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Fisher and Shannon information entropies for a noncentral inversely quadratic plus exponential Mie-type potential

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Abstract

In this work, we determine the Fisher and Shannon entropies, the expectation values and the squeeze state for a noncentral inversely quadratic plus exponential Mie-type potential analytically. The proposed potential is solved under the Schrödinger equation using a special Greene Aldrich approximation to the centrifugal term to obtain a normalised wave function within the framework of the Nikiforov–Uvarov method. Numerical results are obtained for different screening parameters: $\alpha = 0.1$, 0.12 and 0.13 for varying real constant parameter (B). The numerical solutions are obtained only for ground state. The numerical results of Fisher entropy both for position and momentum spaces are in good agreement with existing literature. The normalisation constant, wave function, and probability density plots are carried out using a well designed Mathematica algorithm. The Fourier transform of position space entropy gives the momentum space entropy.

Keywords: Fisher and Shannon entropies, Schrödinger equation, expectation values, Nikiforov–Uvarov method

(Some figures may appear in colour only in the online journal)

1. Introduction

Entropic measures and information theory in general provides a clear understanding of quantum mechanical systems. In recent years, the study of bound state and scattering state solutions for both relativistic and non-relativistic wave equations has a gained wider interest because of their potential applications, particularly in the areas of information entropies and quantum technologies. Information entropy is a significant tool in studying the electronic structure of atoms and molecules. The Fisher information measurement is used as a tool for characterising complex signals of quantum mechanical systems with applications to biology, atomic physics and other related science disciplines [1–13]. Shannon, Fisher and other quantum information entropies usually measure the spread of probability distribution for allowed quantum mechanical states in a D-dimensional space [14–18]. Quantum information theory has a direct relationship with the Heisenberg uncertainty principle, which plays a very significant role in the simultaneous measurement of position and momentum of quantum mechanical particles. In 1948, a new uncertainty relation, based on Shannon entropy, was established as a basic tool for investigating the fundamental limit of signal processing [19, 20]. In this work, we developed a novel potential called a Noncentral Inversely Quadratic plus exponential Mie-type potential to study Fisher and and Shannon entropies, their expectation values, and their squeeze state, using suitable real constant parameters. This potential has applications in signal processing. The wave function and probability density plots obtained in this work reveal some

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significant properties and characteristics of quantum systems. We have discovered that all odd values of the wave function give the property of a wave function, while even values of the wave function plots give probability density. Larger values of the Shannon entropy are indicative of a more delocalised density while smaller values are associated with localised distribution. This means that Shannon entropy increases with an increase in uncertainty and vice versa [21]. There are other information entropies as investigated by various researchers. Recently Jen-Hao and Yew Ho carried out fantastic work on the benchmark calculations of Renyi, Tsallis and Onicescu information entropies for ground state helium using a correlated Hylleraas wave function [22]. The new Shannon entropy was first introduced by Beckner, Bialynicki-Birula and Mycieslki in 1975 as $S_x + S_p \geqslant D(1 + log\pi)$ [23–25] where D represents the spatial dimension. However, the position S_x and momentum S_p is defined as [26–29].

$$S_x = -\int_{-\infty}^{+\infty} |\Psi(x)|^2 \log\, |\Psi(x)|^2 \mathrm{d}x \tag{1}$$

$$S_p = -\int_{-\infty}^{+\infty} |\phi(p)|^2 \log\, |\phi(p)|^2 \mathrm{d}p \tag{2}$$

where $\Psi(x)$ is a normalised eigen function in a spatial coordinate and $\phi(x)$ is the normalised Fourier transform. In general, the momentum space entropy is obtained by taking the Fourier transform of position space in a spatial coordinate [30, 31]. Equations (1) and (2) give the measure of the spread of a single particle density of position and momentum space respectively. Equation (1) and (2) can also be expressed as

$$S_x = -\int_{-\infty}^{+\infty} \rho(x) \ln \rho(x) \mathrm{d}x \tag{3}$$

$$S_p = -\int_{-\infty}^{+\infty} \rho(p) \ln \rho(p) \mathrm{d}p, \tag{4}$$

where $\rho(x) = |\Psi(x)|^2$ and $\rho(p) = |\Psi(p)|^2$ respectively. Equation (3) can further be expressed as

$$S_x = -\int_{-\infty}^{+\infty} \left(\frac{1}{d^2}\, |P_n(x)|^2 w(x\right) \times \log\left(\frac{1}{d^2}\, |P_n(x)|^2 w(x\right) \mathrm{d}x, \tag{5}$$

where

$$S_x = \log(\mathrm{d}_n^2) + \frac{1}{\mathrm{d}^2}(E_n + I_n). \tag{6}$$

Here, the term E_n and I_n can then be expressed as

$$E_n = -\int_{-\infty}^{+\infty} (|P_n(x)|^2 w(x) \log(|P_n(x)|^2) \mathrm{d}x, \tag{7}$$

$$I_n = -\int_{-\infty}^{+\infty} (|P_n(x)|^2 w(x) \log(w(x) \mathrm{d}x. \tag{8}$$

This work is divided into seven sections. Section 1 gives a brief introduction of the article. In section 2, we provide an overview of the generalised parametric Nikiforov–Uvarov method. In section 3 we solve analytically the radial solution of a Schrödinger wave equation to obtain a normalised wave function and the energy-eigen equation. In section 4 we apply the normalised wave function obtained in section 3 to analytically obtained position and momentum space Shannon entropy. In section 5, we obtain analytical solutions to Fisher position and momentum space entropies. We present the numerical solutions , their expectation values and the squeeze state in section 6. Results and discussion are presented in section 7 while section 8 gives the conclusion of the article. The adopted parameter values for both the wave function and probability density plots were all real constants. We discover that the proposed potential is most suitable for ground state energy. This is a further reason why our plots for wave functions and probability densities are all plotted for $n, l = 0$ using a well designed rigorous mathematica algorithm. We also discover that at higher state, that is for n, $l > 0$, The plots were not suitable in providing solutions to the information measures of Shannon and Fisher entropies for this particular potential. Therefore, this work is limited to ground state energy and wave function. The normalisation constant, the wave function plots, the probability density plots, and all numerical computations were carried out using Mathematica. The graph for the noncentral potential is given in figure 1, while that of the Greene Aldrich approximation and special Greene Aldrich approximation are presented in figures 1(a) and (b) respectively. The combination of the noncentral and exponential Mie-type potential is significant in the study of vibrational and rotational energies of diatomic molecules and their degeneracies for a particular quantum state [32]. In most cases, the topological properties of atoms of diatomic molecules, including their chemical functional groups, are evaluated using Density Functional theory(DFT) and pseudopotential formalism [33]. Inversely quadratic potential is a long range potential and in combination with other exponential type potentials may be used in finding the shape of organic molecules such as cyclic polyenes and benzene [22].

2. The generalised parametric Nikiforov–Uvarov method

The NU method was presented by Nikiforov and Uvarov [31] and has been employed to solve second order differential equations such as Schrödinger wave equations (SWE), Klein–Gordon equations (KGE), Dirac equations (DE), etc. The Schrödinger wave equation is given as

$$\frac{\mathrm{d}^2\Psi(r)}{\mathrm{d}r^2} + \frac{2\mu}{\hbar^2}\left[E_{nl} - V(r) - \frac{\hbar^2 l(l+1)}{2\mu r^2}\right]\Psi(r) = 0, \tag{9}$$

correspondingly, for the purpose of this work, we shall be considering the exact solution of a Schrödinger equation where the orbital angular quantum number $l = 0$, hence equation (9) reduces to

$$\frac{\mathrm{d}^2\Psi(r)}{\mathrm{d}r^2} + \frac{2\mu}{\hbar^2}[E_{nl} - V(r)]\Psi(r) = 0. \tag{10}$$

Equation (10) can be solved by transforming it into an hypergeometric type equation using the transformation $s = s(x)$ and its resulting equation is expressed as:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{11}$$

