and T^2 are not zero, the helicity σ of a massless particle would have a continuous value without taking account of the topology of the Lorentz group.[2] However, in the spinor realization of ISO(2), the eigenvalues of $A^1 \equiv T^1_{\rm S}$ and $A^2 \equiv T^2_{\rm S}$ are zero automatically. (Here "S" stands for the spinor representation.) So a continuous value of the helicity σ of a massless fermionic particle can be avoided, without even considering the topology of the Lorentz group.

It is obvious for a given Lorentz transformation, the Wigner rotation cannot be uniquely defined. For a fixed "standard" D-momentum k^μ, one may choose two different standard Lorentz transformations $L(p)$ and $\tilde{L}(p)$, in the sense that $L(p)^\mu{}_\nu k^\nu = \tilde{L}(p)^\mu{}_\nu k^\nu = p^\mu$ but $L(p) \neq \tilde{L}(p)$. The resulting two Wigner rotations satisfy

$$\tilde{W}(\Lambda,p) = S(\Lambda p)W(\Lambda,p)S^{-1}(p)\,, \tag{2}$$

where $S(p) \equiv \tilde{L}^{-1}(p)L(p)$. The above equation may be useful in studying gauge fields: Here $S(p)$ may have a connection with the gauge transformation of U(1) gauge field in D dimensions. As an example, we discuss the relation between the electromagnetic gauge invariance and the non-uniqueness of Wigner rotation in four-dimensional spacetime (see Sec. 4).

The results of this paper may be useful in studying theories in the higher dimensions, such as superstring theory or M-theory.

Our paper is organized as follows. In Sec. 2, we work out the Wigner rotations for massive particles in D dimensions, and discuss the special case of $D = 4$. In Sec. 3, we derive the Wigner rotations for massless particles in D dimensions. In Sec. 4, we investigate the relation between the Wigner rotation and the U(1) or abelian gauge symmetry in $4D$. We summarize our conventions and some

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‡For a particle of unit mass, $k^\mu = (0,0,\ldots,0,1)$; For a massless particle, $k^\mu = (0,\ldots,0,1,1)$, with "1" standing for unit energy.

useful identities in Appendix A. In Appendix B, we verify that the little group elements for massive particles belong to SO$(D-1)$, and In Appendix C, we verify that some little group elements for massless particles belong to SO$(D-2)$.

2 Wigner Rotations for Massive Particles

2.1 *D* Dimensions

For a particle of mass M in D dimensions, we choose the standard vector as $k^\mu=(0,0,\ldots,0,M)$. The spinor representation of the "standard boost" can be constructed as follows

$$L_{\rm S}(\eta)={\rm e}^{\eta_i\Sigma^{i0}}\,. \tag{3}$$

Here $\Sigma^{i0}=(1/4)[\gamma^0,\gamma^i]$ is the set of boost generators (our conventions are summarized in Appendix A), and η^i the set of rapidities; The subscript "S" stands for spinor representation. The relation between $L_{\rm S}(\eta)$ and $L(\eta)$[§] is the standard one:

$$L_{\rm S}(\eta)\gamma^\mu L_{\rm S}^{-1}(\eta)=L_\nu{}^\mu(\eta)\gamma^\nu\,. \tag{4}$$

Using $(2\Sigma^{i0})^2=1$ (no sum), one can convert (3) into the form

$$L_{\rm S}(\eta)=\cosh(\eta/2)+\sinh(\eta/2)\hat\eta^i(2\Sigma^{i0})\,, \tag{5}$$

where $\hat\eta^i\equiv\eta^i/\eta$ and $\eta\equiv|\vec\eta|=\sqrt{(\eta^i)^2}$. Substituting (5) into (4), we find that

$$\begin{aligned}&L_i{}^j(\eta)=\delta^{ij}+(\cosh\eta-1)\hat\eta^i\hat\eta^j\,,\\&L_0{}^i(\eta)=L_i{}^0(\eta)=-\hat\eta^i\sinh\eta\,,\\&L_0{}^0(\eta)=\cosh\eta\,.\end{aligned} \tag{6}$$

Substituting

$$\hat\eta^i=\hat p^i,\quad \sinh\eta=|\vec p|/M \tag{7}$$

into Eq. (6),

$$\begin{aligned}&L_i{}^j(p)=\delta^{ij}+(\gamma-1)\hat p^i\hat p^j\,,\\&L_0{}^i(p)=L_i{}^0(p)=-\hat p^i\sqrt{\gamma^2-1}\,,\\&L_0{}^0(p)=\gamma\,,\end{aligned} \tag{8}$$

where $\gamma\equiv\sqrt{|\vec p|^2/M^2+1}=p^0/M$. We see that $L(\eta)$ or $L(p)$ does carry the D-momentum from k^μ to p^μ. Since now, we do not distinguish $L(\eta)$ and $L(p)$. It can be seen that if $D=4$, the standard boost (6) is exactly the same as the one in Ref. [2].

For a given general Lorentz transformation Λ, we denote its spinor counterpart as $\Lambda_{\rm S}$;[¶] They satisfy the equation

$$\Lambda_{\rm S}\gamma^\mu\Lambda_{\rm S}^{-1}=\Lambda_\nu{}^\mu\gamma^\nu\,. \tag{9}$$

Then the Wigner rotation in the spinor space reads

$$W_{\rm S}(\Lambda,\eta)=L_{\rm S}^{-1}(\eta_\Lambda)\Lambda_{\rm S}L_{\rm S}(\eta)\,. \tag{10}$$

Here η_Λ must be defined such that $L(\eta_\Lambda)$ transforms p^μ into $(\Lambda p)^\mu$, i.e.,

$$\hat\eta^i_\Lambda=\widehat{(\Lambda p)}^i,\qquad \sqrt{\big((\Lambda p)^i\big)^2}=M\sinh(\eta_\Lambda)\,. \tag{11}$$

This can be fulfilled by requiring that

$$L_{\rm S}(\eta_\Lambda)\gamma^0L_{\rm S}^{-1}(\eta_\Lambda)=\Lambda_{\rm S}L_{\rm S}(\eta)\gamma^0L_{\rm S}^{-1}(\eta)\Lambda_{\rm S}^{-1}\,. \tag{12}$$

On one hand,

$$\Lambda_{\rm S}L_{\rm S}(\eta)\gamma^0L_{\rm S}^{-1}(\eta)\Lambda_{\rm S}^{-1}=\Lambda_\nu{}^\mu L_\mu{}^0(\eta)\gamma^\nu\,. \tag{13}$$

On the other hand, in analogy to Eq. (5), we have

$$L_{\rm S}(\eta_\Lambda)=\cosh(\eta_\Lambda/2)+\sinh(\eta_\Lambda/2)\hat\eta^i_\Lambda(2\Sigma^{i0})\,. \tag{14}$$

So

$$\begin{aligned}L_{\rm S}(\eta_\Lambda)\gamma^0L_{\rm S}^{-1}(\eta_\Lambda)&=L_\nu{}^0(\eta_\Lambda)\gamma^\nu\\&=(\cosh(\eta_\Lambda)+\sinh(\eta_\Lambda)\hat\eta^i_\Lambda(2\Sigma^{i0}))\gamma^0\\&=\cosh(\eta_\Lambda)\gamma^0-\sinh(\eta_\Lambda)\hat\eta^i_\Lambda\gamma^i\,.\end{aligned} \tag{15}$$

Comparing (13) and (15) gives

$$\begin{aligned}&\cosh(\eta_\Lambda)=(\Lambda L)_0{}^0=\Lambda_0{}^0\cosh(\eta)-\Lambda_0{}^i\hat\eta_i\sinh(\eta)\,,\\&\hat\eta^j_\Lambda\sinh(\eta_\Lambda)=-(\Lambda L)_j{}^0=\Lambda_j{}^i\hat\eta_i\sinh(\eta)-\Lambda_j{}^0\cosh(\eta)\,,\end{aligned} \tag{16}$$

where we have used Eq. (6), and for readability, we have written $\Lambda_\mu{}^\rho L_\rho{}^\nu$ as $(\Lambda L)_\mu{}^\nu$.

The inverse transformation reads

$$L_{\rm S}^{-1}(\eta_\Lambda)=\cosh(\eta_\Lambda/2)-\sinh(\eta_\Lambda/2)\hat\eta^i_\Lambda(2\Sigma^{i0})\,. \tag{17}$$

It is possible to recast it into the following form:

$$L_{\rm S}^{-1}(\eta_\Lambda)=\frac{(\Lambda_{\rm S}^\dagger)^{-1}L_{\rm S}^{-2}(\eta)\Lambda_{\rm S}^\dagger+1}{2\cosh(\eta_\Lambda/2)}\,. \tag{18}$$

To see this, let us evaluate $\Lambda_{\rm S}L_{\rm S}^2\Lambda_{\rm S}^\dagger$ first. Using Eqs. (5), (9), and $\Lambda_{\rm S}^{-1}=\gamma^0\Lambda_{\rm S}^\dagger(\gamma^0)^{-1}$, we find that

$$\begin{aligned}\Lambda_{\rm S}L_{\rm S}^2\Lambda_{\rm S}^\dagger&=\Lambda_{\rm S}[\cosh(\eta)+\sinh(\eta)\hat\eta^i(2\Sigma^{i0})]\Lambda_{\rm S}^\dagger\\&=[\Lambda_0{}^0\cosh(\eta)-\Lambda_0{}^i\hat\eta^i\sinh(\eta)]+[\Lambda_j{}^i\hat\eta^i\sinh(\eta)\\&\quad-\Lambda_j{}^0\cosh(\eta)](2\Sigma^{j0})\,.\end{aligned} \tag{19}$$

Using the above result, it is not difficult to compute $(\Lambda_{\rm S}^\dagger)^{-1}L_{\rm S}^{-2}(\eta)\Lambda_{\rm S}^\dagger$:

$$\begin{aligned}(\Lambda_{\rm S}^\dagger)^{-1}L_{\rm S}^{-2}(\eta)\Lambda_{\rm S}^\dagger&=(\Lambda_{\rm S}L_{\rm S}^2\Lambda_{\rm S}^\dagger)^{-1}=\gamma^0(\Lambda_{\rm S}L_{\rm S}^2\Lambda_{\rm S}^\dagger)^\dagger(\gamma^0)^{-1}\\&=[\Lambda_0{}^0\cosh(\eta)-\Lambda_0{}^i\hat\eta^i\sinh(\eta)]-[\Lambda_j{}^i\hat\eta^i\\&\quad\times\sinh(\eta)-\Lambda_j{}^0\cosh(\eta)](2\Sigma^{j0})\,.\end{aligned} \tag{20}$$

Plugging the above equation into Eq. (18), and using Eq. (16), we find that Eq. (18) is exactly the same as Eq. (17).

Plugging Eq. (17) into Eq. (10) gives the spinor representation of the general Wigner rotation for massive particles in D dimension:

$$\begin{aligned}W_{\rm S}(\Lambda,\eta)&=\frac{(\Lambda_{\rm S}^\dagger)^{-1}L_{\rm S}^{-1}(\eta)+\Lambda_{\rm S}L_{\rm S}(\eta)}{2\cosh(\eta_\Lambda/2)}\\&=\frac{\gamma^0\Lambda_{\rm S}L_{\rm S}(\eta)(\gamma^0)^{-1}+\Lambda_{\rm S}L_{\rm S}(\eta)}{\sqrt{2(1+[\Lambda L(p)]_0{}^0)}}\,,\end{aligned} \tag{21}$$

where we have written the denominator as

$$2\cosh(\eta_\Lambda/2)=\sqrt{2(\cosh(\eta_\Lambda)+1)}=\sqrt{2(1+[\Lambda L(p)]_0{}^0)}\,. \tag{22}$$

It is easy to check that

$$W_{\rm S}^\dagger(\Lambda,\eta)=W_{\rm S}^{-1}(\Lambda,\eta)\,. \tag{23}$$

So according to our convention in Appendix A, $W_{\rm S}(\Lambda,\eta)$ must furnish a unitary representation of SO$(D-1)$.

[§] In this paper, Lorentz transformations without the subscript "S", such as $L(p)$, Λ, R, and $W(\Lambda,p)$ are in the vector representation.

[¶] If $D\le4$, it is relatively easy to work out the explicit expression of $\Lambda_{\rm S}$ for a given general Λ. (See Subsec. 2.2)

The general Wigner rotation or the little group element $W(\Lambda,\eta)$ can be worked out via the equation:

$$W_{\rm S}(\Lambda,\eta)\gamma^\mu W_{\rm S}^{-1}(\Lambda,\eta)=W_\nu{}^\mu(\Lambda,\eta)\gamma^\nu\,. \tag{24}$$

First of all, if $\gamma^\mu=\gamma^0$, it is easy to verify that

$$W_{\rm S}(\Lambda,\eta)\gamma^0 W_{\rm S}^{-1}(\Lambda,\eta)=\gamma^0\,, \tag{25}$$

that is,

$$W_0{}^0(\Lambda,\eta)=1\,,\quad W_i{}^0(\Lambda,\eta)=0\,. \tag{26}$$

Secondly, if $\gamma^\mu=\gamma^i$, using Eqs. (4), (9), and the commutation relations in Appendix A, we obtain

$$W_{\rm S}(\Lambda,\eta)\gamma^i W_{\rm S}^{-1}(\Lambda,\eta)=W_\nu{}^i(\Lambda,\eta)\gamma^\nu=W_j{}^i(\Lambda,\eta)\gamma^j$$
$$=\Big(-\frac{[\Lambda L(\eta)]_0{}^i[\Lambda L(\eta)]_j{}^0}{1+[\Lambda L(\eta)]_0{}^0}+[\Lambda L(\eta)]_j{}^i\Big)\gamma^j\,. \tag{27}$$

In summary,

$$\begin{aligned}&W_0{}^0(\Lambda,p)=1\,,\\&W_i{}^0(\Lambda,p)=W_0{}^i(\Lambda,p)=0\,,\\&W_j{}^i(\Lambda,p)=-\frac{[\Lambda L(p)]_0{}^i[\Lambda L(p)]_j{}^0}{1+[\Lambda L(p)]_0{}^0}+[\Lambda L(p)]_j{}^i\,.\end{aligned} \tag{28}$$

We see that once the explicit expression for Λ is known, one can calculate $W_j{}^i(\Lambda,p)$ immediately, without having to work out the explicit expression of $\Lambda_{\rm S}$.

