

Block-coherence measures and coherence measures based on positive-operator-valued measures

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Received 12 October 2021, revised 13 December 2021

Accepted for publication 14 December 2021

Published 22 February 2022



Abstract

We study block-coherence measures based on the resource theory of block-coherence and coherence measures based on positive-operator-valued measures (POVM). Several block-coherence measures are presented, including the block-coherence measure based on maximum relative entropy, the one-shot block-coherence cost under maximally block-incoherent operations, and the coherence measure based on coherent rank. Their relationships are obtained. Moreover, we describe the deterministic coherence dilution process by constructing block-incoherent operations. Based on the POVM coherence resource theory, we also propose two coherence measures and analyze their relationship.

Keywords: coherence measure, block-coherence measure, positive-operator-valued measure

1. Introduction

Quantum coherence is an important ingredient in quantum information processing [1]. Baumgratz *et al* proposed the theoretical framework (BCP framework) of the resource theory of quantum coherence in 2014 [2]. This framework comprises three basic elements: a set of free states that do not contain resource, a corresponding set of free operations that map an arbitrary free state to a free state, and a metric functional [2].

In the resource theory of quantum coherence, the free states are incoherent states, which can be diagonalized under a fixed reference basis [2]. Free operations (incoherent operations) are some specified classes of physically realizable operations [2]. According to different operational capabilities and physical relevance, the sets of free operations may be: the maximally incoherent operation [3, 4], the dephasing-covariant incoherent operation [3, 5, 6], the incoherent operation [2], the strictly incoherent operation [7, 8], and the physically implementable incoherent operation [6]. In order to quantify coherence, many coherence measures are proposed in the resource theory of coherence, such as the l_1 -norm coherence measure [2], relative entropy coherence measure [2],

coherence of formation [7, 9], coherence concurrence [10], etc. Coherence measures of different meanings help us better quantify and understand coherence [2–14].

An interesting problem in the resource theory of quantum coherence is the transformation of states via free operations [14, 15], especially the transformation between an arbitrary state ρ and a maximally coherent state [15]. In particular, the process of converting a given state ρ to the maximally coherent state by an incoherent operation is referred to as coherence distillation [15–18]. In contrast to distillation, the dilution process converts the maximally coherent state into the desired target state [15, 19, 20]. The processes of asymptotic dilution and distillation are performed under the independent and identically distributed assumption [15, 17–19], which ignores the possible correlation between the different state preparations. Therefore, in order to relax the assumption, it is necessary to consider the one-shot scenario, where only one copy of the state is supplied [15, 17–19].

The resource theory of block-coherence was introduced in [4]. Here, we adopt the framework proposed in [21]. In the resource theory of block-coherence, the block-incoherent states can be considered to be generated by a von Neumann measurement $\mathcal{P} = \{P_i\}$, $i = 1, 2, \dots, d$, i.e., the block-

incoherent state $\sigma = \sum_{i=1}^d P_i \rho P_i$ for the state $\rho \in \mathcal{S}$, where \mathcal{S} denotes the set of quantum states on the Hilbert space \mathcal{H} , the rank of the orthogonal projector P_i is arbitrary, and the orthogonal projectors form a complete set, i.e., $\sum_{i=1}^d P_i = \mathbb{I}$ [4, 21–23].

In 2019, Bischof *et al* [21] established the positive-operator-valued measures (POVM) coherence resource theory. It employs the Naimark extension to define the POVM coherence via block-coherence in a larger Hilbert space, where the quantum states act through an embedded channel in the d' -dimensional ($d' > d$) Hilbert space \mathcal{H}' (Naimark space), and a POVM \mathbf{E} is extended to the projective measurement \mathbf{P} of the Naimark space \mathcal{H}' [21, 22, 24]. We will give detailed description of the resource theory of block-coherence and the POVM coherence resource theory in the second section of this paper.

In this paper, we study the block-coherence measures based on the resource theory of block-coherence and the coherence measures based on the POVM coherence resource theory, and then analyze the relationship between these block-coherence measures.

This paper is divided into five sections. In section 2, we introduce some main concepts, and review the resource theory of block-coherence and POVM coherence resource theory. In section 3, we propose two block-coherence measures and the one-shot block-coherence cost in the framework of the resource theory of block-coherence, and analyze their relationship. We illustrate the problem of deterministic coherence dilution by constructing a block-incoherent operation. In section 4, a POVM-based coherence measure and the one-shot block-coherence cost under the maximally POVM-incoherent operations are presented and analyzed.

2. Background

2.1. Block-coherence theoretical framework

In 2006, Åberg introduced the general measurement method of the degree of superposition of mixed quantum states and applied it to the orthogonal decomposition of Hilbert space, and thus created the resource theory of block-coherence. Similar to the theoretical framework of BCP, the resource theory of block-coherence also consists of three elements: the set of block-incoherent states, the set of block-incoherent operations, and the block-coherence measures [4, 21].