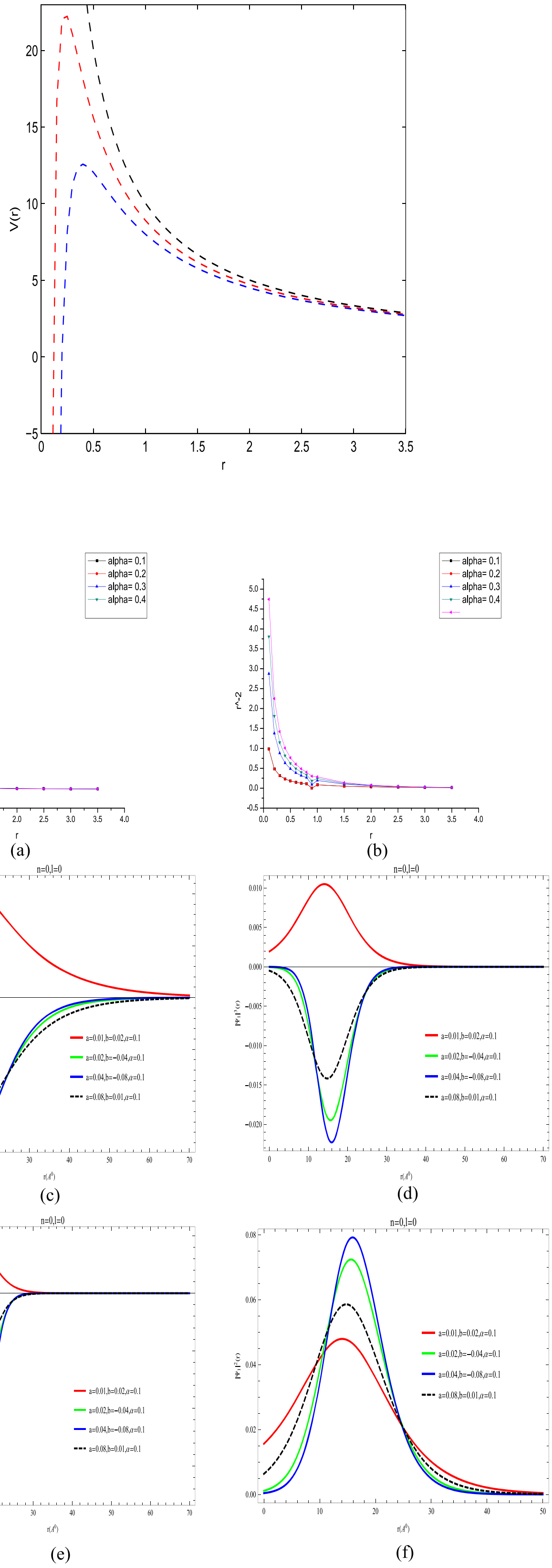

Figure 1. The graph of the Noncentral potential. (a) The centrifugal term $\frac{1}{r^2}$ for standard Greene Aldrich approximation with varying $\alpha = 0.1$ to 0.5. (b) The centrifugal term $\frac{1}{r^2}$ for special Greene Aldrich approximation with varying $\alpha = 0.1$ to 0.5. (c) The normalised wave function plot 1. (d) The normalised wave function plot 2. (e) The normalised wave function plot 3. (f) The probability density function plot 1. (g) The probability density function plot 2. (h) The probability density function plot 3. (i) The normalised wave function plot 1(a). (j) The normalised wave function plot 2(a). (k) The normalised wave function plot 3(a). (l) The probability density function plot 1(b). (m) The probability density function plot 2(b). (n) The probability density function plot 3(b).

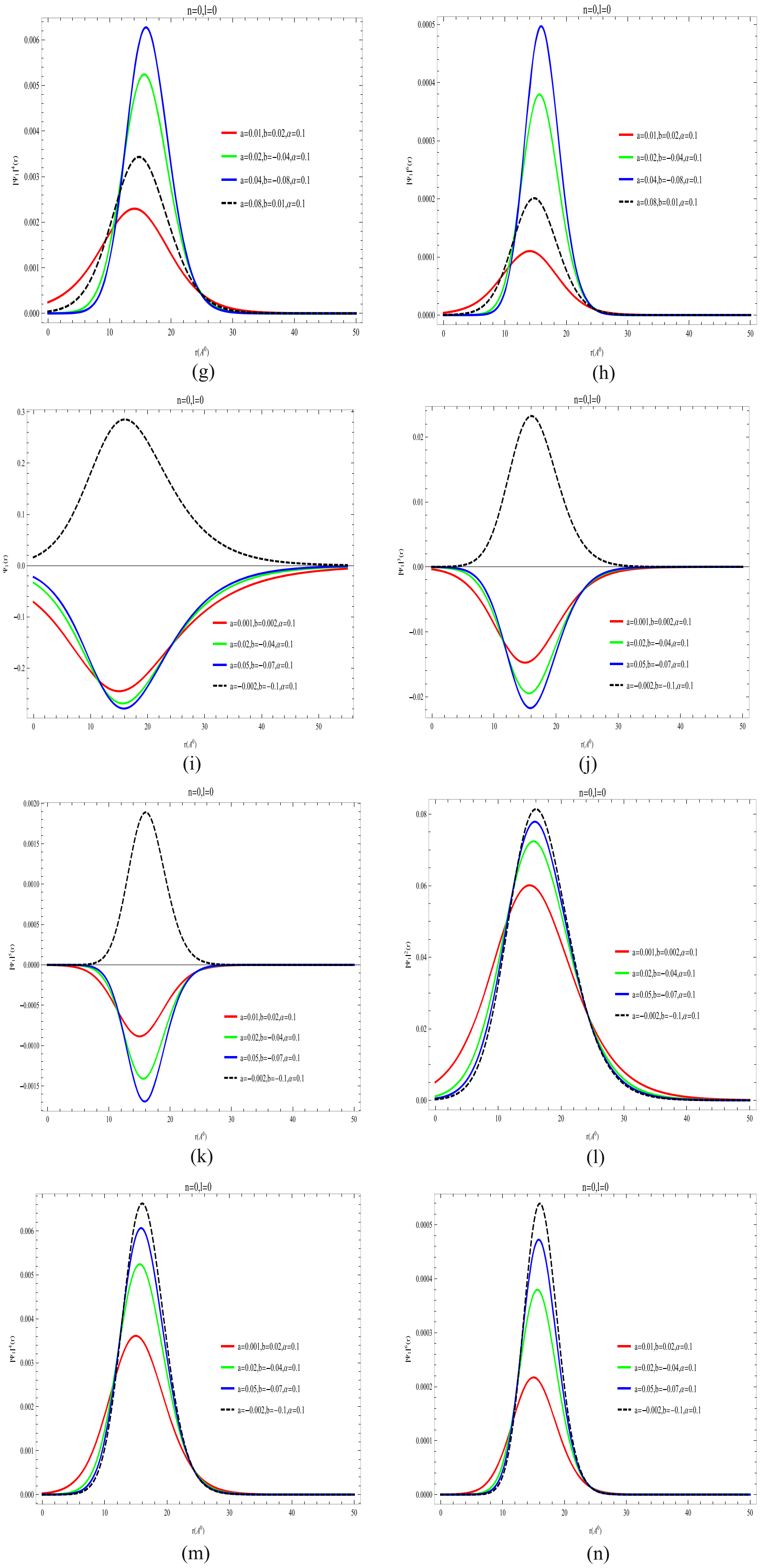

Figure 1. (Continued.)

Table 1. Numerical results for Fisher information for $n = 0$, with various values of B, with $A = 0.5$ eV for the noncentral potential (NCP) for $\alpha = 0.1$.

B	I_r	I_p	$I_r I_p \geqslant 36.0$	$\min(I_r I_p)$
0.001	0.96355	97.0367	93.4997	36.0
0.002	1.01546	96.2569	97.7455	36.0
0.003	1.06619	95.4925	101.814	36.0
0.004	1.11572	94.7429	105.707	36.0
0.005	1.16403	94.0077	109.428	36.0
0.006	1.21111	93.2865	112.981	36.0
0.007	1.25695	92.5789	116.367	36.0
0.008	1.30155	91.8843	119.592	36.0
0.009	1.34489	91.2025	122.657	36.0
0.010	1.38696	90.5330	125.566	36.0
0.011	1.46732	89.8755	128.322	36.0
0.012	1.42777	89.2295	130.928	36.0
0.013	1.50559	88.5949	133.387	36.0
0.014	1.54259	87.9711	135.703	36.0
0.015	1.57832	87.3580	137.879	36.0
0.016	1.61279	86.7553	139.918	36.0
0.017	1.64598	86.1625	141.822	36.0
0.018	1.67791	85.5795	143.595	36.0
0.019	1.70858	85.0061	145.240	36.0
0.020	1.73800	84.4418	146.760	36.0
0.021	1.76617	83.8865	148.157	36.0
0.022	1.79309	83.3400	149.436	36.0
0.023	1.81877	82.8020	150.598	36.0
0.024	1.84322	82.2723	151.646	36.0
0.025	1.86645	81.7507	152.584	36.0
0.026	1.86647	81.2369	153.413	36.0
0.027	1.90928	80.7309	154.138	36.0
0.028	1.92889	80.2323	154.759	36.0
0.029	1.94732	79.7410	155.281	36.0
0.030	1.96456	79.2569	155.705	36.0

Table 2. Numerical results for Fisher information for $n = 0$, with various values of B, with $A = 0.5$ eV for the noncentral potential (NCP) for $\alpha = 0.12$.