Using Eq. (7), a short calculation gives

$$W_j{}^i(\Lambda,p)=\frac{[-\Lambda_0{}^0p^i/M+\Lambda_0{}^i+(\gamma-1)\Lambda_0{}^k\hat p_k\hat p^i](\Lambda p)_j}{M+(\Lambda p)^0}$$
$$-\Lambda_j{}^0p^i/M+(\gamma-1)\Lambda_j{}^k\hat p_k\hat p^i+\Lambda_j{}^i\,. \tag{29}$$

The Wigner rotation (28) can be also derived without relying on Clifford algebra. We begin by writing down the standard boost $L(\Lambda p)$:

$$\begin{aligned}&L^i{}_j(\Lambda p)=\delta^{ij}+(\gamma_\Lambda-1)\widehat{\Lambda p}^i\widehat{\Lambda p}^j\,,\\&L^0{}_i(\Lambda p)=L^i{}_0(\Lambda p)=\widehat{\Lambda p}^i\sqrt{\gamma_\Lambda^2-1}\,,\\&L^0{}_0(\Lambda p)=\gamma_\Lambda\,,\end{aligned} \tag{30}$$

where

$$\begin{aligned}&\gamma_\Lambda=(\Lambda p)^0/M=[\Lambda L(p)]^0{}_0\,,\\&\widehat{\Lambda p}^i=\frac{(\Lambda p)^i}{\sqrt{(\Lambda p)^j(\Lambda p)^j}}=\frac{[\Lambda L(p)]^i{}_0}{\sqrt{\gamma_\Lambda^2-1}}\,.\end{aligned} \tag{31}$$

The inverse transformation $(L^{-1})^\mu{}_\nu(\Lambda p)$ are determined by the fundamental equation

$$(L^{-1})^\mu{}_\nu(\Lambda p)=\eta^{\mu\rho}\eta_{\nu\sigma}L^\sigma{}_\rho(\Lambda p)\,. \tag{32}$$

Substituting Eq. (30) into the above equation gives

$$\begin{aligned}&(L^{-1})^i{}_j(\Lambda p)=\delta^{ij}+\frac{[\Lambda L(p)]^i{}_0[\Lambda L(p)]^j{}_0}{[\Lambda L(p)]^0{}_0+1}\,,\\&(L^{-1})^0{}_i(\Lambda p)=(L^{-1})^i{}_0(\Lambda p)=-[\Lambda L(p)]^i{}_0\,,\\&(L^{-1})^0{}_0(\Lambda p)=[\Lambda L(p)]^0{}_0\,.\end{aligned} \tag{33}$$

Substituting Eq. (33) into the equation

$$W^\mu{}_\nu(\Lambda,p)=(L^{-1})^\mu{}_\rho(\Lambda p)\Lambda^\rho{}_\sigma L^\sigma{}_\nu(p)\,, \tag{34}$$

after a slightly length algebra, one obtains

$$\begin{aligned}&W^0{}_0(\Lambda,p)=1\,,\\&W^i{}_0(\Lambda,p)=W^0{}_i(\Lambda,p)=0\,,\\&W^j{}_i(\Lambda,p)=-\frac{[\Lambda L(p)]^0{}_i[\Lambda L(p)]_0{}^j}{1+[\Lambda L(p)]^0{}_0}+[\Lambda L(p)]^j{}_i\,,\end{aligned} \tag{35}$$

which are in agreement with Eq. (28).

Using $\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\eta^{\rho\sigma}=\eta^{\mu\nu}$ and $L_\rho{}^\mu L_\sigma{}^\nu\eta^{\rho\sigma}=\eta^{\mu\nu}$, it is not difficult to verify that

$$W^k{}_i(\Lambda,p)W^k{}_j(\Lambda,p)=\delta_{ij}\,. \tag{36}$$

(For a detailed proof, see Appendix B.) Namely, the little group is indeed SO$(D-1)$. Eqs. (36) are also consistent with the fact that $W^0{}_i(\Lambda,p)=0$ (see the second line of Eq. (35)). Notice that

$$\eta_{\mu\nu}W^\mu{}_i(\Lambda,p)W^\nu{}_j(\Lambda,p)=-W^0{}_i(\Lambda,p)W^0{}_j(\Lambda,p)$$
$$+W^k{}_i(\Lambda,p)W^k{}_j(\Lambda,p)=\delta_{ij}\,, \tag{37}$$

which are exactly the same as Eqs. (36) taking account of $W^0{}_i(\Lambda,p)=0$.

We now proceed to discuss two important special cases: Λ is a general pure boost or a general pure rotation.

If $\Lambda_{\rm S}$ is a pure rotation, i.e., $\Lambda_{\rm S}=R_{\rm S}$, then by (A11), one has $(R_{\rm S}^\dagger)^{-1}=R_{\rm S}$. Plugging it into equation (21), and using (5), we are led to

$$W_{\rm S}(R,\eta)=\frac{R_{\rm S}[L_{\rm S}^{-1}(\eta)+L_{\rm S}(\eta)]}{2\cosh(\eta_\Lambda/2)}=\frac{\cosh(\eta/2)}{\cosh(\eta_\Lambda/2)}R_{\rm S}$$
$$=R_{\rm S}\,. \tag{38}$$

In the last equity, $\cosh(\eta_\Lambda/2)=\cosh(\eta/2)$ can be proved as follows: If $\Lambda=R$, one has $\Lambda_0{}^0=1$ and $\Lambda_0{}^i=0$; Plugging them into the first equation of (16) proves $\cosh(\eta_\Lambda)=\cosh(\eta)$. Using (38), we find that

$$W_{\rm S}(R,\eta)\gamma^iW_{\rm S}^{-1}(R,\eta)=W_\mu{}^i(R,\eta)\gamma^\mu=R_{\rm S}\gamma^iR_{\rm S}^{-1}$$
$$=R_j{}^i\gamma^j\,. \tag{39}$$

Namely, $W_0{}^i(R,\eta)=0$ and $W_j{}^i(R,\eta)=R_j{}^i$. That is

$$W(R,\eta)=R\,. \tag{40}$$

(One can also prove the above equation by substituting $\Lambda=R$ into Eq. (28).) In other words, if Λ is an arbitrary pure rotation R, the Wigner rotation $W(R,\eta)$ is exactly the same as R, independent of the parameter η or momentum p. In $4D$, the above important equation is proved by using a different method.[2] We see that in D dimensions, this equation still holds.

However, we have to emphasize that $W(R,\eta)=R$ is due to the particular "standard boost" (6) or (8). If we use another "standard boost" $\tilde L(p)$, satisfying $\tilde L(p)^\mu{}_\nu k^\nu=L(p)^\mu{}_\nu k^\nu=p^\mu$, but $\tilde L(p)\neq L(p)=(8)$, it is possible that $\tilde W(R,\eta)\neq R$. This can be seen as follows: According to Eq. (2),

$$\tilde W(R,p)=S(\Lambda p)W(R,p)S^{-1}(p)=S(\Lambda p)RS^{-1}(p)\,, \tag{41}$$

where $S(p)\equiv\tilde L^{-1}(p)L(p)$; Generally speaking, $S(\Lambda p)R\,S^{-1}(p)\neq R$.

If $\Lambda_{\rm S}$ is a pure boost, i.e., $\Lambda_{\rm S}=L_{\rm S}(\xi)$, then by (A11), we have $L_{\rm S}^\dagger(\xi)=L_{\rm S}(\xi)$. Plugging this equation into Eq. (21), we obtain

$$W_{\rm S}(\xi,\eta)\equiv W_{\rm S}(\Lambda,\eta)\big|_{\Lambda=L(\xi)}$$
$$=\frac{L_{\rm S}^{-1}(\xi)L_{\rm S}^{-1}(\eta)+L_{\rm S}(\xi)L_{\rm S}(\eta)}{\sqrt{2(1+[L_{\rm S}(\xi)L(\eta)]_0{}^0)}}\,. \tag{42}$$

Using Eqs. (5) and (6), a short calculation gives

$$W_{\rm S}(\xi,\eta)=\cos\left(\frac{\Theta}{2}\right)+\sin\left(\frac{\Theta}{2}\right)\frac{2\hat{\xi}_i\hat{\eta}_j\Sigma^{ij}}{\sqrt{1-(\hat{\xi}\cdot\hat{\eta})^2}}$$

$$=\exp\left(\Theta\frac{\hat{\xi}_i\hat{\eta}_j\Sigma^{ij}}{\sqrt{1-(\hat{\xi}\cdot\hat{\eta})^2}}\right), \tag{43}$$

where Θ is defined via the equation

$$\tan\Big(\frac{\Theta}{2}\Big)=\frac{\sinh(\xi/2)\sinh(\eta/2)\sqrt{1-(\hat{\xi}\cdot\hat{\eta})^2}}{\cosh(\xi/2)\cosh(\eta/2)+(\hat{\xi}\cdot\hat{\eta})\sinh(\xi/2)\sinh(\eta/2)}\,. \tag{44}$$

Note that $W_{\rm S}(\xi,\eta)$ is invariant under the discrete transformation $\eta\to-\xi$ and $\xi\to\eta$, or $\eta\to\xi$ and $\xi\to-\eta$ (see Eq. (43)), i.e.,

$$W_{\rm S}(\xi,\eta)=W_{\rm S}(\eta,-\xi)=W_{\rm S}(-\eta,\xi)\,. \tag{45}$$

Using

$$W_{\rm S}(\xi,\eta)\gamma^i W_{\rm S}^{-1}(\xi,\eta)={W_j}^i(\xi,\eta)\gamma^j\,, \tag{46}$$

and Eq. (A7), we find that

$${W_j}^i(\xi,\eta)=\exp\Big(\Theta\frac{\hat{\xi}_k\hat{\eta}_l\tau^{kl}}{\sqrt{1-(\hat{\xi}\cdot\hat{\eta})^2}}\Big)_j^{\ i}\,, \tag{47}$$

where

$${(\tau^{kl})_j}^i=\delta^{li}\delta^k_j-\delta^{ki}\delta^l_j \tag{48}$$

is the set of $\mathrm{SO}(D-1)$ matrices, defined via Eq. (A7). We see that ${W_j}^i(\xi,\eta)$ is a rotation on the η-ξ plane, possessing the symmetry property ${W_j}^i(\xi,\eta)={W_j}^i(\eta,-\xi)={W_j}^i(-\eta,\xi)$. The explicit expression of ${W_j}^i(\xi,\eta)$ can be worked out by either plugging (44) into (46), or expanding (47) directly:

$$\begin{aligned}{W_j}^i(\xi,\eta)&=\delta^i_j+\sin\Theta\frac{\hat{\xi}_k\hat{\eta}_l}{\sqrt{1-(\hat{\xi}\cdot\hat{\eta})^2}}{(\tau^{kl})_j}^i+2(1-\cos\Theta)\frac{(\hat{\xi}_m\hat{\eta}_n)(\hat{\xi}_k\hat{\eta}_l)}{1-(\hat{\xi}\cdot\hat{\eta})^2}{(\tau^{mn}\tau^{kl})_j}^i\\&=\delta^i_j+\frac{(\cosh\eta-1)(\cosh\xi-1)[2(\hat{\eta}\cdot\hat{\xi})\hat{\eta}^{(i}\hat{\xi}^{j)}-(\hat{\xi}^i\hat{\xi}^j+\hat{\eta}^i\hat{\eta}^j)]}{1+\cosh\eta\cosh\xi+(\hat{\eta}\cdot\hat{\xi})\sinh\eta\sinh\xi}\\&\quad-\frac{2\hat{\eta}^{[i}\hat{\xi}^{j]}[\sinh\eta\sinh\xi+(\cosh\eta-1)(\cosh\xi-1)(\hat{\eta}\cdot\hat{\xi})]}{1+\cosh\eta\cosh\xi+(\hat{\eta}\cdot\hat{\xi})\sinh\eta\sinh\xi}\,,\end{aligned} \tag{49}$$

where $\hat{\eta}^{(i}\hat{\xi}^{j)}=(\hat{\eta}^i\hat{\xi}^j+\hat{\eta}^j\hat{\xi}^i)/2$ and $\hat{\eta}^{[i}\hat{\xi}^{j]}=(\hat{\eta}^i\hat{\xi}^j-\hat{\eta}^j\hat{\xi}^i)/2$. In deriving (49), we have used (44). The Wigner rotation (49) can be also worked out by substituting the pure boost $\Lambda=L(\xi)$ into the general Wigner rotation (28).