The Hilbert space \mathcal{H} is divided into d orthogonal subspaces, and the projective measurement $\mathbf{P} = \{P_i\}$ is performed on the set \mathcal{S} of quantum states, where P_i is the projector of the i th subspace. Block-incoherent states [4, 21–23] are defined as

$$\rho_{\text{BI}} = \sum_i P_i \rho P_i = \Delta[\rho], \quad \rho \in \mathcal{S}, \quad (1)$$

where Δ denotes the block-dephasing operation. The set of block-incoherent states is denoted as $\mathcal{I}_B(\mathcal{H})$.

We refer to the largest class of (free) operations that cannot produce block-coherence as maximally block-incoherent

(MBI) operations. A channel Λ_{MBI} on \mathcal{S} is free operations if it maps any block-incoherent state to a block-incoherent state [4, 21–23], namely,

$$\Lambda_{\text{MBI}}(\mathcal{I}_B(\mathcal{H})) \subseteq \mathcal{I}_B(\mathcal{H}), \quad (2)$$

or equivalently

$$\Lambda_{\text{MBI}} \circ \Delta = \Delta \circ \Lambda_{\text{MBI}} \circ \Delta. \quad (3)$$

A quantum channel Λ is often expressed by the Kraus operators. Let $\{K_n\}$ be a set of Kraus operators on \mathcal{H} , and satisfy the normalization condition $\sum_n K_n^\dagger K_n = \mathbb{I}$. Some Kraus operators have the form

$$K_n = \sum_i P_{f(i)} c_n P_i, \quad (4)$$

where f is the index function, and c_n is the complex matrix. K_n is a block-incoherent Kraus operator, if f is an index permutation.

A real-valued function $\mathcal{C}(\rho, \mathbf{P})$ is called the block-coherence monotone of quantum state ρ with respect to the projective measurement \mathbf{P} , if it satisfies [4, 21–23]:

(B1) Faithfulness: $\mathcal{C}(\rho, \mathbf{P}) \geq 0$ with equality if $\rho \in \mathcal{I}_B(\mathcal{H})$.

(B2) Monotonicity: $\mathcal{C}(\Lambda_{\text{BI}}(\rho), \mathbf{P}) \leq \mathcal{C}(\rho, \mathbf{P})$ for any block-incoherent operation Λ_{BI} .

(B3) Strong monotonicity: $\sum_n p_n \mathcal{C}(\rho_n, \mathbf{P}) \leq \mathcal{C}(\rho, \mathbf{P})$ for any block-incoherent operation $\Lambda_{\text{BI}} = \{K_n\}$ and states ρ , where $p_n = \text{Tr}(K_n \rho K_n^\dagger)$, $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$.

(B4) Convexity: $\mathcal{C}(\sum_n p_n \rho_n, \mathbf{P}) \leq \sum_n p_n \mathcal{C}(\rho_n, \mathbf{P})$ for all states ρ_n , and any probability distribution $\{p_n\}$.

Note that the rank of the above projector P_i is arbitrary, and when the rank of P_i is 1, the resource theory of block-coherence is consistent with the standard resource theory of coherence.

2.2. POVM coherence theoretical framework

The most general quantum measurement refers to the positive-operator-valued measures (POVM) [21]. Let a set $\mathbf{E} = \{E_i\}_{i=1}^n$ of positive-semi definite operators be a POVM on a d -dimensional Hilbert space \mathcal{H} , and $\sum_i E_i = \mathbb{I}$, where E_i is called the POVM element. Suppose $E_i = A_i^\dagger A_i$ for any i , where $\{A_i\}$ is a set of measurement operators for \mathbf{E} , and A_i can be written as $A_i = U_i \sqrt{E_i}$ with any unitary operator U_i .

The i th post-measurement state for a given A_i is $\rho_i = \frac{A_i \rho A_i^\dagger}{\text{Tr}[A_i \rho A_i^\dagger]}$ [21, 22, 24].

The POVM coherence resource theory is established via the Naimark extension [21, 25]. Every POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ on a d -dimensional Hilbert space \mathcal{H} can be extended to a projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ on the Hilbert space \mathcal{H}' , if one can embed the d -dimensional Hilbert space \mathcal{H} into a larger d' -dimensional Hilbert space \mathcal{H}' called the Naimark space, where $d' \geq d$. The general way to embed the original space \mathcal{H} into a larger space \mathcal{H}' is via a direct sum, namely, in the Naimark space \mathcal{H}' , and the corresponding state $\varepsilon(\rho)$ of quantum state ρ in the d -dimensional Hilbert space \mathcal{H} is

$$\varepsilon(\rho) = \rho \oplus 0, \quad (5)$$

and

$$\text{Tr}[E_i \rho] = \text{Tr}[P_i \varepsilon(\rho)] = \text{Tr}[P_i(\rho \oplus 0)], \quad (6)$$

is required for all states ρ in the set \mathcal{S} of quantum states. Here, \oplus denotes the orthogonal direct sum, and 0 is the zero matrix of dimension $d' - d$. Any projective measurement \mathbf{P} that satisfies equation (6) is called a Naimark extension of \mathbf{E} .