B	I_r	I_p	$I_r I_p \geqslant 36.0$	$\min(I_r I_p)$
0.001	1.04867	64.3148	67.4451	36.0
0.002	1.09086	63.8761	69.6802	36.0
0.003	1.13221	63.4448	71.8329	36.0
0.004	1.17271	63.0204	73.9046	36.0
0.005	1.21235	62.6029	75.8964	36.0
0.006	1.25112	62.1922	77.8097	36.0
0.007	1.28902	61.7879	79.6457	36.0
0.008	1.32604	61.3899	81.4057	36.0
0.009	1.36219	60.9981	83.0911	36.0
0.010	1.39746	60.6124	84.7031	36.0
0.011	1.43183	60.2324	86.2429	36.0
0.012	1.46533	59.8582	87.7119	36.0
0.013	1.49793	59.4896	89.1114	36.0
0.014	1.52965	59.1264	90.4427	36.0
0.015	1.56048	58.7685	91.7069	36.0
0.016	1.59042	58.4158	92.9054	36.0
0.017	1.61947	58.0681	94.0393	36.0
0.018	1.64763	57.7253	95.1100	36.0
0.019	1.67491	57.3873	96.1187	36.0
0.020	1.70131	57.0540	97.0666	36.0
0.021	1.72682	56.7254	97.9548	36.0
0.022	1.75146	56.4012	98.7846	36.0
0.023	1.77523	56.0814	99.5571	36.0
0.024	1.79812	55.7658	100.274	36.0
0.025	1.82014	55.4545	100.935	36.0
0.026	1.84130	55.1472	101.543	36.0
0.027	1.86160	54.8440	102.098	36.0
0.028	1.88105	54.5446	102.601	36.0
0.029	1.89964	54.2491	103.054	36.0
0.030	1.91739	53.9574	103.458	36.0

where $\sigma(s)$ and $\tilde{\sigma}(s)$ must be polynomials of at most second degree, $\tilde{\tau}(s)$ is the first degree polynomial, and $\psi(s)$ is a function of the hypergeometric type. The parametric generalisation of the NU method is given by the generalised hypergeometric type equation 12.

$$\psi''(s) + \frac{(c_1 - c_2 s)}{s(1 - c_3 s)}\psi'(s) + \frac{1}{s^2(1 - c_3 s)^2}[-\xi_1 s^2 + \xi_2 s - \xi_3]\psi(s) = 0. \tag{12}$$

Equation (12) has been applied to provide bound state solutions to both relativistic and non-relativistic wave equations with considerable potential such as Hulthen, Eckart, Coulomb, Pseudoharmonic and many others [34–41]. Equation (12) is solved by comparing with equation (11) and the following polynomials are obtained.

$$\tilde{\tau}(s) = (c_1 - c_2 s),\ \sigma(s) = s(1 - c_3 s),\quad \tilde{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3. \tag{13}$$

According to the NU method, the energy eigenvalue equation and eigen wave function, respectively, satisfy the following set of equations:

$$c_2 n - (2n + 1)c_5(2n + 1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n - 1)c_3 + c_7 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0, \tag{14}$$

$$\psi(s) = N_{nl} s^{c_{12}}(1 - c_3 s)^{-c_{12} - \frac{c_{11}}{c_3}} P_n^{\left(c_{10} - 1,\ \frac{c_{11}}{c_3} - c_{10} - 1\right)}(1 - 2c_3 s), \tag{15}$$

where

$$\left.\begin{array}{l} c_4 = \frac{1}{2}(1 - c_1);\ c_5 = \frac{1}{2}(c_2 - c_3);\ c_6 = c_5^2 + \epsilon_1 \\ c_7 = 2c_4 c_5 - \epsilon_2;\ c_8 = c_4^2 + \epsilon_3; \\ c_9 = c_3 c_7 + c_3^2 c_8 + c_6 \\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}; \\ c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) \\ c_{12} = c_4 + \sqrt{c_8};\ c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) \end{array}\right\}. \tag{16}$$

3. The radial solution of the proposed potential in a Schrödinger equation using the Nikiforov–Uvarov method

The noncentral Inversely quadratic plus exponential Mie-type potential is given as

$$V(r) = \frac{A\sin\alpha}{r^2} + \frac{(B - \eta)\mathrm{e}^{-\alpha r}\sin\alpha}{r} + C, \tag{17}$$

where A is the potential depth in electron volts, B, η and C are real constant parameters. α is the screening parameter. The graph of this potential is shown in figure 1.

Substituting equation (17) into (10) gives

$$\frac{\mathrm{d}^2\Psi(r)}{\mathrm{d}r^2}+\frac{2\mu}{\hbar^2}\left[E_{nl}-\frac{A\sin\alpha}{r^2}-\frac{(B-\eta)\mathrm{e}^{-\alpha r}\sin\alpha}{r}-C\right]\Psi(r)=0. \tag{18}$$

To solve equation (18) analytically, we treat $\sin\alpha$ as a constant trignometric function that tends to 1 and define special Greene Aldrich approximation to the centrifugal term as

$$\frac{1}{r^2}=\frac{4\alpha^2\mathrm{e}^{-2\alpha r}\sin\alpha}{(1-\mathrm{e}^{-2\alpha r}\sin\alpha)^2}\Rightarrow\frac{1}{r}=\frac{2\alpha\mathrm{e}^{-\alpha r}\sin\alpha}{(1-\mathrm{e}^{-2\alpha r}\sin\alpha)}. \tag{19}$$

Let us define a special transformation to S-coordinate as

$$s=\mathrm{e}^{-2\alpha r}\sin\alpha. \tag{20}$$

By comparing the graph of a standard Greene Aldrich approximation in figure 1(a) to a special Greene Aldrich approximation in figure 1(b), it can be observed that figure 1(a) converges asymptotically for different values of the screening parameter $\alpha=0.1$ to 0.5. The same thing is also applicable to figure 1(b). This signifies that the special Greene Aldrich approximation could be approximated to the standard Greene Aldrich approximation; this serves as a good approximation to the proposed potential. Substituting equation (19) into 18, and making use of equation (20), equation (18) is reduced to a hypergeometric type equation:

$$\frac{\mathrm{d}^2\Psi(s)}{\mathrm{d}s^2}+\frac{(1-s)}{s(1-s)}\frac{\mathrm{d}\Psi(s)}{\mathrm{d}s}+\frac{1}{s(1-s)}\times\begin{bmatrix}-(\varepsilon^2+\chi_2+\chi_3)s^2+(2\varepsilon^2-\chi_1-\chi_2+2\chi_3)s\\-(\varepsilon^2+\chi_3)\end{bmatrix}\times\Psi(s)=0, \tag{21}$$

where

$$\chi_1=\frac{\mu A}{\hbar^2};\quad\chi_2=\frac{\mu(B-\eta)}{\alpha\hbar^2};\quad\chi_3=\frac{2\mu C}{4\alpha^2\hbar^2};\quad\varepsilon^2=-\frac{2\mu E_{nl}}{4\alpha^2\hbar^2}. \tag{22}$$

Comparing equation (12) to equation (21), and using equation (16), the following parametric constants are obtained:

$$\begin{aligned}&c_1=c_2=c_3=1;\quad c_4=0;\\&c_5=-\frac{1}{2};\quad c_6=\frac{1}{4}+\varepsilon^2;\\&c_7=(-2\varepsilon^2+\chi_1+\chi_2-2\chi_3);\\&c_8=(\varepsilon^2+\chi_3);\quad c_9=\left(\frac{1}{4}+\chi_1+2\chi_2\right);\\&c_{10}=(1+2\sqrt{\varepsilon^2+\chi_3});\\&c_{11}=2+2\left(\sqrt{\frac{1}{4}+\chi_1+2\chi_2}+\sqrt{\varepsilon^2+\chi_3}\right);\\&c_{12}=\sqrt{\varepsilon^2+\chi_3};\\&c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\chi_1+2\chi_2}+\sqrt{\varepsilon^2+\chi_3}\right);\\&\xi_1=(\varepsilon^2+\chi_2+\chi_3);\\&\xi_2=(2\varepsilon^2-\chi_1-\chi_2+2\chi_3);\quad\xi_3=(\varepsilon^2+\chi_3).\end{aligned} \tag{23}$$

Table 3. Numerical results for Fisher information for $n=0$, with various values of B, with $A=0.5$ eV for the noncentral potential (NCP) for $\alpha=0.13$.