2.2 *4 Dimensions*

For four dimensional spacetime, the Lorentz group $\mathrm{SO}(3,1)=\mathrm{SU}(2)\times\mathrm{SU}(2)$. Since the irreducible representation of $\mathrm{SU}(2)$ is well known, it is possible to work out the explicit expressions for the irreducible unitary representations of any dimensionality of the little group $W(\Lambda,\eta)$ (see (83)).

Our first goal is to work out the explicit expression of the spinor representation of the little group (21). We begin by calculating the general Lorentz transformation in spinor space

$$\Lambda_{\rm S}=\exp\left(\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\right). \tag{50}$$

To simplify calculations, we decompose the generators $\Sigma^{\mu\nu}$ and the parameters into the irreducible parts,

$$\Sigma^{\mu\nu}_{\pm}=\frac{1}{2}\Big(\Sigma^{\mu\nu}\pm\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho\sigma}\Sigma_{\rho\sigma}\Big)\,, \tag{51}$$

$$\omega^{\mu\nu}_{\pm}=\frac{1}{2}\Big(\omega^{\mu\nu}\pm\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho\sigma}\omega_{\rho\sigma}\Big)\,, \tag{52}$$

where the totally antisymmetric tensor is defined as $\varepsilon^{0123}=-\varepsilon_{0123}=1$. Notice that they satisfy the duality conditions

$$\Sigma^{\mu\nu}_{\pm}=\pm\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho\sigma}\Sigma_{\pm\rho\sigma}\,, \tag{53}$$

$$\omega^{\mu\nu}_{\pm}=\pm\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho\sigma}\omega_{\pm\rho\sigma}\,. \tag{54}$$

Now the general Lorentz transformation (50) reads

$$\Lambda_{\rm S}=\exp\Big(\frac{1}{2}\omega_{+\mu\nu}\Sigma^{\mu\nu}_{+}\Big)\exp\Big(\frac{1}{2}\omega_{-\mu\nu}\Sigma^{\mu\nu}_{-}\Big)\,. \tag{55}$$

Define

$$\Lambda_{\rm S\pm}\equiv\exp\Big(\frac{1}{2}\omega_{\pm\mu\nu}\Sigma^{\mu\nu}_{\pm}\Big)\,, \tag{56}$$

$$\omega_{\pm}=\sqrt{\omega_{\pm\mu\nu}\omega^{\mu\nu}_{\pm}}\quad\text{and}\quad\hat{\omega}_{\pm\mu\nu}=\frac{\omega_{\pm\mu\nu}}{\omega_{\pm}}. \tag{57}$$

Note that $\Lambda_{\rm S\pm}$ are nothing but the $\mathrm{SL}(2,C)$ matrices. By a direct calculation, we find that

$$\Lambda_{S\pm} = \frac{1}{2}(1\mp i\gamma_5)\cos\frac{\omega_\pm}{2} + \hat{\omega}_{\pm\mu\nu}\Sigma_\pm^{\mu\nu}\sin\frac{\omega_\pm}{2} + \frac{1}{2}(1\pm i\gamma_5), \quad (58)$$

where γ_5 is defined as $\gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$, or $\gamma_5 = (1/4!)\varepsilon_{\mu\nu\rho\sigma} \times\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$. Using the above equations, it is not difficult to work out Λ_S,

$$\Lambda_S = \Lambda_{S+}\Lambda_{S-} = \frac{1}{2}(1 - i\gamma_5)\cos\frac{\omega_+}{2} + \frac{1}{2}(1 + i\gamma_5)\cos\frac{\omega_-}{2} + \hat{\omega}_{+\mu\nu}\Sigma_+^{\mu\nu}\sin\frac{\omega_+}{2} + \hat{\omega}_{-\mu\nu}\Sigma_-^{\mu\nu}\sin\frac{\omega_-}{2}. \quad (59)$$

The vector counterparts of $\Lambda_{S\pm}$ are defined via the equations

$$\Lambda_{S\pm}\gamma^\nu\Lambda_{S\pm}^{-1} \equiv \Lambda_{\pm\mu}{}^\nu\gamma^\mu. \quad (60)$$

A straightforward computation gives

$$\Lambda_{\pm\mu}{}^\nu = \cos\Big(\frac{\omega_\pm}{2}\Big)\delta_\mu{}^\nu + 2\sin\Big(\frac{\omega_\pm}{2}\Big)\hat{\omega_\pm}_\mu{}^\nu. \quad (61)$$

That is

$$\Lambda_{\pm0}{}^0 = \cos(\omega_\pm/2), \quad \Lambda_{\pm0}{}^i = \Lambda_{\pm i}{}^0 = -2\sin(\omega_\pm/2)\hat{\omega}_{\pm i0},$$
$$\Lambda_{\pm i}{}^j = \cos(\omega_\pm/2)\delta^{ij} \mp 2i\sin(\omega_\pm/2)\varepsilon^{ijk}\hat{\omega}_{\pm k0}. \quad (62)$$

If it is a pure boost, i.e., $\omega_{ij} = 0$ and $\omega_{i0} \to \eta^i$, one has

$$L_{\pm0}{}^0(\eta) = \cosh(\eta/2), \; L_{\pm0}{}^i(\eta) = L_{\pm i}{}^0(\eta) = -\sinh(\eta/2)\hat{\eta}^i,$$
$$L_{\pm i}{}^j(\eta) = \cosh(\eta/2)\delta^{ij} \mp i\sinh(\eta/2)\varepsilon^{ijk}\hat{\eta}^k. \quad (63)$$

The standard Lorentz transformation (6) can be also derived using $L_\mu{}^\nu(\eta) = \big(L_-L_+\big)_\mu{}^\nu(\eta)$ and (63). The vector counterpart of Λ_S is given by

$$\Lambda_\mu{}^\nu = (\Lambda_-\Lambda_+)_\mu{}^\nu = \cos\Big(\frac{\omega_+}{2}\Big)\cos\Big(\frac{\omega_-}{2}\Big)\delta_\mu{}^\nu + 2\cos\Big(\frac{\omega_+}{2}\Big)\sin\Big(\frac{\omega_-}{2}\Big)\hat{\omega}_{-\mu}{}^\nu + 2\cos\Big(\frac{\omega_-}{2}\Big)\sin\Big(\frac{\omega_+}{2}\Big)\hat{\omega}_{+\mu}{}^\nu + 4\sin\Big(\frac{\omega_+}{2}\Big)\sin\Big(\frac{\omega_-}{2}\Big)(\hat{\omega}_+\hat{\omega}_-)_\mu{}^\nu. \quad (64)$$

Alternatively, using the relation between SL$(2, C)$ and the 4D Lorentz group, one can calculate $\Lambda_\mu{}^\nu$ as follows

$$\Lambda_{S\pm}(\overline{\gamma}_\pm\gamma^\nu)\Lambda_{S\mp}^{-1} = \Lambda_\mu{}^\nu(\overline{\gamma}_\pm\gamma^\mu), \quad (65)$$

where

$$\overline{\gamma}_\pm \equiv \frac{1}{2}(1 \mp i\gamma_5). \quad (66)$$

We now would like to work out the spinor little group (21). We expect that it takes the "standard" form

$$W_S(\Lambda, \eta) = \frac{\gamma^0\Lambda_S L_S(\eta)(\gamma^0)^{-1} + \Lambda_S L_S(\eta)}{\sqrt{2(1 + [\Lambda L(p)]_0^0)}} = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\hat{\Theta}_i(2\Sigma_i), \quad (67)$$

where

$$\Sigma_i \equiv \frac{1}{2}\varepsilon_{ijk}\Sigma^{jk}, \quad \Theta_i \equiv \frac{1}{2}\varepsilon_{ijk}\Theta^{jk}, \quad \hat{\Theta}_i \equiv \Theta_i/\sqrt{\Theta^2}, \quad (68)$$

and $\Theta^i = \Theta^i(\Lambda, \eta)$ is a function of $\Lambda_\mu{}^\nu$ and η^i. To determine Θ^i, let us first calculate $\Lambda_S L_S(\eta)$. According to Eq. (59), it must take the general form

$$\Lambda_S L_S(\eta) = \frac{1}{2}(1 - i\gamma_5)\cos\frac{\alpha_+}{2} + \frac{1}{2}(1 + i\gamma_5)\cos\frac{\alpha_-}{2} + \hat{\alpha}_{+\mu\nu}\Sigma_+^{\mu\nu}\sin\frac{\alpha_+}{2} + \hat{\alpha}_{-\mu\nu}\Sigma_-^{\mu\nu}\sin\frac{\alpha_-}{2}, \quad (69)$$

where the new parameters $\hat{\alpha}_{\pm\mu\nu} = \hat{\alpha}_{\pm\mu\nu}(\omega, \eta)$ and $\alpha_\pm = \alpha_\pm(\omega, \eta)$ are functions of $\omega_{\mu\nu}$ and η_i, to be determined later. The definitions and properties of $\hat{\alpha}_{\pm\mu\nu}$ and $\alpha_\pm$ are similar to that of $\hat{\omega}_{\pm\mu\nu}$ and $\omega_\pm$ (see Eqs. (52), (54), and (57)). Inserting Eq. (69) into the first equation of (67),

$$W_S(\Lambda, \eta) = \frac{(\cos(\alpha_+/2) + \cos(\alpha_-/2)) - 2i(\sin(\alpha_+/2)\hat{\alpha}_{+i0} + \sin(\alpha_-/2)\hat{\alpha}_{+i0})(2\Sigma_i)}{\sqrt{2(1 + [\Lambda L(p)]_0{}^0)}}. \quad (70)$$

We now must determine the relations of $\alpha_{\pm\mu\nu}$ between $\omega_{\pm\mu\nu}$ and η_i. According to Eq. (61), the vector representations of $\Lambda_\pm L_\pm(\eta)$ are given by

$$(\Lambda_\pm L_\pm(\eta))_\mu{}^\nu = \cos\Big(\frac{\alpha_\pm}{2}\Big)\delta_\mu{}^\nu + 2\sin\Big(\frac{\alpha_\pm}{2}\Big)\hat{\alpha}_{\pm\mu}{}^\nu. \quad (71)$$

Substituting Eqs. (62) and (63) into Eq. (71), one obtains

$$(\Lambda_\pm L_\pm(\eta))_0{}^0 = \cos\frac{\alpha_\pm}{2} = \cos\frac{\omega_\pm}{2}\cosh\frac{\eta}{2} + 2\sin\frac{\omega_\pm}{2}\sinh\frac{\eta}{2}(\hat{\omega}_\pm\cdot\hat{\eta}), \quad (72)$$

$$(\Lambda_\pm L_\pm(\eta))_0{}^i = -2\sin\frac{\alpha_\pm}{2}\hat{\alpha}_{\pm i0} = -\cos\frac{\omega_\pm}{2}\sinh\frac{\eta}{2}\eta_i - 2\sin\frac{\omega_\pm}{2}\cosh\frac{\eta}{2}\hat{\omega}_{\pm i0} \mp 2i\sin\frac{\omega_\pm}{2}\sinh\frac{\eta}{2}(\hat{\omega}_\pm\times\hat{\eta})_i, \quad (73)$$

where $\hat{\omega}_\pm\cdot\hat{\eta} \equiv \hat{\omega}_{\pm i0}\hat{\eta}_i$ and $(\hat{\omega}_\pm\times\hat{\eta})_i = \varepsilon_{ijk}\omega_{\pm j0}\hat{\eta}_k$. Using the above two equations, all terms in the numerator of Eq. (70) can be expressed in terms of $\omega_{\pm\mu\nu}$ and η_i.

Using Eqs. (72) and (73), we see that Eq. (70) also takes the following form:

$$W_S(\Lambda, \eta) = \frac{(\Lambda_+L_+(\eta))_0{}^0 + (\Lambda_-L_-(\eta))_0{}^0}{\sqrt{2(1 + [\Lambda L(p)]_0{}^0)}} + i\frac{(\Lambda_+L_+(\eta))_0{}^i + (\Lambda_-L_-(\eta))_0{}^i}{\sqrt{2(1 + [\Lambda L(p)]_0{}^0)}}(2\Sigma_i). \quad (74)$$

Here

$$\sqrt{2(1 + (\Lambda L)_0{}^0)} = \sqrt{2[1 + \Lambda_0{}^0\cosh(\eta) - \Lambda_0{}^i\hat{\eta}_i\sinh(\eta)]}, \quad (75)$$

(See the first equation of Eq. (16)).