The embedding into a larger-dimensional Hilbert space can also be realized via the canonical Naimark extension [21, 25]: one attaches a probe, initially in the state $|1\rangle\langle 1|$, via the tensor product $\varepsilon(\rho) = \rho \otimes |1\rangle\langle 1|$ [21]. A canonical Naimark extension projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ of the POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ is described by a unitary matrix V satisfying [21, 22]

$$P_i := V^\dagger(\mathbb{I} \otimes |i\rangle\langle i|)V, \quad (7)$$

and

$$\text{Tr}[E_i \rho] = \text{Tr}[P_i(\rho \otimes |1\rangle\langle 1|)], \quad (8)$$

for every state ρ in the quantum state set \mathcal{S} .

A state ρ is called a POVM-incoherent state [21, 22, 24], if

$$E_i \rho E_j = 0, \quad \text{for all } i \neq j, \quad (9)$$

or equivalently

$$A_i \rho A_j^\dagger = 0, \quad \text{for all } i \neq j. \quad (10)$$

The set of POVM-incoherent states is denoted as \mathcal{I}_{PI} .

A channel Λ is called a POVM-incoherent (PI) operation with respect to the POVM $\mathbf{E} = \{E_i\}_{i=1}^n$, if it admits a Kraus decomposition $\Lambda(\rho) = \sum_l K_l \rho K_l^\dagger$ such that all operators K_l with respect to a canonical Naimark extension projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ of the POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ satisfies

$$K_l \rho K_l^\dagger \otimes |1\rangle\langle 1| = K_l'(\rho \otimes |1\rangle\langle 1|)(K_l')^\dagger, \quad (11)$$

for all $l \in \{1, 2, \dots, n\}$, where $\{K_l'\}$ is a set of the block-incoherent operations on the extended Hilbert space \mathcal{H}' [22].

The POVM-based coherence measure $\mathcal{C}(\rho, \mathbf{E})$ of a state ρ with respect to a POVM $\mathbf{E} = \{E_i\}_{i=1}^n$ is defined as the block-coherence measure $\mathcal{C}(\varepsilon(\rho), \mathbf{P})$ of $\varepsilon(\rho)$ with respect to the Naimark extension \mathbf{P} of \mathbf{E} [21, 22, 24], namely,

$$\mathcal{C}(\rho, \mathbf{E}) := \mathcal{C}(\varepsilon(\rho), \mathbf{P}), \quad (12)$$

where the function \mathcal{C} on the right side denotes any unitary-covariant block-coherence measure.

The POVM-based coherence measure $\mathcal{C}(\rho, \mathbf{E})$ with respect to a POVM measurement $\mathbf{E} = \{E_i\}_{i=1}^n$ should satisfy:

(P1) Faithfulness: $\mathcal{C}(\rho, \mathbf{E}) \geq 0$ with equality if $\rho \in \mathcal{I}_{\text{PI}}$.

(P2) Monotonicity: $\mathcal{C}(\Lambda_{\text{PI}}(\rho), \mathbf{E}) \leq \mathcal{C}(\rho, \mathbf{E})$ for any POVM-incoherent operation Λ_{PI} .

(P3) Strong monotonicity: $\sum_l p_l \mathcal{C}(\rho_l, \mathbf{E}) \leq \mathcal{C}(\rho, \mathbf{E})$ for any POVM-incoherent operation $\Lambda_{\text{PI}} = \{K_l\}$, where $p_l = \text{Tr}(K_l \rho K_l^\dagger)$, $\rho_l = \frac{K_l \rho K_l^\dagger}{p_l}$.

(P4) Convexity: $\mathcal{C}(\sum_i p_i \rho_i, \mathbf{E}) \leq \sum_i p_i \mathcal{C}(\rho_i, \mathbf{E})$ for all states ρ_i , and the probability distribution $\{p_i\}$.

2.3. Max-relative entropy and the coherent rank

In the theoretical framework of BCP, the max-relative entropy between quantum state $\rho \geq 0$ and quantum state $\sigma \geq 0$ is defined as [26, 27]

$$D_{\max}(\rho \parallel \sigma) = \log_2 \min \{ \lambda \mid \rho \leq \lambda \sigma \}. \quad (13)$$

One equivalent definition of $D_{\max}(\rho \parallel \sigma)$ [27] is

$$D_{\max}(\rho \parallel \sigma) := \log_2 \min \{ \lambda \mid \text{Tr}[P_+^\lambda(\rho - \lambda \sigma)] = 0 \}, \quad (14)$$

where P_+^λ is the projector of $\rho - \lambda \sigma$ with positive eigenvalues.

The coherent rank \mathcal{C}_r of a pure state $|\varphi\rangle = \sum_{i=1}^R \varphi_i |i\rangle$ (not necessarily normalized) is defined as the number of terms with $\varphi_i \neq 0$ [7, 28], i.e.,

$$\mathcal{C}_r(\varphi) = R, \quad (15)$$

if $\varphi_i \neq 0$ for $i = 1, 2, \dots, R$.