B	I_r	I_p	$I_rI_p\geqslant 36.0$	$\min(I_rI_p)$
0.001	1.08189	53.5829	57.9710	36.0
0.002	1.09086	53.2419	59.6565	36.0
0.003	1.13221	52.9061	61.2833	36.0
0.004	1.17271	52.5754	62.8524	36.0
0.005	1.21235	52.2496	64.3646	36.0
0.006	1.25112	51.9287	65.8206	36.0
0.007	1.28902	51.6126	67.2214	36.0
0.008	1.32604	51.3010	68.5679	36.0
0.009	1.36219	50.9939	69.8610	36.0
0.010	1.39746	50.6912	71.1014	36.0
0.011	1.43183	50.3928	72.2901	36.0
0.012	1.46533	50.0986	73.4279	36.0
0.013	1.49793	49.8084	74.5157	36.0
0.014	1.52965	49.5223	75.5544	36.0
0.015	1.56048	49.2400	76.5447	36.0
0.016	1.59042	48.9616	77.4876	36.0
0.017	1.61947	48.6869	78.3838	36.0
0.018	1.64763	48.4159	79.2343	36.0
0.019	1.67491	48.1484	80.0398	36.0
0.020	1.70131	47.8843	80.8011	36.0
0.021	1.72682	47.6237	81.5191	36.0
0.022	1.75146	47.3665	82.1946	36.0
0.023	1.77523	47.1125	82.8283	36.0
0.024	1.79812	46.8617	83.4210	36.0
0.025	1.82014	46.6140	83.9736	36.0
0.026	1.84130	46.3694	84.4868	36.0
0.027	1.86160	46.1277	84.9613	36.0
0.028	1.88105	45.8890	85.3979	36.0
0.029	1.89964	45.6533	85.7974	36.0
0.030	1.91739	45.4203	86.1604	36.0

Table 4. Numerical results for uncertainty relation in the ground eigenstates for various values of B, with n, $l = 0$, $A = 0.5$ eV $\alpha = 0.1$.

B	$\langle r^2\rangle$	$\langle r\rangle$	$\Delta\ (r)$	$(\Delta r)^2$	$\langle p^2\rangle$	$\Delta\ (p)$	$\Delta(r)\Delta(p) \geqslant \frac{\hbar}{2}$	$(\Delta r)^2(\Delta p)^2$	min $\{(\Delta r)^2(\Delta p)^2\}$
0.001	24.2592	3.40874	3.55523	2.90712	0.96354	0.490803	1.74492	3.04474	0.250000
0.002	24.0642	3.39462	3.54130	2.89748	1.01546	0.503851	1.78429	3.18368	0.250000
0.003	23.8731	3.38073	3.52757	2.88797	1.06619	0.516283	1.82123	3.31687	0.250000
0.004	23.6857	3.36707	3.51406	2.87857	1.11572	0.528139	1.85591	3.44440	0.250000
0.005	23.5019	3.35362	3.50074	2.86929	1.16403	0.539452	1.88848	3.56635	0.250000
0.006	23.3216	3.34038	3.48762	2.86013	1.21111	0.550253	1.91907	3.68284	0.250000
0.007	23.1447	3.32735	3.47469	2.85107	1.25695	0.560570	1.94781	3.79395	0.250000
0.008	22.9711	3.31451	3.46195	2.84213	1.30155	0.570427	1.97479	3.89979	0.250000
0.009	22.8006	3.30187	3.44939	2.83329	1.34488	0.579846	2.00011	4.00046	0.250000
0.010	22.6332	3.28941	3.43700	2.82456	1.38696	0.588847	2.02387	4.09604	0.250000
0.011	22.4689	3.27714	3.42479	2.81593	1.42777	0.597447	2.04613	4.18666	0.250000
0.012	22.3074	3.26505	3.41275	2.80740	1.46731	0.605664	2.06698	4.27240	0.250000
0.013	22.1487	3.25312	3.40087	2.28090	1.50558	0.613512	2.08647	4.35337	0.250000
0.014	21.9928	3.24137	3.38914	2.79063	1.54259	0.621005	2.10468	4.42967	0.250000
0.015	21.8395	3.22978	3.37758	2.78239	1.57832	0.628157	2.12165	4.50139	0.250000
0.016	21.6888	3.21835	3.36616	2.77424	1.61278	0.634977	2.13744	4.56864	0.250000
0.017	21.5406	3.20707	3.35495	2.76618	1.64598	0.641479	2.15209	4.63151	0.250000
0.018	21.3949	3.19595	3.34377	2.75821	1.67791	0.647671	2.16567	4.69011	0.250000
0.019	21.2515	3.18497	3.33279	2.75032	1.70858	0.653564	2.17819	4.74452	0.250000
0.020	21.1105	3.17414	3.32195	2.74252	1.73799	0.659166	2.18971	4.79485	0.250000
0.021	20.9716	3.16344	3.31123	2.73480	1.76616	0.664486	2.20027	4.84118	0.250000
0.022	20.8350	3.15288	3.30066	2.72716	1.79308	0.669531	2.20989	4.88362	0.250000
0.023	20.7005	3.14246	3.29020	2.71960	1.81877	0.674309	2.21861	4.92225	0.250000
0.024	20.5681	3.13217	3.27987	2.71212	1.84322	0.678827	2.22647	4.95716	0.250000
0.025	20.4377	3.12200	3.26967	2.70472	1.86645	0.683091	2.23348	4.98845	0.250000
0.026	20.3092	3.11196	3.25958	2.69739	1.88846	0.687108	2.23969	5.01621	0.250000
0.027	20.1827	3.10204	3.24962	2.69013	1.90927	0.690883	2.24511	5.04052	0.250000
0.028	20.0581	3.09224	3.23976	2.68295	1.92889	0.694423	2.24977	5.06146	0.250000
0.029	19.9353	3.08256	3.23003	2.67584	1.94732	0.697731	2.25369	5.07913	0.250000
0.030	19.8142	3.07299	3.22039	2.66879	1.96456	0.700815	2.25690	5.09361	0.250000

By making use of equation (14), the energy-eigen equation for the proposed potential is given as

$$E_{nl} = \frac{-2\alpha^2\hbar^2}{\mu} \times \left\{\frac{\left(n^2 + n + \frac{1}{2}\right) + \left(n + \frac{1}{2}\right)\sqrt{1 + \frac{4\mu A}{\hbar^2} + \frac{8\mu(B-\eta)}{\alpha\hbar^2}} + \frac{\mu A}{\hbar^2} + \frac{\mu(B-\eta)}{\alpha\hbar^2}}{(2n+1) + \sqrt{1 + \frac{4\mu A}{\hbar^2} + \frac{8\mu(B-\eta)}{\alpha\hbar^2}}}\right\}^2 + C. \tag{24}$$

Also, by making use of equation (15), the total wave function is given as

$$\Psi_n(s) = N_n S^{\sqrt{\varepsilon^2+\chi_3}}(1 - s)^{-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\chi_1+2\chi_2}+\sqrt{\varepsilon^2+\chi_3}\right)}$$
$$\times P_n^{\left[(1+2\sqrt{\varepsilon^2+\chi_3}),2+2\left(\sqrt{\frac{1}{4}+\chi_1+2\chi_2}+\sqrt{\varepsilon^2+\chi_3}\right)\right]}(1 - 2s)$$
$$\Rightarrow \Psi_n(s) = N_n(\mathrm{e}^{-2\alpha r}\sin\alpha)^{\sqrt{\frac{2\mu C}{4\alpha^2\hbar^2}-\frac{2\mu E_{nl}}{4\alpha^2\hbar^2}}}$$
$$\times (1 - \mathrm{e}^{-2\alpha r}\sin\alpha)^{-\frac{1}{2}-\left(\sqrt{\frac{1}{4}+\frac{\mu A}{\hbar^2}+2\frac{\mu(B-\eta)}{\alpha\hbar^2}}+\sqrt{\frac{2\mu C}{4\alpha^2\hbar^2}-\frac{2\mu E_{nl}}{4\alpha^2\hbar^2}}\right)}$$
$$\times P_n^{\left[\left(1+2\sqrt{\frac{2\mu C}{4\alpha^2\hbar^2}-\frac{2\mu E_{nl}}{4\alpha^2\hbar^2}}\right),2+2\left(\sqrt{\frac{1}{4}+\frac{\mu A}{\hbar^2}+2\frac{\mu(B-\eta)}{\alpha\hbar^2}}+\sqrt{\frac{2\mu C}{4\alpha^2\hbar^2}-\frac{2\mu E_{nl}}{4\alpha^2\hbar^2}}\right)\right]}$$
$$\times (1 - 2\mathrm{e}^{-2\alpha r}\sin\alpha). \tag{25}$$

The normalised wave function and probability density curves for the first set of real constant parameters are given below.