Using Eqs. (72), (73), and (75), Eq. (67) or (74) can be readily worked out:

$$W_S(\Lambda, \eta) = \cos\frac{\Theta}{2} + \sin\frac{\Theta}{2}\hat{\Theta}_i(2\Sigma_i) = \exp(\Theta\hat{\Theta}_i\Sigma_i), \quad (76)$$

where

$$\cos\frac{\Theta}{2}=\frac{[(\cos(\omega_+/2)+\cos(\omega_-/2))\cosh(\eta/2)+2(\sin(\omega_+/2)(\hat{\omega}_+\cdot\hat{\eta})+\sin(\omega_-/2)(\hat{\omega}_-\cdot\hat{\eta}))\sinh(\eta/2)]}{\sqrt{2[1+\Lambda_0{}^0\cosh(\eta)-\Lambda_0{}^i\hat{\eta}_i\sinh(\eta)]}}\,, \tag{77}$$

$$\sin\frac{\Theta}{2}\hat{\Theta}_i=\frac{1}{\sqrt{2[1+\Lambda_0{}^0\cosh(\eta)-\Lambda_0{}^i\hat{\eta}_i\sinh(\eta)]}}\Big[-\mathrm{i}\Big(\Big(\cos\frac{\omega_+}{2}-\cos\frac{\omega_-}{2}\Big)\sinh(\eta/2)\hat{\eta}_i+2\Big(\sin\frac{\omega_+}{2}\hat{\omega}_{+i0}+\sin\frac{\omega_-}{2}\hat{\omega}_{-i0}\Big)\cosh\frac{\eta}{2}$$
$$+2\mathrm{i}\Big(\sin\frac{\omega_+}{2}\hat{\omega}_{+j0}+\sin\frac{\omega_-}{2}\hat{\omega}_{-j0}\Big)\varepsilon_{ijk}\eta_k\sinh(\eta/2)\Big)\Big]\,. \tag{78}$$

In Eqs. (77) and (78), the set of parameters η^i is related to the momentum $\vec{p}$ and mass M via (7), and the relation between $\Lambda_\mu{}^\nu$ and $\omega_{\mu\nu}$ is given by (64).

Note that Eq. (76) provides a third way to construct the vector representation of the little group (28) in four-dimensional spacetime. Since now $\cos\Theta/2$ and $\sin(\Theta/2)\hat{\Theta}_i$ have been worked out completely, it is not difficult to complete the calculation

$$W_j{}^i(\Lambda,\eta)=(\exp(\Theta\hat{\Theta}_k\tau^k))_j{}^i=\cos\Theta\delta_{ji}+(1-\cos\Theta)\hat{\Theta}_j\hat{\Theta}_i+\sin\Theta\varepsilon_{jik}\Theta^k\,, \tag{79}$$

where $\tau^k=(1/2)\varepsilon^{kij}\tau_{ij}$, with $(\tau_{ij})_{kl}=\delta_{ik}\delta_{jl}-\delta_{jk}\delta_{il}$. Plugging the data of Eqs. (77) and (78) into Eq. (79), a slightly length calculation gives

$$W_j{}^i(\Lambda,\eta)=-\frac{[\Lambda L(\eta)]_0{}^i[\Lambda L(\eta)]_j{}^0}{1+[\Lambda L(\eta)]_0{}^0}+[\Lambda L(\eta)]_j{}^i\,, \tag{80}$$

which is exactly the same as (28) or (29), with $i,j=1,2,3$.

The Wigner rotation in any irreducible representation can be constructed by replacing $\tau_i\to-\mathrm{i}J_i$ in the right-hand side of the first equity of Eq. (79):

$$W^{(j)}_{m'm}(\Lambda,\eta)\equiv W^{(j)}_{m'm}(\Theta(\Lambda,\eta))=(\exp(-\mathrm{i}\Theta\hat{\Theta}_kJ^{(j)}_k))_{m'm}\,. \tag{81}$$

Here the irreducible representations of J_i are the familiar ones,

$$(J^{(j)}_3)_{m'm}=m\hbar\delta_{m'm}\,,\qquad (J^{(j)}_1\pm\mathrm{i}J^{(j)}_2)_{m'm}=\hbar\delta_{m',m\pm1}\sqrt{(j\pm m+1)(j\mp m)}\,, \tag{82}$$

where $m',m=j,j-1,\ldots,-(j-1),-j$. The Wigner's formula for d-function may be useful in calculating $W^{(j)}_{m'm}(\Lambda,\eta)$. For instance, in the special case of $\hat{\Theta}_k=\hat{y}$ or $\Theta\hat{\Theta}_kJ^{(j)}_k=\Theta J^{(j)}_2$, Eq. (81) is nothing but the the Wigner's d-function:[3]

$$W^{(j)}_{m'm}(\Theta(\Lambda,\eta))=\sum_k(-1)^{k-m+m'}\frac{\sqrt{(j+m)!(j-m)!(j+m')!(j-m')!}}{(j+m-k)!k!(j-k-m')!(j-m+m')!}\Big(\cos\frac{\Theta}{2}\Big)^{2j-2k+m-m'}\Big(\sin\frac{\Theta}{2}\Big)^{2k-m+m'}, \tag{83}$$

where the expressions of $\cos\Theta/2$ and $\sin\Theta/2$ are given by Eqs. (77) and (78).

2.3 *Summary of This Section*

In summary, in D dimensions, the spinor representation of the Wigner rotation is given by

$$W_{\mathrm{S}}(\Lambda,\eta)=\frac{\gamma^0\Lambda_{\mathrm{S}}L_{\mathrm{S}}(\eta)(\gamma^0)^{-1}+\Lambda_{\mathrm{S}}L_{\mathrm{S}}(\eta)}{\sqrt{2(1+[\Lambda L(p)]_0{}^0)}}\,, \tag{84}$$

and the vector representation of the Wigner rotation is given by

$$W_j{}^i(\Lambda,p)=-\frac{[\Lambda L(p)]_0{}^i[\Lambda L(p)]_j{}^0}{1+[\Lambda L(p)]_0{}^0}+[\Lambda L(p)]_j{}^i$$
$$=\frac{[-\Lambda_0{}^0p^i/M+\Lambda_0{}^i+(\gamma-1)\Lambda_0{}^k\hat{p}_k\hat{p}^i](\Lambda p)_j}{M+(\Lambda p)^0}$$
$$-\Lambda_j{}^0p^i/M+(\gamma-1)\Lambda_j{}^k\hat{p}_k\hat{p}^i+\Lambda_j{}^i\,. \tag{85}$$

Here $\Lambda_\mu{}^\nu$ is an arbitrary Lorentz transformation, and $L(p)$ or $L(\eta)$ carries the standard D-momentum $k^\mu=(0,0,\ldots,0,M)$ to p^μ, i.e. $L^\mu{}_\nu(\eta)k^\nu=p^\mu$, with p^μ the D-momentum of the particle of mass M. The explicit expressions of $L(p)$ and $L(\eta)$ are given by Eqs. (6)–(8). And Λ_{S} and $L_{\mathrm{S}}(\eta)$ are spinor counterparts of $\Lambda_\mu{}^\nu$ and $L_\mu{}^\nu(\eta)$, respectively. The explicit expression for $L_{\mathrm{S}}(\eta)$ is given by (5).

3 Wigner Rotations for Massless Particles

3.1 *D Dimensions*

We now turn to the case of massless particles in D dimensions. We define the standard D-vector of energy κ as

$$k^\mu=(0,0,\ldots,\kappa,\kappa)\,. \tag{86}$$

We see that $k_\mu\gamma^\mu=\kappa(-\gamma^0+\gamma^{D-1})$. It is therefore more convenient to work in the light-cone coordinates (our conventions are summarized in Appendix A),

$$\gamma^\pm=\frac{1}{\sqrt{2}}(\pm\gamma^0+\gamma^{D-1})\,,\quad k^\pm=\frac{1}{\sqrt{2}}(k^0+k^{D-1})\,. \tag{87}$$

In the light-cone coordinates, we have

$$k_\mu\gamma^\mu=k_-\gamma^-=\sqrt{2}\kappa\gamma^-\,. \tag{88}$$

The little group $W^{\mu}{}_{\nu}$ preserves k^{μ}, in the sense that $W^{\mu}{}_{\nu}k^{\nu} = k^{\mu}$. In the spinor space, this is equivalent to require that

$$W_{\rm S}\gamma^{-}W_{\rm S}^{-1} = \gamma^{-}\,, \tag{89}$$

where $W_{\rm S}$ is spinor representation of the little group.

We define the "standard Lorentz transformation" in spinor space as follows

$$\begin{aligned} L_{\rm S}(\lambda) &\equiv \exp(\lambda_a\Sigma^{+a})\exp(\lambda_-\Sigma^{+-})\,, \\ &= \cosh\frac{\lambda_-}{2} + {\rm e}^{-\lambda_-/2}\lambda_a\Sigma^{+a} + 2\sinh\frac{\lambda_-}{2}\Sigma^{+-}\,, \end{aligned} \tag{90}$$

where the set of generators is $(\Sigma^{+a}, \Sigma^{+-})$, $a = 1,\ldots, D-2$, with $\Sigma^{+a} = (1/\sqrt{2})(\Sigma^{0a} + \Sigma^{D-1,a})$ and $\Sigma^{+-} = \Sigma^{0,D-1}$, and the parameters are defined as[‖]

$$(\lambda_a, \lambda_-) = (-p_a/p_-, -\ln(p_-/k_-))\,, \tag{91}$$

where $p_- = p^+ \equiv (p^0 + p^{D-1})/\sqrt{2}$. The vector counterpart of (90) $L_{\mu}{}^{\nu}(\lambda)$, defined via the equation

$$L_{\rm S}(\lambda)\gamma^{\mu}L_{\rm S}^{-1}(\lambda) = L_{\nu}{}^{\mu}(\lambda)\gamma^{\nu}\,, \tag{92}$$

is therefore given by

$$L(\lambda) = \exp(\lambda_a\tau^{+a})\exp(\lambda_-\tau^{+-})\,. \tag{93}$$

Here $\tau^{+a} = (1/\sqrt{2})(\tau^{0a} + \tau^{D-1,a})$ and $\tau^{+-} = \tau^{0,D-1}$. The matrix elements of $\tau^{\mu\nu}$ are defined as $(\tau^{\mu\nu})_{\sigma}{}^{\rho} = \delta^{\mu}_{\sigma}\eta^{\nu\rho} - \delta^{\nu}_{\sigma}\eta^{\mu\rho}$ (see Eq. (A9)). The matrix elements of $L(\lambda)$ can be either read off from (92) or calculated directly using (93): In the lightcone coordinate system, they are given by

$$\begin{aligned} &L_a{}^b(\lambda) = \delta_a{}^b\,, \quad L_a{}^-(\lambda) = \frac{p_a}{k_-}\,, \quad L_a{}^+(\lambda) = 0\,, \\ &L_-{}^b(\lambda) = 0\,, \quad L_-{}^-(\lambda) = \frac{p_-}{k_-}\,, \quad L_-{}^+(\lambda) = 0\,, \\ &L_+{}^b(\lambda) = -\frac{p^b}{p_-}\,, \quad L_+{}^-(\lambda) = \frac{p_+}{k_-}\,, \quad L_+{}^+(\lambda) = \frac{k_-}{p_-}\,. \end{aligned} \tag{94}$$

It is straightforward to verify that $L(\lambda)$ does bring k^{μ} to p^{μ}.

The Wigner rotation in spinor space is defined as

$$W_{\rm S}(\Lambda, \lambda) = L_{\rm S}^{-1}(\lambda_{\Lambda})\Lambda_{\rm S}L_{\rm S}(\lambda)\,. \tag{95}$$

Here $\Lambda_{\rm S}$ is the general Lorentz transformation in spinor space, and

$$\begin{aligned} L_{\rm S}^{-1}(\lambda_{\Lambda}) &= \exp(-\lambda_{\Lambda-}\Sigma^{+-})\exp(-\lambda_{\Lambda a}\Sigma^{+a}) \\ &= \cosh\frac{\lambda_{\Lambda-}}{2} - {\rm e}^{-\lambda_{\Lambda-}/2}\lambda_{\Lambda a}\Sigma^{+a} - 2\sinh\frac{\lambda_{\Lambda-}}{2}\Sigma^{+-}\,, \end{aligned} \tag{96}$$

where the set of parameters λ_{Λ} is defined such that $L(\lambda_{\Lambda})$ transforms k^{μ} into $\Lambda^{\mu}{}_{\nu}p^{\nu} \equiv (\Lambda p)^{\mu}$, i.e.,

$$(\lambda_{\Lambda a}, \lambda_{\Lambda -}) = \Big(-\frac{(\Lambda p)_a}{(\Lambda p)_-}, -\ln\frac{(\Lambda p)_-}{k_-}\Big)\,. \tag{97}$$

(The matrix elements of $L(\lambda_{\Lambda})$ are given by Eq. (107).)

The general Wigner rotation $W_{\nu}{}^{\mu}(\Lambda, \lambda)$ can be read off from the following equation:

$$W_{\rm S}(\Lambda, \lambda)\gamma^{\mu}W_{\rm S}^{-1}(\Lambda, \lambda) = W_{\nu}{}^{\mu}(\Lambda, \lambda)\gamma^{\nu}\,, \tag{98}$$

where in the light-cone coordinates $\gamma^{\mu} = (\gamma^a, \gamma^-, \gamma^+)$.