3. Block-coherence measures

Based on the max-relative entropy, we first define a block-coherence measure, which is a generalization of the coherence measure in [19].

Definition 1. The block-coherence measure $\mathcal{C}_{\max}(\rho, \mathbf{P})$ of a quantum state ρ with respect to the projective measurement \mathbf{P} is defined as

$$\mathcal{C}_{\max}(\rho, \mathbf{P}) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho \parallel \sigma). \quad (16)$$

Then, we have the following result.

Proposition 1. The block-coherence measure $\mathcal{C}_{\max}(\rho, \mathbf{P})$ is a block-coherence monotone under MBI operations and quasi-convex.

Proof. First, we show that $\mathcal{C}_{\max}(\rho, \mathbf{P}) \geq 0$ with the equality if and only if $\rho \in \mathcal{I}_B(\mathcal{H})$.

By the definition, we derive [26, 27]

$$\begin{aligned} \mathcal{C}_{\max}(\rho, \mathbf{P}) &= \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho \parallel \sigma) \\ &= \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} \log_2 \min \{ \lambda \mid \rho \leq \lambda \sigma \}. \end{aligned} \quad (17)$$

Since $\rho \leq \lambda \sigma$, we have $\text{Tr}(\lambda \sigma - \rho) \geq 0$. So $\lambda \geq 1$ holds. Hence,

$$\mathcal{C}_{\max}(\rho, \mathbf{P}) \geq 0. \quad (18)$$

From [26, 27], one has that $D_{\max}(\rho \parallel \sigma) = 0$ if and only if $\rho = \sigma$. Then, when

$$\mathcal{C}_{\max}(\rho, \mathbf{P}) = \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho \parallel \sigma) = 0, \quad (19)$$

ρ must be a block-incoherent state. This implies that $\mathcal{C}_{\max}(\rho, \mathbf{P})$ satisfies (B1).

Second, we can prove that for any MBI operation with $\{K_n\}$, $C_{\max}(\rho, \mathbf{P})$ satisfies (B2). According to [26], we know that the max-relative entropy $D_{\max}(\rho||\sigma)$ are monotonic under completely positive trace-preserving map (CPTP) Λ . Hence,

$$D_{\max}(\Lambda(\rho)||\Lambda(\sigma)) \leq D_{\max}(\rho||\sigma). \tag{20}$$

As any MBI operation Λ_{MBI} with $\{K_n\}$ is a CPTP map, we have

$$\min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\Lambda_{\text{MBI}}(\rho)||\Lambda_{\text{MBI}}(\sigma)) \leq \min_{\sigma \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\rho||\sigma). \tag{21}$$

Therefore,

$$C_{\max}(\sum_n K_n \rho K_n^\dagger, \mathbf{P}) \leq C_{\max}(\rho, \mathbf{P}). \tag{22}$$

It means that $C_{\max}(\rho, \mathbf{P})$ satisfies (B2).

Next we will show that $C_{\max}(\rho, \mathbf{P})$ is quasi-convex, i.e.,

$$C_{\max}(\sum_n p_n \rho_n, \mathbf{P}) \leq \max_n C_{\max}(\rho_n, \mathbf{P}), \tag{23}$$

where $p_n = \text{Tr}(K_n \rho K_n^\dagger)$, $\rho_n = \frac{K_n \rho K_n^\dagger}{p_n}$.

For any mixture of state $\rho = \sum p_n \rho_n$, we can construct a block-incoherent state $\sigma = \sum p_n \sigma_n$, where every σ_n is a block-incoherent state. Another equivalent definition of the max-relative entropy $D_{\max}(\rho||\sigma)$ [26] is

$$D_{\max}(\rho||\sigma) := \log_2 \min\{\lambda | \text{Tr}[P_+^\lambda(\rho - \lambda\sigma)] = 0\}, \tag{24}$$

where P_+^λ is the projector of $\rho - \lambda\sigma$ with positive eigenvalues. Note that

$$\begin{aligned} \text{Tr}[P(A - B)] &\leq \text{Tr}\{[A \geq B](A - B)\}, \\ \text{Tr}[P(A - B)] &\geq \text{Tr}\{[A \leq B](A - B)\}, \end{aligned} \tag{25}$$

in [27, 29, 30] for self-adjoint operators A, B and any positive-operator $0 \leq P \leq I$, then we get [27]

$$\begin{aligned} 0 &\leq \text{Tr}[P_+^\lambda(\rho - \lambda\sigma)] = \sum_n p_n \text{Tr}[P_+^\lambda(\rho_n - \lambda\sigma_n)] \\ &\leq \sum_n p_n \text{Tr}[P_+^{\lambda_n}(\rho_n - \lambda\sigma_n)], \end{aligned} \tag{26}$$

where $P_+^{\lambda_n}$ is the projector of $\rho_n - \lambda\sigma_n$ with positive eigenvalues. Let $\lambda = \max\{\lambda_n\}$, where for each n , λ_n is defined by $\log_2 \lambda_n = C_{\max}(\rho_n, \mathbf{P})$.