The graph for the second set of adopted parameters, with specific adjustable screening parameters, were also plotted for wave function and probability density for the sake of comparison, as shown in the figures below.

Figures 1(c), (d) and (e) are wave function plots for the first set of real constant parameters for Ψ, $|\Psi|^2$, and $|\Psi|^3$, respectively, while figures 1(i), (j) and (k) represent the wave function plots Ψ, $|\Psi|^2$, and $|\Psi|^3$, respectively, for the second set of real constant parameters. However,by observation, the wave function plots for these two sets of real parameters are almost the same, and this enables us to propose a generalised conclusion that odd value wave functions have the same properties, especially when the wave function is continuous, having continuous partial derivatives. A careful observation of the probability density in figures 1(f), (g) and (h) for the

Table 5. Numerical results for uncertainty relation in the ground eigenstates for various values of B, with n, $l = 0$, $A = 0.5$ eV $\alpha = 0.12$.

B	$\langle r^2\rangle$	$\langle r\rangle$	$\Delta\,(r)$	$(\Delta r)^2$	$\langle p^2\rangle$	$\Delta\,(p)$	$\Delta(r)\Delta(p) \geqslant \frac{\hbar}{2}$	$(\Delta r)^2\,(\Delta p)^2$	$\min\{(\Delta r)^2(\Delta p)^2\}$
0.001	16.0787	2.76177	2.90712	8.45134	0.262168	0.512023	1.48851	2.21567	0.250000
0.002	15.9690	2.75202	2.89748	8.39540	0.272716	0.522222	1.51313	2.28956	0.250000
0.003	15.7551	2.74242	2.88797	8.34035	0.283053	0.532027	1.53648	2.36076	0.250000
0.004	15.6507	2.73294	2.87857	8.28616	0.293177	0.541458	1.55863	2.42931	0.250000
0.005	15.6507	2.72358	2.86929	8.23283	0.303086	0.550532	1.57964	2.49526	0.250000
0.006	15.5480	2.71435	2.86013	8.18032	0.312779	0.559266	1.59957	2.55864	0.250000
0.007	15.4477	2.70524	2.85107	8.12862	0.322255	0.567674	1.61848	2.61948	0.250000
0.008	15.3475	2.69625	2.84213	8.07770	0.331511	0.575769	1.63641	2.67785	0.250000
0.009	15.2495	2.68737	2.83329	8.02755	0.340548	0.583564	1.65341	2.73376	0.250000
0.010	15.1531	2.67861	2.82456	7.97815	0.349364	0.591070	1.66951	2.78728	0.250000
0.011	15.1531	2.66995	2.81593	7.92947	0.357959	0.598296	1.68476	2.83842	0.250000
0.012	14.9646	2.66140	2.81593	7.92947	0.366332	0.605253	1.69919	2.88725	0.250000
0.013	12.2407	2.65296	2.81593	5.20251	0.374483	0.611950	1.39580	1.94825	0.250000
0.014	14.7816	2.64461	2.79063	7.78763	0.382412	0.618395	1.72571	2.97809	0.250000
0.015	14.6921	2.63637	2.78239	7.74170	0.390119	0.624595	1.73787	3.02019	0.250000
0.016	14.6039	2.62822	2.77423	7.69641	0.397604	0.630558	1.74932	3.06012	0.250000
0.017	14.5176	2.62017	2.76617	7.65175	0.404867	0.636291	1.76010	3.09794	0.250000
0.018	14.4313	2.61221	2.75821	7.60771	0.411908	0.641800	1.77022	3.13367	0.250000
0.019	14.3468	2.60434	2.75032	7.56426	0.418728	0.647091	1.77971	3.16737	0.250000
0.020	14.2635	2.59656	2.74252	7.52140	0.425327	0.652171	1.78859	3.19906	0.250000
0.021	14.1813	2.58887	2.73479	7.47912	0.431706	0.657043	1.79688	3.22878	0.250000
0.022	14.1003	2.58126	2.72715	7.43740	0.437866	0.661714	1.80460	3.25658	0.250000
0.023	14.0203	2.57374	2.71960	7.39623	0.443806	0.666187	1.81177	3.28249	0.250000
0.024	13.9415	2.56629	2.71212	7.35560	0.449529	0.670469	1.81839	3.30656	0.250000
0.025	13.8636	2.55893	2.70471	7.31549	0.455035	0.674563	1.82450	3.32881	0.250000
0.026	13.7868	2.55165	2.69738	7.27590	0.460325	0.678472	1.83010	3.34928	0.250000
0.027	13.7110	2.54444	2.69013	7.23681	0.465401	0.682202	1.83522	3.36801	0.250000
0.028	13.6362	2.53731	2.68294	7.19821	0.470262	0.685756	1.83985	3.38505	0.250000
0.029	13.5623	2.53025	2.67583	7.16010	0.474911	0.689137	1.84402	3.40041	0.250000
0.030	13.4893	2.52327	2.66879	7.12245	0.479349	0.692350	1.84774	3.41414	0.250000

first set of parameter and (l), (m) and (n) for the second set of parameter shows that even values of the wave function represent the probability density for a continuous wave function having continuous partial derivatives.

4. Position and momentum space for Shannon entropy

In order to determine both position and momentum space for Shannon entropy, one needs to calculate the probability density from the wave function. Meanwhile, the wave function must be normalised. First, we simplify equation (25) for easy normalisation. Assuming that $\sigma_1 = \sqrt{\varepsilon^2 + \chi_3}$ and $\sigma_2 = \sqrt{\frac{1}{4} + \chi_1 + 2\chi_2}$ Then equation (25) reduces to

$$\Psi_n(s) = N_n S^{\sigma_1}(1 - s)^{-\frac{1}{2}-(\sigma_2+\sigma_1)} P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1 - 2s). \tag{26}$$

The probability density is the square of the wave function. Hence, squaring equation (26) gives

$$\rho(x) = \rho(r) = N_n^2 S^{2\sigma_1}(1 - s)^{-1-2(\sigma_2+\sigma_1)} \times |P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1 - 2s)|^2. \tag{27}$$

From (20), the integration boundaries from $(-\infty, +\infty)$ in r-dimension change to (0,1) in s-dimension and by making use of equation (3), the Shannon entropy for position space for the noncentral potential is given as

$$S_x^{NCP} = -\frac{N_n^2}{2\alpha}\int_0^1 \times \begin{bmatrix} (S^{2\sigma_1-1}(1 - s)^{-1-2(\sigma_2+\sigma_1)}|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1 - 2s)|^2) \\ \times \log N_n^2(S^{2\sigma_1-1}(1 - s)^{-1-2(\sigma_2+\sigma_1)}|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1 - 2s)|^2) \end{bmatrix} \mathrm{d}s. \tag{28}$$

Table 6. Numerical results for uncertainty relation in the ground eigenstates for various values of B, with n, $l = 0$, $A = 0.5$ eV $\alpha = 0.13$.