First of all, it is not difficult to verify that Eq. (89) is obeyed,

$$W_{\rm S}(\Lambda, \lambda)\gamma^{-}W_{\rm S}^{-1}(\Lambda, \lambda) = \gamma^{-}\,. \tag{99}$$

The above equation implies that

$$W_b{}^-(\Lambda, \lambda) = W_+{}^-(\Lambda, \lambda) = 0\,, \quad W_-{}^-(\Lambda, \lambda) = 1\,. \tag{100}$$

Secondly, after a length calculation, one obtains

$$\begin{aligned} W_{\rm S}(\Lambda, \lambda)\gamma^{a}W_{\rm S}^{-1}(\Lambda, \lambda) &= [(\Lambda_b{}^a + \lambda^a\Lambda_b{}^+) + (\Lambda_-{}^a \\ &+ \lambda^a\Lambda_-{}^+)\lambda_{\Lambda b}]\gamma^b + {\rm e}^{\lambda_{\Lambda}}(\Lambda_-{}^a + \lambda^a\Lambda_-{}^+)\gamma^-\,. \end{aligned} \tag{101}$$

It can be seen that

$$\begin{aligned} &W_+{}^a(\Lambda, \lambda) = 0\,, \\ &W_b{}^a(\Lambda, \lambda) = (\Lambda_b{}^a + \lambda^a\Lambda_b{}^+) + (\Lambda_-{}^a + \lambda^a\Lambda_-{}^+)\lambda_{\Lambda b} \\ &\quad = -\frac{[\Lambda L(\lambda)]_a{}^-[\Lambda L(\lambda)]_-{}^b}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_a{}^b \\ &\quad = \frac{1}{p_-(\Lambda p)_-}((p_-\Lambda_b{}^a - p^a\Lambda_b{}^+)(\Lambda p)_- \\ &\qquad -(p_-\Lambda_-{}^a - p^a\Lambda_-{}^+)(\Lambda p)_b), \\ &W_-{}^a(\Lambda, \lambda) = {\rm e}^{\lambda_{\Lambda}}(\Lambda_-{}^a + \lambda^a\Lambda_-{}^+) = \frac{[\Lambda L(\lambda)]_-{}^a}{[\Lambda L(\lambda)]_-{}^-}\,. \end{aligned} \tag{102}$$

In calculating Eqs. (102), we have used Eqs. (94) and (97). (The relation between the standard Lorentz transformation $L(\lambda)$ and the momentum p^{μ} is given by Eq. (94).)

Finally, we consider the following equation

$$W_{\rm S}(\Lambda, \lambda)\gamma^{+}W_{\rm S}^{-1}(\Lambda, \lambda) = W_{\nu}{}^{+}(\Lambda, \lambda)\gamma^{\nu}\,. \tag{103}$$

We find that the results are

$$\begin{aligned} &W_+{}^+(\Lambda, \lambda) = 1\,, \\ &W_-{}^+(\Lambda, \lambda) = \frac{[\Lambda L(\lambda)]_-{}^+}{[\Lambda L(\lambda)]_-{}^-}\,, \\ &W_b{}^+(\Lambda, \lambda) = -\frac{[\Lambda L(\lambda)]_-{}^+[\Lambda L(\lambda)]_b{}^-}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_b{}^+\,, \end{aligned} \tag{104}$$

where $L(p)$ is defined by Eq. (94).

Note that the matrix elements in Eqs. (104) are *not* independent quantities, in the sense that they can be expressed in terms of the other matrix elements by using the Lorentz transformation

$$W_{\mu}{}^{\rho}W_{\nu}{}^{\sigma}\eta_{\rho\sigma} = \eta_{\mu\nu}\,. \tag{105}$$

For instance, using $W_b{}^{\rho}W_-{}^{\sigma}\eta_{\rho\sigma} = \eta_{b-} = 0$, we obtain that

$$\begin{aligned} W_b{}^+(\Lambda, \lambda) &= -W_b{}^a(\Lambda, \lambda)W_-{}^a(\Lambda, \lambda) \\ &= -\frac{[\Lambda L(\lambda)]_-{}^+[\Lambda L(\lambda)]_b{}^-}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_b{}^+\,, \end{aligned} \tag{106}$$

‖For a massless particle of unit energy, $k_- = \sqrt{2}\kappa = \sqrt{2}$.

which is exactly the same as the last equation of Eq. (104). On the other hand, the elements in Eq. (100) are either 0 or 1, so the only "non-trivial" elements are $W_+{}^a(\Lambda,\lambda)$ and $W_b{}^a(\Lambda,\lambda)$.

Here is another way to calculate the little group element $W_\mu{}^\nu(\Lambda,\lambda)$. First, one can obtain $L_\mu{}^\nu(\lambda_\Lambda)$ by replacing $p^\mu\to(\Lambda p)^\mu$ and $\lambda\to\lambda_\Lambda$ in Eq. (94),

$$\begin{aligned}&L_a{}^b(\lambda_\Lambda)=\delta_a^b\,,\quad L_a{}^-(\lambda_\Lambda)=\frac{(\Lambda p)_a}{k_-}\,,\quad L_a{}^+(\lambda_\Lambda)=0\,,\\&L_-{}^b(\lambda_\Lambda)=0\,,\quad L_-{}^-(\lambda_\Lambda)=\frac{(\Lambda p)_-}{k_-}\,,\quad L_-{}^+(\lambda_\Lambda)=0\,,\\&L_+{}^b(\lambda_\Lambda)=-\frac{(\Lambda p)^b}{(\Lambda p)_-}\,,\quad L_+{}^-(\lambda_\Lambda)=\frac{(\Lambda p)_+}{k_-}\,,\\&L_+{}^+(\lambda_\Lambda)=\frac{k_-}{(\Lambda p)_-}\,.\end{aligned}\tag{107}$$

Second, using the fundamental conditions $L_\mu{}^\rho(\lambda_\Lambda)L_\mu{}^\sigma(\lambda_\Lambda)\eta_{\rho\sigma}=\eta_{\mu\nu}$, it is not difficult to determine the inverse of $L_\mu{}^\nu(\lambda_\Lambda)$,

$$(L^{-1})_\mu{}^\nu(\lambda_\Lambda)=\eta_{\mu\rho}\eta^{\nu\sigma}L_\sigma{}^\rho(\lambda_\Lambda)\,.\tag{108}$$

A straightforward computation gives

$$\begin{aligned}&(L^{-1})_a{}^b(\lambda_\Lambda)=\delta_a^b\,,\quad(L^{-1})_a{}^-(\lambda_\Lambda)=-\frac{(\Lambda p)_a}{(\Lambda p)_-}\,,\\&(L^{-1})_a{}^+(\lambda_\Lambda)=0\,,\quad(L^{-1})_-{}^b(\lambda_\Lambda)=0\,,\\&(L^{-1})_-{}^-(\lambda_\Lambda)=\frac{\kappa_-}{(\Lambda p)_-}\,,\quad(L^{-1})_-{}^+(\lambda_\Lambda)=0\,,\\&(L^{-1})_+{}^b(\lambda_\Lambda)=\frac{(\Lambda p)^b}{\kappa_-}\,,\quad(L^{-1})_+{}^-(\lambda_\Lambda)=\frac{(\Lambda p)_+}{\kappa_-}\,,\\&(L^{-1})_+{}^+(\lambda_\Lambda)=\frac{(\Lambda p)_-}{\kappa_-}\,.\end{aligned}\tag{109}$$

Finally, one can calculate all matrix elements $W_\mu{}^\nu(\Lambda,\lambda)$ by substituting Eqs. (94) and (109) into the equation

$$W(\Lambda,\lambda)=L^{-1}(\lambda_\Lambda)\Lambda L(\lambda)\,.\tag{110}$$

For instance, using Eq. (109), we find that

$$\begin{aligned}W_b{}^a(\Lambda,\lambda)&=(L^{-1})_b{}^+(\lambda_\Lambda)[\Lambda L(\lambda)]_+{}^a+(L^{-1})_b{}^-(\lambda_\Lambda)[\Lambda L(\lambda)]_-{}^a+(L^{-1})_b{}^c(\lambda_\Lambda)[\Lambda L(\lambda)]_c{}^a\\&=0-\frac{(\Lambda p)_b}{(\Lambda p)_-}[\Lambda L(\lambda)]_-{}^a+\delta_b{}^c[\Lambda L(\lambda)]_c{}^a=-\frac{[\Lambda L(\lambda)]_a{}^-[\Lambda L(\lambda)]_-{}^b}{[\Lambda L(\lambda)]_-{}^-}+[\Lambda L(\lambda)]_a{}^b\,,\end{aligned}\tag{111}$$

which is exactly the same as the second equation of Eq. (102). In the last line, we have used Eq. (94).

By a length but direct calculation, one can show that

$$W_a{}^c(\Lambda,\lambda)W_b{}^c(\Lambda,\lambda)=\delta_{ab}\,.\tag{112}$$

(For a detailed proof, see Appendix C.) So $W_b{}^a(\Lambda,\lambda)$ must be the elements of the SO$(D-2)$ subgroup. Hence the group elements $W_b{}^a(\Lambda,\lambda)$ are the most important result of this section. Eq. (112) also follows from

$$\eta_{\mu\nu}W_a{}^\mu(\Lambda,\lambda)W_b{}^\nu(\Lambda,\lambda)=\delta_{ab}\,,\tag{113}$$

and $W_b{}^-(\Lambda,\lambda)=0$ (see Eq. (100)).

However, we still need to show that the little group is ISO$(D-2)$. Using (101) and $(\gamma^-)^2=0$, one obtains immediately

$$W_{\rm S}(\Lambda,\lambda)A^aW_{\rm S}^{-1}(\Lambda,\lambda)=W_b{}^a(\Lambda,\lambda)A^b\,,\tag{114}$$

where $A^a=\Sigma^{-a}$ (see Eq. (A19)). On the other hand,

$$W_{\rm S}(\Lambda,\lambda)\Sigma^{ab}W_{\rm S}^{-1}(\Lambda,\lambda)=W_c{}^a(\Lambda,\lambda)W_d{}^b(\Lambda,\lambda)\Sigma^{cd}+(W_-{}^a(\Lambda,\lambda)W_c{}^b(\Lambda,\lambda)-W_-{}^b(\Lambda,\lambda)W_c{}^a(\Lambda,\lambda))A^c\,.\tag{115}$$

After defining

$$a^a(\Lambda,\lambda)\equiv W_-{}^b(\Lambda,\lambda)W_a{}^b(\Lambda,\lambda)=-W_a{}^+(\Lambda,\lambda)\,,\tag{116}$$

(See Eq. (106)) Eq. (115) can be written as

$$W_{\rm S}(\Lambda,\lambda)\Sigma^{ab}W_{\rm S}^{-1}(\Lambda,\lambda)=W_c{}^a(\Lambda,\lambda)W_d{}^b(\Lambda,\lambda)(\Sigma^{cd}+a^c(\Lambda,\lambda)A^d-a^d(\Lambda,\lambda)A^c)\,.\tag{117}$$

Equations (114) and (117) are the standard transformation law of the set of generators of ISO$(D-2)$, with the spinor group parameterized as

$$W_{\rm S}(\Lambda,\lambda)=\exp(a^a(\Lambda,\lambda)A^a)\exp\Big(\frac{1}{2}\Theta_{cd}(\Lambda,\lambda)\Sigma^{cd}\Big)\,.\tag{118}$$

Here the set of parameters $\Theta_{cd}(\Lambda,\lambda)$ is defined via the equation

$$\exp\Big(\frac{1}{2}\Theta_{cd}(\Lambda,\lambda)\tau^{cd}\Big)_a{}^b=W_a{}^b(\Lambda,\lambda)\,,\tag{119}$$

with $(\tau^{cd})_a{}^b=\delta_a^c\delta^{db}-\delta_a^d\delta^{cb}$.

It is interesting to note that in our construction, the spinor representation matrices of the translation operators A^a satisfy

$$(A^a)^2 = 0, \quad \text{(no sum)} \tag{120}$$

where we have used Eq. (A19). So the eigenvalues of A^a are *zero* automatically, without even considering the topology of the Lorentz group.[2]

Equation (118) suggests that the general representation of the little group takes the form

$$W_{(R)}(\Lambda,\lambda) = \exp(a^a(\Lambda,\lambda)T^a_{(R)})\exp\Big(\frac{1}{2}\Theta_{cd}(\Lambda,\lambda)J^{cd}_{(R)}\Big)\,, \tag{121}$$

with $T^a_{(R)}$ and $J^{cd}_{(R)}$ furnishing a representation R of the generators of the ISO$(D-2)$ group. However, to avoid continuous degree of freedom of massless particles, we require that the physical states are eigenstates of $T^a_{(R)}$, but all eigenvalues are zero.[2]

3.2 *Summary of This Section*

In D dimensions, the vector representation of the SO$(D-2)$ part of the Wigner little group ISO$(D-2)$ is given by

$$W_b{}^a(\Lambda,\lambda) = -\frac{[\Lambda L(\lambda)]_a{}^-[\Lambda L(\lambda)]_-{}^b}{[\Lambda L(\lambda)]_-{}^-} + [\Lambda L(\lambda)]_a{}^b = \frac{1}{p_-(\Lambda p)_-}((p_-\Lambda_b{}^a - p^a\Lambda_b{}^+)(\Lambda p)_- - (p_-\Lambda_-{}^a - p^a\Lambda_-{}^+)(\Lambda p)_b), \tag{122}$$

and the translation part is defined as

$$a^a(\Lambda,p) = -W_a{}^+(\Lambda,\lambda) = \frac{[\Lambda L(\lambda)]_-{}^+[\Lambda L(\lambda)]_a{}^-}{[\Lambda L(\lambda)]_-{}^-} - [\Lambda L(\lambda)]_a{}^+ = \sqrt{2}\kappa\Big(\frac{\Lambda_-{}^+(\Lambda p)^a}{(\Lambda p)_-p_-} - \frac{\Lambda_a{}^+}{p_-}\Big)\,. \tag{123}$$

Here $\Lambda_\mu{}^\nu$ is an arbitrary Lorentz transformation, and the "standard Lorentz transformation" $L(\lambda)$ carries the standard D-momentum $k^\mu = (0,\ldots,0,\kappa,\kappa)$ to p^μ, i.e. $L^\mu{}_\nu(\lambda)k^\nu = p^\mu$, with p^μ the D-momentum of any massless particle. The matrix $L_\mu{}^\nu(\lambda)$ is defined by Eq. (94).