For this choice of λ , there is $\text{Tr}[P_+^\lambda(\rho - \lambda\sigma)] = 0$, and hence, $\log_2 \lambda \geq C_{\max}(\rho, \mathbf{P})$, i.e.,

$$C_{\max}(\sum_n p_n \rho_n, \mathbf{P}) \leq \max_n C_{\max}(\rho_n, \mathbf{P}). \tag{27}$$

So $C_{\max}(\rho, \mathbf{P})$ is the quasi-convex.

Suppose that \mathcal{H} is a d -dimensional Hilbert space and $\mathbf{P} = \{P_k\}$ is a projective measurement. A maximally block-coherent state is defined by [24]

$$|\psi_N\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^N |k\rangle, \tag{28}$$

where $|k\rangle$ is a state in the space $P_k \mathcal{H}$. Here, the rank of projective measurement P_k is arbitrary and the number of P_k in the projective measurement \mathbf{P} is $N (N \leq d)$.

Obviously, for an maximally block-coherent state $|\psi_N\rangle$, we have

$$C_{\max}(\psi_N, \mathbf{P}) = \log_2 N, \tag{29}$$

namely, the value of $C_{\max}(\psi_N, \mathbf{P})$ depends on the number N of projectors in the space.

The one-shot scenario is the most general conversion case, where the conversion is from an initial state to a final state. The one-shot block-coherence dilution process converts the MBI state $|\psi_N\rangle$ into the desired state ρ via the MBI operation [15, 19, 20].

Now we define a block-coherence measure as the one-shot block-coherence cost of quantifying block-coherence dilution.

Definition 2. Let MBI denote the set of the MBI operations. For a given state ρ and $\epsilon \geq 0$, the one-shot block-coherence cost under the MBI operations is defined as

$$C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = \min_{\Lambda \in \text{MBI}} \{\log_2 N | F[\Lambda(|\psi_N\rangle), \rho] \geq 1 - \epsilon\}, \tag{30}$$

where $F(\rho, \varsigma) = (\text{Tr}[\sqrt{\sqrt{\rho}\varsigma\sqrt{\rho}}])^2$ is the fidelity between two quantum states ρ and ς , and $|\psi_N\rangle$ is the maximally block-coherent state.

Since errors are allowed in one-shot scenarios, in the presence of the error ϵ , we use

$$C^\epsilon(\rho) := \min_{\rho': F(\rho, \rho') \geq 1 - \epsilon} C(\rho') \tag{31}$$

to characterize the coherence measure of state ρ [19]. That is, in order to define the coherence cost with a certain error ϵ , one can use a smoothing to the measure $C(\rho)$ by minimizing over states ρ' satisfying $F(\rho, \rho') \geq 1 - \epsilon$ to smooth the measure $C(\rho)$.

Next, we discuss the relationship between the coherence measure $C_{\max}^\epsilon(\rho, \mathbf{P})$ and the one-shot block-coherence cost $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$.

Theorem 1. For $\epsilon > 0$, the coherence measures satisfy

$$C_{\max}^\epsilon(\rho, \mathbf{P}) \leq C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) \leq C_{\max}^\epsilon(\rho, \mathbf{P}) + 1. \tag{32}$$

Proof. Now we prove the first inequality of equation (32). For quantum state ρ and the projective measurement $\mathbf{P} = \{P_i\}$, let $\log_2 N' = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$, where the rank of P_i is arbitrary. Since there is an operation Λ_{MBI} such that $F[\Lambda_{\text{MBI}}(|\psi_{N'}\rangle), \rho] \geq 1 - \epsilon$ by the definition of $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$, then

$$\begin{aligned} C_{\max}^\epsilon(\rho, \mathbf{P}) &\leq C_{\max}(\Lambda_{\text{MBI}}(\psi_{N'}), \mathbf{P}) \\ &= \min_{\delta \in \mathcal{I}_B(\mathcal{H})} D_{\max}(\Lambda_{\text{MBI}}(\psi_{N'})||\delta) \\ &\leq D_{\max}(\Lambda_{\text{MBI}}(\psi_{N'})||\Lambda_{\text{MBI}}(\sigma)) \\ &\leq D_{\max}(\psi_{N'}||\sigma) \\ &= \log_2 \min\{\lambda | \psi_{N'} \leq \lambda\sigma\}. \end{aligned} \tag{33}$$

Here, σ is a block-incoherent state and $\psi_{N'} = |\psi_{N'}\rangle\langle\psi_{N'}|$. We calculate the critical value of λ in the case of $\psi_{N'} = \lambda\sigma$. It is easy to get $\lambda = N'$. Hence,

$$C_{\max}^\epsilon(\rho, \mathbf{P}) \leq D_{\max}(\psi_{N'}||\sigma) = \log_2 N' = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}), \tag{34}$$

which means

$$C_{\max}^{\epsilon}(\rho, \mathbf{P}) \leq C_{\text{MBI}}^{\epsilon}(\rho, \mathbf{P}), \quad (35)$$

as required.