B	$\langle r^2\rangle$	$\langle r\rangle$	$\Delta\ (r)$	$(\Delta r)^2$	$\langle p^2\rangle$	$\Delta\ (p)$	$\Delta(r)\Delta(p) \geqslant \frac{\hbar}{2}$	$(\Delta r)^2\ (\Delta p)^2$	$\min\{(\Delta r)^2(\Delta p)^2\}$
0.001	13.3957	2.51467	2.65936	7.07218	0.270473	0.520070	1.38305	1.91283	0.250000
0.002	13.3105	2.50638	2.65114	7.02855	0.280120	0.529264	1.40315	1.96884	0.250000
0.003	13.2265	2.49819	2.64302	6.98557	0.283053	0.538131	1.42229	2.02292	0.250000
0.004	13.1438	2.49011	2.63500	6.94322	0.298868	0.546688	1.44052	2.07511	0.250000
0.005	13.0624	2.48212	2.62707	6.90148	0.307966	0.554947	1.45788	2.12543	0.250000
0.006	12.9822	2.47423	2.61923	6.86035	0.316879	0.562920	1.47442	2.17390	0.250000
0.007	12.9031	2.46644	2.61147	6.81979	0.325606	0.570619	1.49016	2.22057	0.250000
0.008	12.8252	2.45875	2.60381	6.77981	0.334145	0.578053	1.50514	2.26544	0.250000
0.009	12.7485	2.45114	2.59623	6.74039	0.342497	0.585232	1.51940	2.30856	0.250000
0.010	12.6728	2.44362	2.58873	6.70151	0.350660	0.592165	1.53295	2.34995	0.250000
0.011	12.5982	2.43619	2.58131	6.66317	0.358633	0.598860	1.54584	2.38963	0.250000
0.012	12.5246	2.42885	2.57398	6.62535	0.366417	0.605324	1.55809	2.42764	0.250000
0.013	12.4521	2.42158	2.56672	6.58804	0.374012	0.611565	1.56971	2.46400	0.250000
0.014	12.3806	2.41441	2.55954	6.55122	0.381416	0.617589	1.58074	2.49874	0.250000
0.015	12.3100	2.40731	2.55243	6.51489	0.388630	0.623402	1.59119	2.53188	0.250000
0.016	12.2404	2.40028	2.54540	6.47904	0.395655	0.629011	1.60108	2.56346	0.250000
0.017	12.1717	2.39334	2.53844	6.44365	0.402489	0.634420	1.61044	2.59350	0.250000
0.018	12.1040	2.38647	2.53155	6.40872	0.409134	0.639636	1.61927	2.62203	0.250000
0.019	12.0371	2.37967	2.52473	6.37424	0.415589	0.644662	1.62760	2.64907	0.250000
0.020	11.9711	2.37295	2.51797	6.34019	0.421856	0.649504	1.63543	2.67465	0.250000
0.021	11.9059	2.36630	2.51129	6.30657	0.427933	0.654166	1.64280	2.69879	0.250000
0.022	11.8416	2.35971	2.50467	6.27337	0.433822	0.658652	1.64971	2.72153	0.250000
0.023	11.7781	2.35320	2.49812	6.24058	0.439524	0.662966	1.65617	2.74289	0.250000
0.024	11.7154	2.34675	2.49162	6.20819	0.445039	0.667112	1.66219	2.76289	0.250000
0.025	11.6535	2.34036	2.48519	6.17619	0.450367	0.671094	1.66780	2.78155	0.250000
0.026	11.5923	2.33404	2.47883	6.14458	0.455510	0.674915	1.67300	2.79892	0.250000
0.027	11.5319	2.32779	2.47252	6.11335	0.460467	0.678577	1.67780	2.8150	0.250000
0.028	11.4723	2.32159	2.46627	6.08248	0.465241	0.682086	1.68221	2.82982	0.250000
0.029	11.4133	2.31546	2.46008	6.05198	0.469831	0.685442	1.68624	2.84341	0.250000
0.030	11.3551	2.30938	2.45394	6.02184	0.474240	0.688651	1.68991	2.85579	0.250000

Equation (28) can also be reduced to

$$S_x^{NCP} = -\frac{N_n^2}{2\alpha}\int_0^1 \times \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \\ \times \begin{pmatrix} \log(N_n^2) + \log(S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ +\log(|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \end{pmatrix} \end{bmatrix} \mathrm{d}s. \tag{29}$$

Equation (29) can further be separated into three separate integrals as shown in equation (30)

$$\begin{aligned} S_x^{NCP} &= -\log(N_n^2) \\ &\times \frac{N_n^2}{2\alpha}\int_0^1 \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \end{bmatrix} \mathrm{d}s \\ &-\frac{N_n^2}{2\alpha}\int_0^1 \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \\ \times \log(S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \end{bmatrix} \mathrm{d}s \\ &-\frac{N_n^2}{2\alpha}\int_0^1 \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \\ \times \log(|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \end{bmatrix} \mathrm{d}s. \end{aligned} \tag{30}$$

The first integral of equation (30) signifies normalisation with a unique value of one as shown in equation (31)

$$\frac{N_n^2}{2\alpha}\int_0^1 [(S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2)]\mathrm{d}s = 1. \tag{31}$$

Equation (30) finally reduces to

$$S_x^{NCP} = -\log N_n^2 + \frac{N_n^2}{2\alpha}[E(P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s) + I(P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)] \tag{32}$$

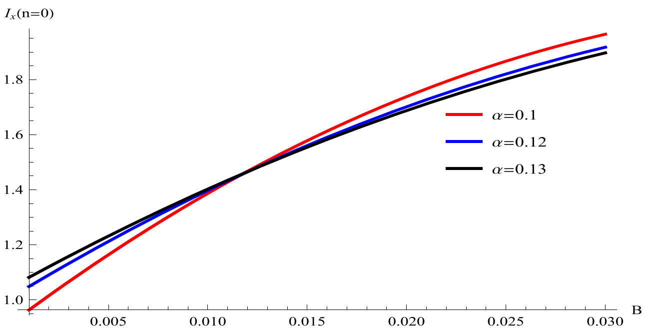

Figure 2. Fisher Position entropy for $n = 0$.

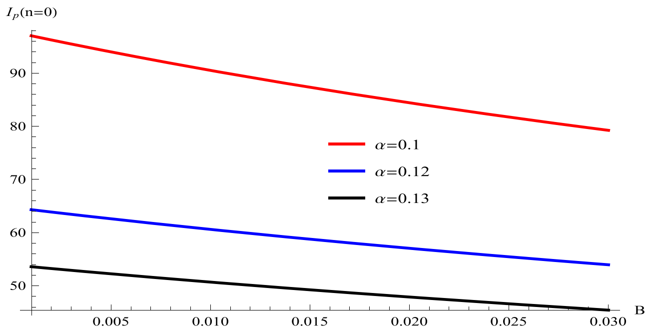

Figure 3. Fisher momentum entropy for $n = 0$.

where E and I are entropic integrals with the following expressions:

$$E[P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)] = -\int_0^1 \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \\ \times \log(|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \end{bmatrix} \mathrm{d}s$$

$$I[P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)] = -\int_0^1 \begin{bmatrix} (S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \\ \times (|P_n^{[(1+2\sigma_1),2+2(\sigma_2+\sigma_1)]}(1-2s)|^2) \\ \times \log(S^{2\sigma_1-1}(1-s)^{-1-2(\sigma_2+\sigma_1)}) \end{bmatrix} \mathrm{d}s. \quad (33)$$

Guerrero and Aptekarev define entropic integrals using digamma functions [42, 43] as

$$\begin{aligned} E[P_n^{(\alpha,\beta)}(x)] &= -\int_a^b w(\alpha,\beta)\,|P_n^{(\alpha,\beta)}(x)|^2 \\ &\quad\times \log|P_n^{(\alpha,\beta)}(x)|^2 \mathrm{d}x \\ &= \log(\pi) - 1 - (\alpha+\beta)\log 2 + o(1), \end{aligned} \quad (34)$$

In order to express E_n as a digamma function, we consider equation (32) such that

$$\alpha = 2\sigma_1 + 1;\ \ 2 + 2(\sigma_1 + \sigma_2);\ \ x = 1 - 2s. \quad (36)$$

Then, by making use of equation (34), the following can be obtained:

$$\begin{aligned} I|P_n^{(\alpha,\beta)}(x)| &= -(1+2\sigma_1)\psi(n+2\sigma_1+2) \\ &- (2+2\sigma_1+2\sigma_2)\psi(n+2+2\sigma_1+2\sigma_2) \\ &+ (4\sigma_1+2\sigma_2+3)\left[-\log 2 + \frac{1}{(2n+4\sigma_1+2\sigma_2+4)}\right. \\ &\left.+ 2\psi(2n+4\sigma_1+2\sigma_2+3) - \psi(n+4\sigma_1+2\sigma_2+4)\right]. \end{aligned} \quad (37)$$