The general representation of the little group for massless particles is given by Eq. (121), where the parameters Θ_{cd} and a^a defined by (119) and (116), respectively.

4 Applications to Gauge Theory in 4D

In this section, we discuss the applications of Wigner rotation to the 4D gauge theory. We begin by constructing the Wigner rotation for massless particles such as photons. In 4D, it is relatively easier to determine the angle of Wigner rotation $\Theta(\Lambda,\lambda)$,

$$\sin(\Theta(\Lambda,\lambda)) = W_1{}^2(\Lambda,\lambda) = -W_2{}^1(\Lambda,\lambda)\,,$$
$$\cos(\Theta(\Lambda,\lambda)) = W_1{}^1(\Lambda,\lambda) = W_2{}^2(\Lambda,\lambda)\,, \tag{124}$$

where the matrix elements $W_b{}^a(\Lambda,\lambda)$ $(a,b=1,2)$ are given by the second equation of Eq. (102). According to Eq. (116), the set of parameters of the translation part of ISO(2) is

$$a^a(\Lambda,p) = -W_a{}^+(\Lambda,\lambda)\,, \tag{125}$$

whose values can be read off from Eqs. (106) and (107).

It is interesting to consider a different "standard Lorentz transformation". For instance, let us try

$$\tilde{L}(p) = \exp(-\phi\tau^{12})\exp(-\theta\tau^{13})\exp(\lambda\tau^{03})\,, \tag{126}$$

with the parameters relating to the momentum $\vec{p}$ as follows

$$\hat{p}^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\,, \quad |\vec{p}| = \kappa e^{-\lambda}\,. \tag{127}$$

This $\tilde{L}(p)$ is adopted from the textbook,[2] but rewritten in terms of our notation. It can be seen that $\tilde{L}(p)^\mu{}_\nu k^\nu = L(p)^\mu{}_\nu k^\nu = p^\mu$ but $\tilde{L}(p) \neq L(p)$. (Our $L(p)$ is defined by Eqs. (93) and (91)) Now the "new" little group reads

$$\tilde{W}(\Lambda,p) = \tilde{L}^{-1}(\Lambda p)\Lambda\tilde{L}(p)\,. \tag{128}$$

According to Eq. (2), we must have

$$\tilde{W}(\Lambda,p) = S(\Lambda p)W(\Lambda,p)S^{-1}(p)\,. \tag{129}$$

Note that

$$S(p) = \tilde{L}^{-1}(p)L(p) \tag{130}$$

is itself a little group, since

$$S^\mu{}_\nu(p)k^\nu = (\tilde{L}^{-1})^\mu{}_\rho(p)L^\rho{}_\nu(p)k^\nu$$
$$= (\tilde{L}^{-1})^\mu{}_\rho(p)p^\rho = k^\mu. \tag{131}$$

In light-cone coordinates, we can decompose Eq. (129) into the following two essential parts

$$\tilde{W}_b{}^a(\Lambda,p) = S_b{}^c(\Lambda p)W_c{}^d(\Lambda,p)(S^{-1})_d{}^a(p)\,, \tag{132}$$

$$\tilde{a}^a(\Lambda,p) = S_a{}^b(\Lambda p)a^b(\Lambda,p) - S_a{}^+(\Lambda p) - S_a{}^b(\Lambda p)$$
$$\times W_b{}^c(\Lambda,p)(S^{-1})_c{}^+(p)\,. \tag{133}$$

In deriving Eq. (133), we have used the definition $\tilde{a}^a(\Lambda,p) = -\tilde{W}_a{}^+(\Lambda,p)$. Equations (132) and (133) also hold in D-dimensions.

We now would like to work out $\tilde{W}_b{}^a(\Lambda,p)$ $(a,b=1,2)$. Inserting Eq. (127) into Eq. (126), a direct calculation gives $\tilde{L}^{\mu}{}_{\nu}(p)$: (We set $\kappa = 1$)

$$\tilde{L}^{i}{}_{0}(p) = \frac{p_0^2 - 1}{2p_0^2} p^i\,, \quad \tilde{L}^{0}{}_{0}(p) = \frac{p_0^2 + 1}{2p^0}\,,$$

$$\tilde{L}^{i}{}_{3}(p) = \frac{p_0^2 + 1}{2p_0^2} p^i\,, \quad \tilde{L}^{0}{}_{3}(p) = \frac{p_0^2 - 1}{2p^0}\,,$$

$$\tilde{L}^{a}{}_{1}(p) = \frac{p_3 p^a}{p^0\sqrt{p_0^2 - p_3^2}}\,, \quad \tilde{L}^{a}{}_{2}(p) = \frac{-\varepsilon_{ab}p^b}{\sqrt{p_0^2 - p_3^2}}\,,$$

$$\tilde{L}^{3}{}_{1}(p) = -\sqrt{1 - \frac{p_3^2}{p_0^2}}\,,$$

$$\tilde{L}^{3}{}_{2}(p) = \tilde{L}^{0}{}_{2}(p) = \tilde{L}^{0}{}_{1}(p) = 0\,, \tag{134}$$

where $\varepsilon_{ab} = -\varepsilon_{ba}$ and $\varepsilon_{12} = 1$, and $i = 1, 2, 3$. One can obtain $\tilde{L}^{\mu}{}_{\nu}(\Lambda p)$ from the above equation by simply replacing p^μ by $(\Lambda p)^\mu$. The inverse transformation matrix $(\tilde{L}^{-1})^{\mu}{}_{\nu}(p_\Lambda)$ can be calculated by using the equation $(\tilde{L}^{-1})^{\mu}{}_{\nu}(p_\Lambda) = \eta^{\mu\rho}\eta_{\nu\sigma}\tilde{L}^{\sigma}{}_{\rho}(p_\Lambda)$; Its expression is

$$(\tilde{L}^{-1})^{0}{}_{i}(p_\Lambda) = -\frac{(p_\Lambda^0)^2 - 1}{2(p_\Lambda^0)^2} p_\Lambda^i\,, \quad (\tilde{L}^{-1})^{0}{}_{0}(p_\Lambda) = \frac{(p_\Lambda^0)^2 + 1}{2p_\Lambda^0}\,, \quad (\tilde{L}^{-1})^{3}{}_{i}(p_\Lambda) = \frac{(p_\Lambda^0)^2 + 1}{2(p_\Lambda^0)^2} p_\Lambda^i,$$

$$(\tilde{L}^{-1})^{3}{}_{0}(p_\Lambda) = -\frac{(p_\Lambda^0)^2 - 1}{2p_\Lambda^0}\,, \quad (\tilde{L}^{-1})^{1}{}_{a}(p_\Lambda) = \frac{p_\Lambda^3 p_\Lambda^a}{p_\Lambda^0\sqrt{(p_\Lambda^0)^2 - (p_\Lambda^3)^2}}\,, \quad (\tilde{L}^{-1})^{2}{}_{a}(p_\Lambda) = \frac{-\varepsilon_{ab}p_\Lambda^b}{\sqrt{(p_\Lambda^0)^2 - (p_\Lambda^3)^2}}\,,$$

$$(\tilde{L}^{-1})^{1}{}_{3}(p_\Lambda) = -\sqrt{1 - \frac{(p_\Lambda^3)^2}{(p_\Lambda^0)^2}}\,, \quad (L^{-1})^{2}{}_{3}(p_\Lambda) = (\tilde{L}^{-1})^{2}{}_{0}(p_\Lambda) = (\tilde{L}^{-1})^{1}{}_{0}(p_\Lambda) = 0\,, \tag{135}$$

where p_Λ^μ stands for $(\Lambda p)^\mu$.

In terms of matrix elements, the Wigner rotation (128) reads

$$\tilde{W}^{\mu}{}_{\nu}(\Lambda, p) = (\tilde{L}^{-1})^{\mu}{}_{\rho}(p_\Lambda)\Lambda^{\rho}{}_{\sigma}\tilde{L}^{\sigma}(p)\,. \tag{136}$$

Substituting Eqs. (134) and (135) into the above equation, we find that

$$\tilde{W}^{1}{}_{1}(\Lambda, p) \equiv \cos(\tilde{\Theta}(\Lambda, p)) = \frac{\widehat{p}_\Lambda^3\widehat{p}_\Lambda^a[-\Lambda^{a}{}_{3}(1 - \widehat{p}_3^2) + \Lambda^{a}{}_{b}\widehat{p}^b\widehat{p}^3] - [1 - (\widehat{p}_\Lambda^3)^2][-\Lambda^{3}{}_{3}(1 - \widehat{p}_3^2) + \Lambda^{3}{}_{b}\widehat{p}^b\widehat{p}^3]}{\sqrt{[1 - (\widehat{p}_\Lambda^3)^2](1 - \widehat{p}_3^2)}}\,, \tag{137}$$

$$\tilde{W}^{1}{}_{2}(\Lambda, p) \equiv \sin(\tilde{\Theta}(\Lambda, p)) = \frac{\varepsilon_{ab}\widehat{p}_b(\Lambda^{3}{}_{a} - \Lambda^{0}{}_{a}\widehat{p}_\Lambda^3)}{\sqrt{[1 - (\widehat{p}_\Lambda^3)^2](1 - \widehat{p}_3^2)}}\,, \tag{138}$$

where the unit vector $\widehat{p}^i = p^i/|\vec{p}|$ is the direction of the momentum $\vec{p}$, and $\widehat{p}_\Lambda^i$ has a similar definition. Since $\tilde{W}^{a}{}_{b}(\Lambda, p)$ is an SO(2) matrix, we have $\tilde{W}^{2}{}_{2}(\Lambda, p) = \tilde{W}^{1}{}_{1}(\Lambda, p)$ and $\tilde{W}^{2}{}_{1}(\Lambda, p) = -\tilde{W}^{1}{}_{2}(\Lambda, p)$.

Similarly, using Eqs. (134), (135), and (136), the translation part of ISO(2)

$$\tilde{a}^a(\Lambda, p) = -\tilde{W}_{a}{}^{+}(\Lambda, p)\,, \tag{139}$$

(see Eq. (116)) can be worked out, as well. However, since we do not need the explicit expression for $\tilde{a}^a(\Lambda, p)$, we do not present it here.

It is interesting to verify Eqs. (132) and (133). One can calculate $S(p) = \tilde{L}^{-1}(p)L(p)$ using the definition of $L(p)$ (94) and $(\tilde{L}^{-1})^{\mu}{}_{\nu}(p) = \eta^{\mu\rho}\eta_{\nu\sigma}\tilde{L}^{\sigma}{}_{\rho}(p)$, with $\tilde{L}^{\sigma}{}_{\rho}(p)$ defined by (134). And $S^{-1}(\Lambda p) = L^{-1}(\Lambda p)\tilde{L}(\Lambda p)$ can be calculated in a similar way. We have verified Eqs. (132) and (133) in the case of infinitesimal Lorentz transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + (\delta\omega)^{\mu}{}_{\nu}\,, \tag{140}$$

under the condition that $(p^0)^2 - (p^3)^2 \neq 0$.

We now apply our results to the U(1) gauge theory in $4D$. In the interaction picture, the physical or on-shell gauge field** in $4D$ takes the form (see Chapter 5 of Ref. [2])

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3p}{\sqrt{2p_0}}\sum_{\sigma=\pm1}\Big[e_\mu(\vec{p},\sigma)\mathrm{e}^{\mathrm{i}p\cdot x}a(\vec{p},\sigma) + e_\mu^*(\vec{p},\sigma)\mathrm{e}^{-\mathrm{i}p\cdot x}a^\dagger(\vec{p},\sigma)\Big]\,. \tag{141}$$

Here the polarization vector $e^\mu(\vec{p},\sigma) = L(p)^{\mu}{}_{\nu}e^\mu(\vec{k},\sigma)$, with the standard Lorentz transformation $L(p)^{\mu}{}_{\nu}$ defined by Eq. (94). Following the convention of Ref. [2], we specify the polarization vectors as

$$e^\mu(\vec{k}, \pm1) = (1, \pm\mathrm{i}, 0, 0)/\sqrt{2}\,,$$

where $\vec{k}$ is the standard momentum.