Next, we prove the second inequality of equation (32). Suppose that the state ρ' reaches the minimum, then

$$\begin{aligned} C_{\max}^{\epsilon}(\rho, \mathbf{P}) &= C_{\max}(\rho', \mathbf{P}) \\ &= D_{\max}(\rho' || \delta) \\ &= \log_2 \lambda. \end{aligned} \quad (36)$$

Let $N'' = \lceil \lambda \rceil$, then $\rho' \leq N'' \delta$. Consider the following mapping:

$$\begin{aligned} \Lambda(\omega) &= \frac{1}{N'' - 1} (N'' \text{Tr}[\psi_{N''} \circ \omega] - 1) \rho' \\ &\quad + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) \delta, \end{aligned} \quad (37)$$

where $\psi_{N''} \circ \omega = |\psi_{N''}\rangle\langle\psi_{N''}| \omega$, $\text{Tr}[\psi_{N''} \circ \omega] = \langle\psi_{N''} | \omega | \psi_{N''}\rangle$. For all $\delta = \sum_{i=1}^{N''} P_i \delta P_i \in \mathcal{I}_B(\mathcal{H})$, we have $\text{Tr}[\psi_{N''} \circ \delta] = \frac{1}{N''}$ and $\Lambda(\delta) = \delta \in \mathcal{I}_B(\mathcal{H})$. So $\Lambda \in \text{MBI}$. On the other hand, it is easy to obtain $\Lambda(\psi_{N''}) = \rho'$. One can also write the mapping as

$$\begin{aligned} \Lambda(\omega) &= \frac{N''}{N'' - 1} \text{Tr}[\psi_{N''} \circ \omega] \rho' - \frac{1}{N'' - 1} \rho' \\ &\quad + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) \delta \\ &= \frac{1}{N'' - 1} \text{Tr}[\psi_{N''} \circ \omega] \rho' - \frac{1}{N'' - 1} \rho' + \text{Tr}[\psi_{N''} \circ \omega] \rho' \\ &\quad + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) \delta \\ &= \frac{N''}{N'' - 1} (\text{Tr}[\psi_{N''} \circ \omega] - 1) \frac{1}{N''} \rho' \\ &\quad + \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) \delta + \text{Tr}[\psi_{N''} \circ \omega] \rho' \\ &= \frac{N''}{N'' - 1} (1 - \text{Tr}[\psi_{N''} \circ \omega]) (\delta - \frac{1}{N''} \rho') + \text{Tr}[\psi_{N''} \circ \omega] \rho'. \end{aligned} \quad (38)$$

Since $\delta \geq \frac{1}{N''} \rho'$, Λ is an entirely positive MBI operation, which maps $|\psi_{N''}\rangle$ to ρ' , then we have

$$\begin{aligned} C_{\text{MBI}}^{\epsilon}(\rho, \mathbf{P}) &= \log_2 N'' \leq \log_2(1 + \lambda) \\ &\leq \log_2 \lambda + 1 = C_{\max}^{\epsilon}(\rho, \mathbf{P}) + 1. \end{aligned} \quad (39)$$

Therefore, equation (32) holds.

Now, we discuss the deterministic block-coherence dilution between the maximally block-coherent state and the pure block-coherent state under the block-incoherent operation [20, 31–34].

Suppose that $|\phi\rangle = \sum_i \phi_i |i\rangle$ is a block-coherent pure state in the quantum state set \mathcal{S} , where $\{\phi_i\}_{i=1,2,\dots}$ are non-negative real numbers, and all complex phases have been eliminated by block-incoherent operations. The block-coherence dilution process is $|\psi_N\rangle \xrightarrow{\text{BI}} |\phi\rangle$, where $|\psi_N\rangle$ is an maximally block-coherent state, and the BI stands for block-incoherent operations. Then, the deterministic block-coherence dilution can be described as

$$\Lambda_{\text{BI}}(|\psi_N\rangle\langle\psi_N|) = \sum_n K_n |\psi_N\rangle\langle\psi_N| K_n^{\dagger} = |\phi\rangle\langle\phi|. \quad (40)$$

Here, Λ_{BI} is a block-incoherent operation composed by the Kraus operators $\{K_n\}$. We choose the block-incoherent Kraus operators [24]

$$K_n = \sum_i P_{\pi(i)} c_n^i P_i, \quad (41)$$

satisfying $\sum_n (K_n)^{\dagger} (K_n) = \mathbb{I}$, where $P_{\pi(i)}$ is the $\pi(i)$ th projective measurement, π is a permutation of the indices i , and c_n^i are the complex numbers. So equation (40) can be rewritten as

$$\begin{aligned} \Lambda_{\text{BI}}(|\psi_N\rangle\langle\psi_N|) &= \sum_{i,j,n} P_{\pi(i)} c_n^i P_i |\psi_N\rangle\langle\psi_N| P_j c_n^{j*} P_{\pi(j)} \\ &= \sum_{i,j} \phi_i \phi_j^* |i\rangle\langle j|. \end{aligned} \quad (42)$$