Substituting equations (35) and (37) into (32) gives the position space Shannon entropy for the noncentral potential as

$$S_n^{NCP} = -\log N_n^2 + \frac{N_n^2}{2\alpha} \times \begin{Bmatrix} \log(\pi) - (4\sigma_1+2\sigma_2+3) - 1 + o(1) - (1+2\sigma_1)\psi(n+2\sigma_1+2) \\ -(2+2\sigma_1+2\sigma_2)\psi(n+2+2\sigma_1+2\sigma_2) \\ \times \begin{bmatrix} -\log 2 + \dfrac{1}{(2n+4\sigma_1+2\sigma_2+4)} + 2\psi(2n+4\sigma_1+2\sigma_2+3) \\ -\psi(n+4\sigma_1+2\sigma_2+4) \end{bmatrix} \end{Bmatrix}. \quad (38)$$

$$\begin{aligned} I|P_n^{(\alpha,\beta)}(x)| &= -\int_{-1}^1 w(\alpha,\beta)\,|P_n^{(\alpha,\beta)}(x)|^2 \log w(\alpha,\beta)\mathrm{d}x \\ &= -\alpha\psi(n+\alpha+1) - \beta\psi(n+\beta+1) + (\alpha+\beta) \\ &\times \left[-\log 2 + \frac{1}{(2n+\alpha+\beta+1)} + 2\psi(2n+\alpha+\beta+1)\right. \\ &\left.- \psi(n+\alpha+\beta+1)\right]. \end{aligned} \quad (35)$$

To obtain the normalised wave function for the position space Shannon entropy, the normalisation constant can be evaluated using

$$\int_{-\infty}^{+\infty} \Psi(x)\Psi^*(x)\mathrm{d}x = \int_{-\infty}^{\infty} |\Psi(x)|^2 \mathrm{d}x = 1. \quad (39)$$

Substituting equation (26) into (39), and using Mathematica software, the normalisation constant for $n = 0$ is obtained as

$$N_0^{NCP} = \frac{1.41421}{\left(\frac{\Gamma(2\sigma_1)(-\sin(\alpha))^{-2\sigma_1}(-\sin^2(\alpha))^{2\sigma_1}\ {}_2F_1(2\sigma_1, 2\sigma_1+2\sigma_2+1.; 2\sigma_1+1; \sin(\alpha))}{\alpha\Gamma(2\sigma_1+1)}\right)^{0.5}}. \quad (40)$$

Note that if the Jacobi polynomial for $n = 0$ is unity, then the total normalised ground state wave function for position space is given as

$$\Psi_0^{NCP}(r) = \left[\frac{1.41421}{\left(\frac{\Gamma(2\sigma_1)(-\sin(\alpha)^{-2\sigma_1}(-\sin^2(\alpha)^{2\sigma_1}\ {}_2F_1(2\sigma_1, 2\sigma_1+2\sigma_2+1.; 2\sigma_1+1; \sin(\alpha)}{\alpha\Gamma(2\sigma_1+1)}\right)^{0.5}}\right] \times S^{\sigma_1}(1-s)^{-\frac{1}{2}-(\sigma_2+\sigma_1)}. \quad (41)$$

The corresponding normalised wave function in momentum space is derived by taking the Fourier transform of the position space wave function. The Fourier transform is given as

$$\Psi_0^{\mathrm{NCP}}(P) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \Psi_0^{\mathrm{NCP}}(r)\mathrm{e}^{-\mathrm{i}pr}\mathrm{d}r. \quad (42)$$

The corresponding normalised momentum space wave function is then given as

$$\Psi_0^{\mathrm{NCP}}(P) = \frac{0.282095(-\sin(\alpha)^{-\sigma_1}(-\sin^2(\alpha)^{\sigma_1}\Gamma\left(\frac{ip}{2\alpha}+\sigma_1\right)\ {}_2F_1\left(\frac{ip}{2\alpha}+\sigma_1, \sigma_1+\sigma_2+0.5; \frac{ip}{2\alpha}+\sigma_1+1; \sin(\alpha)\right)}{\alpha\Gamma\left(\frac{ip}{2\alpha}+\sigma_1+1\right)\left(\frac{\Gamma(2\sigma_1)(-\sin(\alpha)^{-2\sigma_1}(-\sin^2(\alpha)^{2\sigma_1}\ {}_2F_1(2\sigma_1, 2\sigma_1+2\sigma_2+1.; 2\sigma_1+1; \sin(\alpha)}{\alpha\Gamma(2\sigma_1+1)}\right)^{0.5}}. \quad (43)$$

Substituting equation (43) into equation (4) gives the Shannon momentum entropy, which gives a highly complicated integral of a regularised confluent hypergeometric type as shown in equation (44)

$$S_n^{NCP}(p) = -\int_0^\infty [\Psi_0^{NCP}(p)]\log[\Psi_0^{NCP}(p)]\mathrm{d}p. \quad (44)$$

5. Position and momentum space for Fisher entropy

Fisher, in 1925, defines position space Fisher information as

$$I_x = \int\frac{[\rho_n(x)]^2}{\rho(x)}\mathrm{d}x = 4\int[\Psi'(r)]^2\mathrm{d}r = 4\langle p^2\rangle. \quad (45)$$

The Fisher information for the noncentral potential is given as

$$I_x^{NCP} = 4\int_0^\infty [\Psi'^{NCP}(r)]^2\mathrm{d}r. \quad (46)$$

From (20), if α is very small such that $\sin\alpha \to 1$, then by expressing equation (45) in terms of (20), the position space information entropy reduces to

$$I_x^{NCP} = -\frac{2}{\alpha}\int_0^\infty \frac{1}{s}[\Psi'^{NCP}(s)]^2\mathrm{d}s = \frac{2}{\alpha}\int_{-1}^{+1}\frac{2}{(1-x)}[\Psi'^{NCP}(x)]^2\mathrm{d}x. \quad (47)$$

In order to carry out the derivative of equation (26), which involves the Jacobi polynomial, it should be noted that the derivative of classical orthogonal polynomials of the same family but with different parameters can be done using orthogonality Pearson's relations. This is applicable to hypergeometric functions of Jacobi, Laguerre and Hermite polynomials as [44]

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) = nP_{n-1}^{(\alpha+1,\beta+1)}(x);\ \frac{\mathrm{d}}{\mathrm{d}x}L_n^{(\alpha)}(x) = nL_{n-1}^{(\alpha+1)}(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}H_n(x) = nH_{n-1}(x). \quad (48)$$

Differentiating equation (26) with the help of the above gives

$$\Psi'^{NCP} = \frac{\mathrm{d}\Psi^{NCP}}{\mathrm{d}x} = N_n\left\{\begin{array}{l}\left[\left(\frac{1-x}{2}\right)^{\sigma_1}\left(\frac{1+x}{2}\right)^{-\frac{1}{2}-(\sigma_2+\sigma_1)} nP_{n-1}^{[(2+2\sigma_1),(3+2(\sigma_2+\sigma_1)]}(x)\right]\\ -\left[\left(\frac{1}{4}+\frac{1}{2}(\sigma_2+\sigma_1)\right)\left(\frac{1-x}{2}\right)^{\sigma_1}\left(\frac{1+x}{2}\right)^{-\frac{3}{2}-(\sigma_2+\sigma_1)} P_n^{[(1+2\sigma_1),(2+2(\sigma_2+\sigma_1)]}(x)\right]\\ -\left[\left(\frac{\sigma_1}{2}\right)\left(\frac{1-x}{2}\right)^{\sigma_1-1}\left(\frac{1+x}{2}\right)^{-\frac{1}{2}-(\sigma_2+\sigma_1)} P_n^{[(1+2\sigma_1),(2+2(\sigma_2+\sigma_1)]}(x)\right]\end{array}\right\}, \quad (49)$$

where $P_n^{(\alpha,\beta)}(x)$, $L_n^{(\alpha)}(x)$ and $H_n(x)$ are Jacobi, Laguerre and Hermite polynomials, respectively. Substituting equation (48) into (46) gives the Fisher entropy for position space as

$$I_x^{NCP} = \frac{2N_n^2}{\alpha}\int_{-1}^{+1}\frac{2}{(1-x)}\times\left\{\begin{array}{l}\left[\left(\frac{1-x}{2}\right)^{\sigma_1}\left(\frac{1+x}{2}\right)^{-\frac{1}{2}-(\sigma_2+\sigma_1)} nP_{n-1}^{[(2+2\sigma_1),(3+2(\sigma_2+\sigma_1))]}(x)\right]\\ -\left[\left(\frac{1}{4}+\frac{1}{2}(\sigma_2+\sigma_1)\right)\left(\frac{1-x}{2}\right)^{\sigma_1}\left(\frac{1+x}{2}\right)^{-\frac{3}{2}-(\sigma_2+\sigma_1)} P_n^{[(1+2\sigma_1),(2+2(\sigma_2+\sigma_1))]}(x)\right]\\ -\left[\left(\frac{\sigma_1}{2}\right)\left(\frac{1-x}{2}\right)^{\sigma_1-1}\left(\frac{1+x}{2}\right)^{-\frac{1}{2}-(\sigma_2+\sigma_1)} P_n^{[(1+2\sigma_1),(2+2(\sigma_2+\sigma_1))]}(x)\right]\end{array}\right\}^2 \mathrm{d}x. \quad (50)$$