In $4D$, the vector representation of Eq. (121) reads

$$W^{\mu}{}_{\nu}(\Lambda, p) = \exp(a^a(\Lambda, p)\tau^{-a})^{\mu}{}_{\rho}\exp(\Theta(\Lambda, p)\tau^3)^{\rho}{}_{\nu}\,, \tag{142}$$

**That is, the gauge field of helicity ±1 satisfies the equation of motion $\partial^\nu\partial_\nu A_\mu(x) = 0$.

where $(\tau^{-a})^{\mu}{}_{\nu} = (1/\sqrt{2})(-\tau^{0a}+\tau^{3a})^{\mu}{}_{\nu}$, $(\tau^3)^{\mu}{}_{\nu} = (\tau^{12})^{\mu}{}_{\nu}$, and $(\tau^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu}$. From now on, the letter a will be reserved for the creation and annihilation operators, and following the convention of Ref. [2], we will denote the translation parameters of ISO(2) as α and β, namely,

$$a^a(\Lambda,p) = (\alpha(\Lambda,p),\beta(\Lambda,p))\,. \tag{143}$$

Under an arbitrary Lorentz transformation Λ, the creation and annihilation operators transform as[2]

$$U(\Lambda)a(\vec{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\mathrm{e}^{-\mathrm{i}\sigma\Theta(\Lambda,p)}a(\vec{p}_\Lambda,\sigma)\,, \tag{144}$$

$$U(\Lambda)a^\dagger(\vec{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\mathrm{e}^{\mathrm{i}\sigma\Theta(\Lambda,p)}a^\dagger(\vec{p}_\Lambda,\sigma)\,. \tag{145}$$

Here $\vec{p}_\Lambda$ stands for $\Lambda^i{}_\mu p^\mu$ or $(\Lambda p)^i$. On the other hand, under the Lorentz transformation Λ,

$$\begin{aligned}\Lambda^\mu{}_\nu e^\nu(\vec{p},\pm1) &= L^\mu{}_\nu(\Lambda p)(L^{-1}(\Lambda p)\Lambda L(p))^\nu{}_\rho e^\rho(\vec{k},\pm1) = L^\mu{}_\nu(\Lambda p)W^\nu{}_\rho(\Lambda,p)e^\rho(\vec{k},\pm1)\\ &= \mathrm{e}^{\pm\mathrm{i}\Theta(\Lambda,p)}\Big(e^\mu(\vec{p}_\Lambda,\pm1) + \frac{\alpha(\Lambda,p)\pm\beta(\Lambda,p)}{|\vec{k}|}(\Lambda p)^\mu\Big)\,.\end{aligned} \tag{146}$$

In the last line, we have used Eq. (142). That is, the polarization vectors cannot transform as a true Lorentz vector,[2]

$$\mathrm{e}^{-(\pm\mathrm{i}\Theta(\Lambda,p))}e_\mu(\vec{p},\pm1) = \Lambda^\nu{}_\mu e_\nu(\vec{p}_\Lambda,\pm1) + \frac{\alpha(\Lambda,p)\pm\mathrm{i}\beta(\Lambda,p)}{|\vec{k}|}p_\mu\,. \tag{147}$$

Or, according to Weinberg's notation,[2]

$$\mathrm{e}^\mu(\vec{p}_\Lambda,\pm1)\mathrm{e}^{\pm\mathrm{i}\Theta(\Lambda,p)} = \Lambda^\mu{}_\nu e^\nu(\vec{p},\pm1) + (\Lambda p)^\mu\Omega_\pm(\Lambda,p)\,. \tag{148}$$

Here $\Omega_\pm(\Lambda,p) \equiv -\mathrm{e}^{\pm\mathrm{i}\Theta(\Lambda,p)}[\alpha(\Lambda,p)\pm\mathrm{i}\beta(\Lambda,p)]/|\vec{k}|$.

So under the Lorentz transformation,

$$U(\Lambda)A_\mu(x)U^{-1}(\Lambda) = \Lambda^\nu{}_\mu A_\nu(\Lambda x) + \partial_\mu\Omega(x,\Lambda)\,, \tag{149}$$

where

$$\Omega(x,\Lambda) = -\frac{\mathrm{i}}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3p}{\sqrt{2p_0}}\sum_{\sigma=\pm1}\Big[\frac{\alpha+\mathrm{i}\beta}{|\vec{k}|}\mathrm{e}^{\mathrm{i}p\cdot(\Lambda x)}a(\vec{p},\sigma) - \frac{\alpha-\mathrm{i}\beta}{|\vec{k}|}\mathrm{e}^{-\mathrm{i}p\cdot(\Lambda x)}a^\dagger(\vec{p},\sigma)\Big]\,. \tag{150}$$

The second term of the right hand side of Eq. (149) is a typical gauge transformation: It is this sense that the generators T^1 and T^2†† of the translation part of the little group ISO(2) generate the gauge transformation.[2] On the other hand, if we calculate everything using

$$\tilde{W}^\mu{}_\nu(\Lambda,p) = \exp(\tilde{a}^a(\Lambda,p)\tau^{-a})^\mu{}_\rho\exp(\tilde{\Theta}(\Lambda,p)\tau^3)^\rho{}_\nu\,, \tag{151}$$

where

$$\tilde{a}^a(\Lambda,p) = (\tilde{\alpha}(\Lambda,p),\tilde{\beta}(\Lambda,p))\,, \tag{152}$$

in stead of $W(\Lambda,p)$ (see Eq. (142)), the angle Θ in Eqs. (144) and (145) must be replaced by $\tilde{\Theta}$, and α and β in Eq. (150) must be replaced by $\tilde{\alpha}$ and $\tilde{\beta}$. (One can transform the set of parameters (α,β,Θ) into $(\tilde{\alpha},\tilde{\beta},\tilde{\Theta})$ by using Eqs. (132) and (133).) After making these replacements, the only change in Eq. (149) is that $\Omega(x,\Lambda)$ gets replaced by

$$\tilde{\Omega}(x,\Lambda) = -\frac{\mathrm{i}}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3p}{\sqrt{2p_0}}\sum_{\sigma=\pm1}\Big[\frac{\tilde{\alpha}+\mathrm{i}\tilde{\beta}}{|\vec{k}|}\mathrm{e}^{\mathrm{i}p\cdot(\Lambda x)}a(\vec{p},\sigma) - \frac{\tilde{\alpha}-\mathrm{i}\tilde{\beta}}{|\vec{k}|}\mathrm{e}^{-\mathrm{i}p\cdot(\Lambda x)}a^\dagger(\vec{p},\sigma)\Big]\,, \tag{153}$$

namely,

$$U(\Lambda)A_\mu(x)U^{-1}(\Lambda) = \Lambda^\nu{}_\mu A_\nu(\Lambda x) + \partial_\mu\tilde{\Omega}(x,\Lambda)\,. \tag{154}$$

This is the result calculated by using Eq. (151). We see that Eqs. (149) and (154) are only up to a gauge transformation, which is due to the *difference* between two "standard Lorentz transformation", defined by Eq. (130). Or in other words, two different "standard Lorentz transformations" can generate a gauge transformation.

However, no matter that we apply (149) or (154) to the gauge field, the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms like a true tensor under arbitrary Lorentz transformations, i.e.,

$$U(\Lambda)F_{\mu\nu}(x)U^{-1}(\Lambda) = \Lambda^\rho{}_\mu\Lambda^\sigma{}_\mu F_{\rho\sigma}(\Lambda x)\,, \tag{155}$$

even we choose two different "standard Lorentz trans-

††In Chapter 2 of Ref. [2], the generators T^1 and T^2 are denoted as A and B, respectively.

formations". But one cannot construct an interesting gauge theory such as quantum electrodynamics using $F_{\mu\nu}$ alone.[2] In order to construct an interesting theory, one must introduce a conserved current J^μ into the theory,[2] and allow the gauge field A_μ to couple it as follows

$$J^\mu A_\mu \,. \tag{156}$$

Here J_μ is conserved in the sense that

$$\partial_\mu J^\mu = 0 \,, \tag{157}$$

and under a Lorentz transformation, it transforms as

$$U(\Lambda) J^\mu(x) U^{-1}(\Lambda) = \Lambda_\nu{}^\mu J^\nu(\Lambda x) \,. \tag{158}$$

Using (149) or (154), we learn that

$$\begin{aligned} & U(\Lambda)[J^\mu A_\mu](x) U^{-1}(\Lambda) \\ & = [J^\mu A_\mu](\Lambda x) + \Lambda_\rho{}^\mu J^\rho(\Lambda x)\partial_\mu \Omega(x) \\ & = [J^\mu A_\mu](\Lambda x) + \partial_\mu(\Lambda_\rho{}^\mu J^\rho(\Lambda x)\Omega(x)) \,. \end{aligned} \tag{159}$$

In the second line, we have used $\Lambda_\rho{}^\mu \partial_\mu J^\rho(\Lambda x) = \partial'_\rho J^\rho(x') = 0$ to rewrite the second term as a total derivative, where $x'^\rho = (\Lambda x)^\rho$. According to the action principle, the total derivative term does not affect the equations of motion, so the theory is Lorentz invariant.

Let us consider an explicit example. We introduce a physical spin 1/2 field $\psi(x)$ into the theory and construct the following current

$$J^\mu(x) = \mathrm{i}e\bar\psi(x)\gamma^\mu\psi(x), \tag{160}$$

where e the is the U(1) charge, such as the electric charge, and $\bar\psi(x) \equiv \mathrm{i}\psi^\dagger(x)\gamma^0$. Using the Dirac equation

$$(\gamma^\mu\partial_\mu + m)\psi(x) = 0 \,, \tag{161}$$

where m is the mass of the spin 1/2 particle, it is easy to verify that $\partial_\mu J^\mu = 0$. The field $\psi(x)$ has the Lorentz transformation law[2]

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda_{\mathrm{S}}^{-1}\psi(\Lambda x) \,. \tag{162}$$

Using Eqs. (162), (A11), and (9), one can easily verify that J^μ, defined by (160), satisfies Eq. (158).

The Lagrangian density, by which one can derive all equations of motion, is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar\psi(\gamma^\mu\partial_\mu + m)\psi + J^\mu A_\mu \,. \tag{163}$$

Using Eq. (160), it can be rewritten as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar\psi(\gamma^\mu D_\mu + m)\psi \,, \tag{164}$$

where

$$D_\mu\psi = (\partial_\mu - \mathrm{i}eA_\mu)\psi \tag{165}$$

is the covariant derivative of the spinor field. We now see that the action

$$S = \int \mathrm{d}^4x\mathcal{L} \tag{166}$$

is invariant under the Lorentz transformations (149) and (162), or (154) and (162); And the action is also invariant under the gauge transformations

$$\delta A_\mu(x) = \partial_\mu\epsilon(x) \,, \qquad \delta\psi(x) = \mathrm{i}e\epsilon(x)\psi(x). \tag{167}$$

Here $\epsilon(x)$, depending on x^μ, is the parameter of U(1) gauge transformation. It would be interesting to study the relation between the Wigner rotation and gauge invariance for theories in the higher dimensional spacetime ($D > 4$).

Appendix A: Conventions and Useful Identities

In this appendix, we introduce our conventions for the gamma matrices and Clifford algebra of SO($D-1,1$), and Lorentz transformations. The set of gamma matrices satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \,, \tag{A1}$$

where $\eta_{00} = -1$ and $\eta_{ij} = \delta_{ij}$. We will use $\eta^{\mu\nu}$ ($\eta_{\mu\nu}$) to raise (lower) indices. For instance, $\gamma^\mu = \eta^{\mu\nu}\gamma_\nu$. The gamma matrices obey the reality conditions

$$\gamma^{0\dagger} = -\gamma^0, \quad \gamma^{i\dagger} = \gamma^i \,. \tag{A2}$$

The set of generators of SO($D-1,1$) are defined as

$$\Sigma^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu] \,. \tag{A3}$$

It is convenient to decompose the generators into the two sets,

$$\Sigma^{i0} = \frac{1}{4}[\gamma^i, \gamma^0] \,, \tag{A4}$$

$$\Sigma^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] \,. \tag{A5}$$

They obey the reality conditions

$$\begin{aligned} \Sigma^{i0\dagger} &= -\gamma^0\Sigma^{i0}(\gamma^0)^{-1} = \Sigma^{i0} \,, \\ \Sigma^{ij\dagger} &= -\gamma^0\Sigma^{ij}(\gamma^0)^{-1} = -\Sigma^{ij} \,, \end{aligned} \tag{A6}$$

and satisfy the commutation relations

$$[\Sigma^{\mu\nu}, \gamma^\rho] = \eta^{\nu\rho}\gamma^\mu - \eta^{\mu\rho}\gamma^\nu \equiv (\tau^{\mu\nu})^\rho_\sigma\gamma^\sigma \,, \tag{A7}$$

$$[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] = \eta^{\nu\rho}\Sigma^{\mu\sigma} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho} + \eta^{\mu\sigma}\Sigma^{\nu\rho} \,, \tag{A8}$$

$$\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\} = \frac{1}{2}(\gamma^{\mu\nu\rho\sigma} + \eta^{\nu\rho}\eta^{\mu\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma}) \,,$$

where $\gamma^{\mu\nu\rho\sigma} \equiv \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]} = (1/4!)(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \text{permutations})$, and

$$(\tau^{\mu\nu})_\sigma{}^\rho = \delta^\mu_\sigma\eta^{\nu\rho} - \delta^\nu_\sigma\eta^{\mu\rho} \,. \tag{A9}$$

We parameterize the general Lorentz transformation Λ_{S} in spinor space as follows

$$\Lambda_{\mathrm{S}} = \exp\Big(\frac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\Big) \,, \tag{A10}$$

where the set of parameters $\omega_{\mu\nu}$ is a real antisymmetric tensor, and the subscript "S" stands for spinor representation. Equations (A6) imply that $\Lambda_{\rm S}$ obeys the pseudo-reality condition

$$\gamma^0\Lambda_{\rm S}^{\dagger}(\gamma^0)^{-1}=\Lambda_{\rm S}^{-1}\,. \tag{A11}$$

The rotation and boost are given by

$$R_{\rm S}={\rm e}^{(1/2)\omega_{ij}\Sigma^{ij}} \quad \text{and} \quad L_{\rm S}={\rm e}^{\omega_{i0}\Sigma^{i0}}\,, \tag{A12}$$

where ω_{i0} is the set of rapidities.