For the sake of simplicity here we only discuss the case of $N = d$. In this case, deterministic coherence dilution means that the coefficients between the initial state $|\psi_d\rangle$ (maximally block-coherent state) and the final state $|\phi\rangle$ satisfy the majorization relation [31], i.e.,

$$\begin{aligned} \mu(\psi_d) &= (|\psi_d|^2, |\psi_d|^2, \dots, |\psi_d|^2)^{\text{T}} \\ \prec \mu(\phi) &= (\phi_1^2, \phi_2^2, \dots, \phi_d^2)^{\text{T}}, \end{aligned} \quad (43)$$

where $|\psi_d|^2 = \frac{1}{d}$.

According to the protocol [32] of the deterministic transformations of the coherent states, in the case $\frac{1}{\sqrt{d}} \geq \phi_d$, $\frac{1}{\sqrt{d}} \leq \phi_i$, $i = 1, 2, \dots, d - 1$, the set of d permutations turns out to be

$$\begin{aligned} \{U^i |i = 1, 2, \dots, d\} \\ = \{I_d, |1\rangle \leftrightarrow |d\rangle, |2\rangle \leftrightarrow |d\rangle, \dots, |d-1\rangle \leftrightarrow |d\rangle\}. \end{aligned} \quad (44)$$

The probability distribution is [32]

$$p^1 = 1 - \sum_{i=2}^d p^i, \quad p^i = \frac{\phi_{i-1}^2 - \psi_{i-1}^2}{\phi_{i-1}^2 - \phi_d^2}. \quad (45)$$

The set of Kraus operators of the incoherent operation [32] is

$$\{K^i = U^i \sqrt{p^i} \sum_{j=1}^d c_{ij} \sqrt{d} |j\rangle\langle j|, i = 1, 2, \dots, d\}, \quad (46)$$

where c_{ij} is the (ij) -entry element of the $d \times d$ matrix c satisfying

$$U^i (c_{i1}, c_{i2}, \dots, c_{id})^{\text{T}} = (\phi_1, \phi_2, \dots, \phi_d)^{\text{T}}. \quad (47)$$

For example, when $d = 4$, $N = 4$, there are $P_1 = |1\rangle\langle 1|$, $P_2 = |2\rangle\langle 2|$, $P_3 = |3\rangle\langle 3|$, and $P_4 = |4\rangle\langle 4|$. The maximally block-coherent state is $|\psi_4\rangle = \frac{1}{2} \sum_{i=1}^4 |i\rangle$. We choose the dilution state $|\phi\rangle = \sum_i \phi_i |i\rangle = \sqrt{0.4} |1\rangle + \sqrt{0.3} |2\rangle + \sqrt{0.28} |3\rangle + \sqrt{0.02} |4\rangle$. So $\mu(\phi) = (0.4, 0.3, 0.28, 0.02)$ and $\mu(\psi_4) = (0.25, 0.25, 0.25, 0.25)$. It is easy to see

$$\begin{aligned} \mu(\psi_4) &= (0.25, 0.25, 0.25, 0.25) \\ \prec \mu(\phi) &= (0.4, 0.3, 0.28, 0.02). \end{aligned} \quad (48)$$

Obviously $|\phi\rangle$ and $|\psi_4\rangle$ satisfy $\frac{1}{\sqrt{d}} \geq \phi_d$, $\frac{1}{\sqrt{d}} \leq \phi_i$, $i = 1, 2, \dots, d - 1$, when $d = 4$. The set of permutations for this

case should be

$$\{|4\rangle, |1\rangle\} \leftrightarrow \{|4\rangle, |2\rangle\} \leftrightarrow \{|4\rangle, |3\rangle\} \leftrightarrow \{|4\rangle\}. \quad (49)$$

The matrix c corresponding to this set of permutations is

$$c = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_4 & \phi_2 & \phi_3 & \phi_1 \\ \phi_1 & \phi_4 & \phi_3 & \phi_2 \\ \phi_1 & \phi_2 & \phi_4 & \phi_3 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{10}} & \frac{\sqrt{7}}{5} & \frac{1}{5\sqrt{2}} \\ \frac{1}{5\sqrt{2}} & \sqrt{\frac{3}{10}} & \frac{\sqrt{7}}{5} & \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} & \frac{1}{5\sqrt{2}} & \frac{\sqrt{7}}{5} & \sqrt{\frac{3}{10}} \\ \sqrt{\frac{2}{5}} & \sqrt{\frac{3}{10}} & \frac{1}{5\sqrt{2}} & \frac{\sqrt{7}}{5} \end{pmatrix}. \quad (50)$$

Thus, the probability distribution is

$$\begin{aligned} p^2 &= \frac{\phi_1^2 - \psi_4^2}{\phi_1^2 - \phi_4^2} = \frac{15}{38}, \\ p^3 &= \frac{\phi_2^2 - \psi_4^2}{\phi_2^2 - \phi_4^2} = \frac{5}{28}, \\ p^4 &= \frac{\phi_3^2 - \psi_4^2}{\phi_3^2 - \phi_4^2} = \frac{3}{26}, \\ p^1 &= 1 - p^2 - p^3 - p^4 = \frac{2153}{6916}. \end{aligned} \quad (51)$$