Equation (49) is a complicated integral. However, in this article we evaluate only for the Fisher and Shannon information entropies for the ground state, that is for $n = 0$. The Fisher information at ground state is given as

$$I_x^{NCP} = -\frac{2N_0^{NCP}}{\alpha}\int_{-1}^{+1}\frac{2}{(1-x)} \times\left\{\begin{array}{l}\left[\left(\frac{1}{4}+\frac{1}{2}(\sigma_2+\sigma_1)\right)\left(\frac{1-x}{2}\right)^{\sigma_1}\left(\frac{1+x}{2}\right)^{-\frac{3}{2}-(\sigma_2+\sigma_1)}\right]\\ +\left[\left(\frac{\sigma_1}{2}\right)\left(\frac{1-x}{2}\right)^{\sigma_1-1}\left(\frac{1+x}{2}\right)^{-\frac{1}{2}-(\sigma_2+\sigma_1)}\right]\end{array}\right\}^2 \mathrm{d}x. \quad (51)$$

The integral of equation (50) is carried using mathematica. Hence, the solution to equation (50) expressed in terms of regularised hypergeometric function is given as

$$I_x^{NCP} = -\frac{2N_0^{NCP}}{\alpha}\times\left[\begin{array}{l}\left(\begin{array}{l}2^{3+\sigma_2}\Gamma(-2(\sigma_2+\sigma_1))\,{}_2F_1(1,\ 3-2\sigma_1,\ 1-2\sigma_1-2\sigma_2,\ -1)\\ +\Gamma(-2+2\sigma_1)\,{}_2F_1\left(1,\ 1+2\sigma_1+2\sigma_2,\ -1+2\sigma_1,\ -1\right)\end{array}\right)\sigma_1\\ +\left(\frac{1}{4}+\frac{1}{2}(\sigma_2+\sigma_1)\right)\\ \times 2^{\frac{3}{2}-\sigma_1+\sigma_2}\times\left(\begin{array}{l}\frac{{}_2F_1\left(1,\ \frac{1}{2}+\sigma_1+\sigma_2,\ 1+2\sigma_1,\ -1\right)}{2\sigma_1}\\ +2\times\left(\frac{{}_2F_1\left(1,\ 1-2\sigma_1,\ \frac{3}{2}-\sigma_1-\sigma_2,\ -1\right)}{(1-2\sigma_1-2\sigma_2)}\right)\end{array}\right)\end{array}\right]. \quad (52)$$

Equation (52) is the position space Fisher entropy for the proposed noncentral potential. However, taking the Fourier transform of the position space Fisher entropy gives the corresponding momentum space Fisher entropy. This is obtained by substituting equation (43) into (45) which gives a more complicated integral.

6. Numerical computations for Fisher information entropy

In this section, we carry out numerical computations for both position and momentum Fisher entropies. we calculated the expectation values of $\langle r\rangle$, $\langle r^2\rangle$, $\langle p\rangle$, and $\langle p^2\rangle$. We calculated the Heinsenberg uncertainties in position and momentum using the variance relations $\langle r\rangle = \sqrt{\langle r^2\rangle - \langle r\rangle^2}$ and $\langle p\rangle = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$. The Heinsenberg uncertainty principle is the product of uncertainties of position and momentum, which is expressed in terms of Planck's constant as $\Delta(r)\Delta(p) \geqslant \frac{\hbar}{2\pi}$. Meanwhile the expectation values are calculated numerically using the following expressions:

$$\langle r\rangle = \int_0^\infty (\Psi_0^{NCP}(r) r (\Psi_0^{NCP}(r)\mathrm{d}r,$$
$$\langle r^2\rangle = \int_0^\infty (\Psi_0^{NCP}(r) r^2 (\Psi_0^{NCP}(r)\mathrm{d}r,$$
$$\langle p\rangle = \int_0^\infty (\Psi_0^{NCP}(p) p (\Psi_0^{NCP}(p)\mathrm{d}p,$$
$$\langle p^2\rangle = \int_0^\infty (\Psi_0^{NCP}(p) p^2 (\Psi_0^{NCP}(p)\mathrm{d}p. \quad (53)$$

The adopted real numerical constants parameters are: $A = 0.5$ eV representing the potential depth, B varies infinitesimally from 0.001 to 0.03 in the step of 0.001, $\eta = 0.01$ and $\hbar = 1$.

7. Results and discussion

First, we demonstrated from figures 1(c) to (n) that in general, odd powers of wave function give the property of a wave function while even powers of wave function represent the probability density for a continuous wave function with continuous partial derivatives. The wave function and probability density curves were plotted for two set of parameters: $(A = 0.01,\ B = 0.02)$, $(A = 0.02,\ B = -0.04)$, $(A = 0.04,\ B = -0.08)$, $(A = 0.08,\ B = 0.01)$ and $(A = 0.01,\ B = 0.02)$, $(A = 0.02,\ B = -0.04)$, $(A = 0.05,\ B = -0.07)$, $(A = -0.002,\ B = -0.1)$.

The numerical computations for position and momentum Fisher information for the three different values of α=0.1, 0.12 and 0.13 give perfect results for values of variation parameter B, because it is expected that $I_r.I_p \geqslant 36.0$ which is shown in tables 1, 2 and 3. The computation of various expectation values for different values of α also gave a good result, which is in agreement with existing literature. Here, it is expected that the Heinsenberg uncertainty relation of the product of position and momentum entropies should be given as as $\Delta(r)\Delta(p) \geqslant \frac{\hbar}{2\pi}$ such that the squeeze state of $\Delta(r)^2\Delta(p)^2$ should be a minimum of 0.250 00. Tables 4, 5 and 6 gives the various expectation values for different values of α and their corresponding squeeze state. This table is in agreement with existing literature with a squeeze state value of more than 0.250 00. The expectation values of $\langle r\rangle$, $\langle r^2\rangle$ decrease with an increase in the variation parameter (B), while $\langle p^2\rangle$ and $\Delta\ (p)$ increase with an increase in the variation parameter (B) for $\alpha = 0.1$, 0.12, and 0.13, respectively, as shown in tables 4, 5 and 6. The graph of position entropy in figure 2 for different values of α increases exponentially. However, the graph of momentum entropy as presented in figure 3 shows a clear distinction and quantisation of momentum theory for different values of α. Figure 4 is the graph of the squeeze state which is the product of position and momentum entropy with variation parameter (B) for different values of α. This graph gives a perfect curve with distinct quantisation for different values of α.

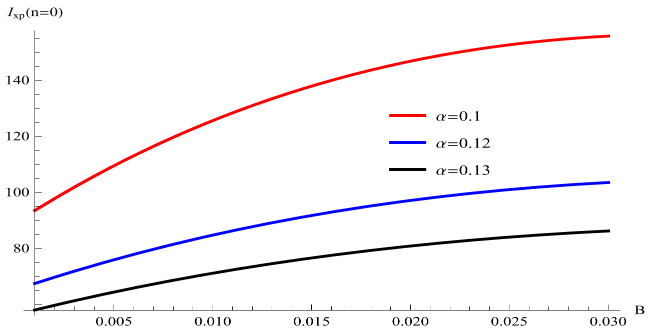

Figure 4. The product of position and momentum entropy for $n = 0$.

8. Conclusion

In this work, we develop a class of noncentral potential (noncentral inversely quadratic plus exponential Mie-type potential) to investigate the quantum information of Fisher and Shannon entropies in a Schrödinger equation. We obtained the normalised wave function and energy-eigen equations by solving the radial solution of the Schrödinger equation via the Nikiforov–Uvarov method. We analytically obtain position and momentum space for both Fisher and Shannon entropies and calculation were carried out with ground state wave function where the principal quantum number n and orbital quantum number l equals zero. The numerical computations were only carried out for Fisher information in position and momentum space, since that of Shannon was a great deal more complicated due to the nature of the potential. All computations were carried out for $\alpha = 0.1$, 0.12 and 0.13 because these are the only set of values that give the expected result. However, the wave function plots and probability density curves were plotted only for $\alpha = 0.1$ because this is the best value of α that gave the best plot. The normalisation constants were obtained using confluent hypergeometric functions with the help of Mathematica. The expectation values obtained with respect to position entropy decreases with an increase in variation parameter (B), whereas that of momentum increases with an increase in this parameter. All the numerical results obtained are in agreement with existing literature, including Heinsenberg uncertainties for position and momentum. The study of these entropies is very significant in terms of their potential applications in signal processing and examining the electronic structures of atoms.

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