To describe massless particles, it is more convenient to introduce the light-cone coordinates in D-dimensional spacetime

$$x^{\pm}=\frac{1}{\sqrt{2}}(\pm x^0+x^{D-1})\,, \tag{A13}$$

and the transverse space-like coordinates x^a, $a=1,2,\ldots,D-2$.

In terms of light-cone coordinates, we have

$$\gamma^{\pm}=\frac{1}{\sqrt{2}}(\pm\gamma^0+\gamma^{D-1})\,, \tag{A14}$$

and the non-vanishing anti-commutators are given by

$$\{\gamma^+,\gamma^-\}=2\eta^{+-}=2\,,\quad \{\gamma^a,\gamma^b\}=2\eta^{ab}\,. \tag{A15}$$

Hence the metric tensor $\eta^{\mu\nu}$ can be decomposed into

$$\eta^{+-}=\eta^{-+}=1,\quad \eta^{ab}=\delta^{ab},\quad \text{and}$$
$$\eta^{++}=\eta^{--}=\eta^{a+}=\eta^{b-}=0\,. \tag{A16}$$

We will use η^{+-} or η_{+-} to raise or lower indices. For instance, $V_-=\eta_{-+}V^+=V^+$. The inner product of two vectors reads

$$\eta^{\mu\nu}V_\mu W_\nu=V^aW^a+V_-V^-+V_+V^+\,. \tag{A17}$$

Using the rules of tensor analysis, one can write down the general Lorentz transformation Λ in the light-cone coordinates. For instance

$$\Lambda_-{}^+=\frac{\partial x^\mu}{\partial x^-}\frac{\partial x^+}{\partial x^\nu}\Lambda_\mu{}^\nu$$
$$=\frac{1}{2}\Big(-\Lambda_0{}^0-\Lambda_0{}^{D-1}+\Lambda_{D-1}{}^0+\Lambda_{D-1}{}^{D-1}\Big)\,. \tag{A18}$$

The set of generators $\Sigma^{\mu\nu}$ is decomposed into

$$A^a\equiv\Sigma^{-a}=\frac{1}{4}[\gamma^-,\gamma^a]\,, \tag{A19}$$

$$\Sigma^{+-}=\frac{1}{4}[\gamma^+,\gamma^-]=\Sigma^{0,D-1}\,, \tag{A20}$$

$$\Sigma^{+a}=\frac{1}{4}[\gamma^+,\gamma^a]\,, \tag{A21}$$

$$\Sigma^{ab}=\frac{1}{4}[\gamma^a,\gamma^b]\,. \tag{A22}$$

Under the above decomposition, the (spinor) algebra of the little group ISO$(D-2)$ reads

$$[A^a,A^b]=0\,, \tag{A23}$$

$$[\Sigma^{ab},A^c]=\delta^{bc}A^a-\delta^{ac}A^b\,, \tag{A24}$$

$$[\Sigma^{ab},\Sigma^{cd}]=\delta^{bc}\Sigma^{ad}-\delta^{ac}\Sigma^{bd}-\delta^{bd}\Sigma^{ac}+\delta^{ad}\Sigma^{bc}\,. \tag{A25}$$

Notice that by the definition of A^a (see Eq. (A19)),

$$(A^a)^2=0\,, \tag{A26}$$

that is, in the spinor representation, the eigenvalues of A^a are zero automatically.

Appendix B: Verifying Little Group SO$(\mathbf{D-1})$

We now try to give a direct verification of Eq. (36), which is essentially the same as the following equation:

$$W_i{}^k(\Lambda,p)W_j{}^k(\Lambda,p)=\delta_{ij}\,. \tag{A27}$$

For readability, we will write $[\Lambda L(p)]_\mu{}^\nu$ as $(\Lambda L)_\mu{}^\nu$. Our main equation for proving (A27) is the fundamental one:

$$\eta_{\rho\sigma}(\Lambda L)_\mu{}^\rho(\Lambda L)_\nu{}^\sigma=\eta_{\mu\nu}\quad\text{or}$$
$$(\Lambda L)_\mu{}^k(\Lambda L)_\nu{}^k=\eta_{\mu\nu}+(\Lambda L)_\mu{}^0(\Lambda L)_\nu{}^0. \tag{A28}$$

Inserting the last equation of (28) into the left-hand side of Eq. (A27) gives

$$\begin{aligned}W_i{}^k(\Lambda,p)W_j{}^k(\Lambda,p)=\frac{1}{[1+(\Lambda L)_0{}^0]^2}&((\Lambda L)_0{}^k(\Lambda L)_0{}^k(\Lambda L)_i{}^0(\Lambda L)_j{}^0-(\Lambda L)_i{}^k(\Lambda L)_0{}^k(\Lambda L)_0{}^0(\Lambda L)_j{}^0-(\Lambda L)_i{}^k(\Lambda L)_0{}^k(\Lambda L)_j{}^0\\&-(\Lambda L)_0{}^k(\Lambda L)_j{}^k(\Lambda L)_i{}^0(\Lambda L)_0{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k(\Lambda L)_0{}^0(\Lambda L)_0{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k(\Lambda L)_0{}^0\\&-(\Lambda L)_0{}^k(\Lambda L)_j{}^k(\Lambda L)_i{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k(\Lambda L)_0{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k)\,.\end{aligned} \tag{A29}$$

The summation of the first term of first line in the bracket of Eq. (A29) and the third term of the second line is

$$\begin{aligned}&((\Lambda L)_0{}^k(\Lambda L)_0{}^k)(\Lambda L)_i{}^0(\Lambda L)_j{}^0+((\Lambda L)_i{}^k(\Lambda L)_j{}^k)(\Lambda L)_0{}^0(\Lambda L)_0{}^0\\&=(\eta_{00}+(\Lambda L)_0{}^0(\Lambda L)_0{}^0)(\Lambda L)_i{}^0(\Lambda L)_j{}^0+(\delta_{ij}+(\Lambda L)_i{}^0(\Lambda L)_j{}^0)(\Lambda L)_0{}^0(\Lambda L)_0{}^0\\&=[\delta_{ij}+2(\Lambda L)_i{}^0(\Lambda L)_j{}^0][(\Lambda L)_0{}^0]^2-(\Lambda L)_i{}^0(\Lambda L)_j{}^0\,.\end{aligned} \tag{A30}$$

Let us now add the second term of first line in the bracket of Eq. (A29) and the second term of the second line,

$$\begin{aligned}&-((\Lambda L)_i{}^kx(\Lambda L)_0{}^k)(\Lambda L)_0{}^0(\Lambda L)_j{}^0-((\Lambda L)_0{}^k(\Lambda L)_j{}^k)(\Lambda L)_i{}^0(\Lambda L)_0{}^0\\&=-(\eta_{i0}+(\Lambda L)_i{}^0(\Lambda L)_0{}^0)(\Lambda L)_0{}^0(\Lambda L)_j{}^0-(\eta_{0j}+(\Lambda L)_0{}^0(\Lambda L)_j{}^0)(\Lambda L)_i{}^0(\Lambda L)_0{}^0=-2(\Lambda L)_i{}^0x(\Lambda L)_j{}^0[(\Lambda L)_0{}^0]^2\,.\end{aligned} \tag{A31}$$

The summation of the rest terms (the first term of second line and all terms of third line) in the big bracket of Eq. (A29) is

$$\begin{aligned}&-(\Lambda L)_i{}^k(\Lambda L)_0{}^k(\Lambda L)_j{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k(\Lambda L)_0{}^0-(\Lambda L)_0{}^k(\Lambda L)_j{}^k(\Lambda L)_i{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k(\Lambda L)_0{}^0+(\Lambda L)_i{}^k(\Lambda L)_j{}^k\\&=-((\Lambda L)_i{}^k(\Lambda L)_0{}^k)(\Lambda L)_j{}^0-((\Lambda L)_0{}^k(\Lambda L)_j{}^k)(\Lambda L)_i{}^0+((\Lambda L)_i{}^k(\Lambda L)_j{}^k)[2(\Lambda L)_0{}^0+1]\\&=-(\eta_{i0}+(\Lambda L)_i{}^0(\Lambda L)_0{}^0)(\Lambda L)_j{}^0-(\eta_{0j}+(\Lambda L)_0{}^0(\Lambda L)_j{}^0)(\Lambda L)_i{}^0\end{aligned}$$

$$+(\delta_{ij}+(\Lambda L)_i{}^0(\Lambda L)_j{}^0)[2(\Lambda L)_0{}^0+1]=-2(\Lambda L)_i{}^0(\Lambda L)_j{}^0(\Lambda L)_0{}^0+(\delta_{ij}+(\Lambda L)_i{}^0(\Lambda L)_j{}^0)[2(\Lambda L)_0{}^0+1]\,. \tag{A32}$$

In deriving (A30), (A31), and (A32), we have used (A28). The big bracket of (A29) is the summation of (A30), (A31), and (A32):

$$(\mathrm{A30})+(\mathrm{A31})+(\mathrm{A32})=\delta_{ij}(1+2(\Lambda L)_0{}^0+[(\Lambda L)_0{}^0]^2)\,. \tag{A33}$$

Replacing the big bracket of (A29) by (A33), the right-hand side of (A29) becomes δ_{ij}. This completes the proof of (A27).

Appendix C: Verifying Little Group SO(D − 2)

We now give a direct proof of Eq. (112). For convenience, we cite it here:

$$W_a{}^c(\Lambda,p)W_b{}^c(\Lambda,p)=\delta_{ab}\,. \tag{A34}$$

We are going to use the fundamental equation

$$\eta^{\rho\sigma}(\Lambda L)_\rho{}^\mu(\Lambda L)_\sigma{}^\nu=\eta^{\mu\nu}\,,\quad \text{or}\quad (\Lambda L)_c{}^\mu(\Lambda L)_c{}^\nu=\eta^{\mu\nu}-(\Lambda L)_+{}^\mu(\Lambda L)_-{}^\nu-(\Lambda L)_-{}^\mu(\Lambda L)_+{}^\nu \tag{A35}$$

to prove (A34), where we have written $[\Lambda L(p)]_\mu{}^\nu$ as $(\Lambda L)_\mu{}^\nu$. Plugging the second line of the second equation of (102) into the left-hand side of (A34),

$$\begin{aligned}W_a{}^c(\Lambda,p)W_b{}^c(\Lambda,p)&=\frac{(\Lambda L)_c{}^-(\Lambda L)_c{}^-(\Lambda L)_-{}^a(\Lambda L)_-{}^b}{[(\Lambda L)_-{}^-]^2}-\frac{(\Lambda L)_c{}^-(\Lambda L)_c{}^b(\Lambda L)_-{}^a+(\Lambda L)_c{}^-(\Lambda L)_c{}^a(\Lambda L)_-{}^b}{(\Lambda L)_-{}^-}\\&\quad+(\Lambda L)_c{}^a(\Lambda L)_c{}^b\,.\end{aligned} \tag{A36}$$

According to Eq. (A35),

$$(\Lambda L)_c{}^-(\Lambda L)_c{}^-=\eta^{--}-(\Lambda L)_+{}^-(\Lambda L)_-{}^--(\Lambda L)_-{}^-(\Lambda L)_+{}^-=-2(\Lambda L)_-{}^-(\Lambda L)_+{}^-\,. \tag{A37}$$

Taking account of (A37), the first line of Eq. (A36) becomes

$$\frac{1}{[(\Lambda L)_-{}^-]^2}\Big((\Lambda L)_c{}^-(\Lambda L)_c{}^-\Big)(\Lambda L)_-{}^a(\Lambda L)_-{}^b=-\frac{2(\Lambda L)_+{}^-(\Lambda L)_-{}^a(\Lambda L)_-{}^b}{(\Lambda L)_-{}^-}\,. \tag{A38}$$

Similarly, one can convert the second of Eq. (A36) into the form:

$$\begin{aligned}&-\frac{1}{(\Lambda L)_-{}^-}[((\Lambda L)_c{}^-(\Lambda L)_c{}^b)(\Lambda L)_-{}^a+((\Lambda L)_c{}^-(\Lambda L)_c{}^a)(\Lambda L)_-{}^b]\\&=-\frac{1}{(\Lambda L)_-{}^-}[(\eta^{-b}-(\Lambda L)_+{}^-(\Lambda L)_-{}^b-(\Lambda L)_-{}^-(\Lambda L)_+{}^b)(\Lambda L)_-{}^a+(a\leftrightarrow b)]\\&=\frac{2(\Lambda L)_+{}^-(\Lambda L)_-{}^a(\Lambda L)_-{}^b}{(\Lambda L)_-{}^-}+(\Lambda L)_+{}^a(\Lambda L)_-{}^b+(\Lambda L)_-{}^a(\Lambda L)_+{}^b\,.\end{aligned} \tag{A39}$$

Inserting (A38) and (A39) into (A36),

$$W_a{}^c(\Lambda,p)W_b{}^c(\Lambda,p)=(\Lambda L)_+{}^a(\Lambda L)_-{}^b+(\Lambda L)_-{}^a(\Lambda L)_+{}^b+(\Lambda L)_c{}^a(\Lambda L)_c{}^b=\delta_{ab}\,. \tag{A40}$$

This completes the proof.

References

[1] E. P. Wigner, Ann. Math. **40** (1939) 149.

[2] S. Weinberg, *"The Quantum Theory of Fields,"* Vol. 1, Cambridge University Press, New York (1995).

[3] E. P. Wigner, *"Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atomspektren"*, Braunschweig, Vieweg Verlag (1931), Translated into English by J. J. Griffin, *"Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra"*, Academic Press, New York (1959).