Then, the Kraus operators are

$$\begin{aligned} K^1 &= U^1 \sqrt{p^1} (2c_{11}|1\rangle\langle 1| + 2c_{12}|2\rangle\langle 2| + 2c_{13}|3\rangle\langle 3| \\ &\quad + 2c_{14}|4\rangle\langle 4|), \\ K^2 &= U^2 \sqrt{p^2} (2c_{21}|1\rangle\langle 1| + 2c_{22}|2\rangle\langle 2| + 2c_{23}|3\rangle\langle 3| \\ &\quad + 2c_{24}|4\rangle\langle 4|), \\ K^3 &= U^3 \sqrt{p^3} (2c_{31}|1\rangle\langle 1| + 2c_{32}|2\rangle\langle 2| + 2c_{33}|3\rangle\langle 3| \\ &\quad + 2c_{34}|4\rangle\langle 4|), \\ K^4 &= U^4 \sqrt{p^4} (2c_{41}|1\rangle\langle 1| + 2c_{42}|2\rangle\langle 2| + 2c_{43}|3\rangle\langle 3| \\ &\quad + 2c_{44}|4\rangle\langle 4|), \end{aligned} \quad (52)$$

where $U^1 = I_4$ is the identity transformation, $U^2 = |1\rangle \leftrightarrow |4\rangle$, $U^3 = |2\rangle \leftrightarrow |4\rangle$, $U^4 = |3\rangle \leftrightarrow |4\rangle$. The Kraus operators can be expressed as

$$K^1 = U^1 \begin{pmatrix} \sqrt{\frac{4306}{8645}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{6459}{17290}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2153}}{5\sqrt{247}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{2153}}{5\sqrt{3458}} \end{pmatrix}, \quad (53)$$

$$K^2 = U^2 \begin{pmatrix} \sqrt{\frac{3}{95}} & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{19}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{42}{95}} & 0 \\ 0 & 0 & 0 & \frac{2\sqrt{3}}{\sqrt{19}} \end{pmatrix}, \quad (54)$$

$$K^3 = U^3 \begin{pmatrix} \sqrt{\frac{2}{7}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{70}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{14}} \end{pmatrix}, \quad (55)$$

$$K^4 = U^4 \begin{pmatrix} \frac{2\sqrt{3}}{\sqrt{65}} & 0 & 0 & 0 \\ 0 & \frac{3}{\sqrt{65}} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{5\sqrt{13}} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{42}}{5\sqrt{13}} \end{pmatrix}, \quad (56)$$

and satisfy $\sum_{i=1}^4 (K^i)^\dagger K^i = \mathbb{I}_4$, where \mathbb{I}_4 is an identity operation.

Next, we define another coherence measure based on coherent rank.

Definition 3. A block-coherence measure based on coherent rank is defined as

$$C_0(\rho, \mathbf{P}) = \min_{\{p_i, |\psi_i\rangle\}} \max_i \log_2 M(|\psi_i\rangle), \quad (57)$$

where \mathbf{P} is the projective measurement, the minimum is taken over all possible pure state decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$, and $M(|\psi_i\rangle)$ is the number of P_j satisfying $\langle \psi_i | P_j | \psi_i \rangle \neq 0$.

We have the following result.

Proposition 2. The quantity $C_0(\rho, \mathbf{P})$ is a coherence monotone under the block-incoherent operation.

Proof. Apparently $C_0(\rho, \mathbf{P}) \geq 0$. Next we prove that the quantity $C_0(\rho, \mathbf{P})$ satisfies $C_0(\rho, \mathbf{P}) = 0 \Leftrightarrow \rho \in \mathcal{I}_B(\mathcal{H})$.

Suppose $C_0(\rho, \mathbf{P}) = 0$ and the corresponding ensemble of ρ is $\{p_i, |\psi_i\rangle\}$, we can deduce for all i , $|\psi_i\rangle\langle \psi_i| = P_j |\psi_i\rangle\langle \psi_i| P_j$, which means that $\rho \in \mathcal{I}_B(\mathcal{H})$. Conversely, suppose $\rho = \sum_i P_i \delta P_i = \sum_{i,j=1}^N \delta_j P_i |\psi_j\rangle\langle \psi_j| P_i$, we can choose $\{\delta_j, P_i, |\psi_j\rangle\}$ as a decomposition, which leads to $C_0(\rho, \mathbf{P}) = 0$. Hence, $C_0(\rho, \mathbf{P})$ satisfies (B1).

Then, we prove that for any block-incoherent operation with $\{K_n\}$, there is

$$C_0(\sum_n K_n \rho K_n^\dagger, \mathbf{P}) \leq C_0(\rho, \mathbf{P}). \quad (58)$$

Before we prove the above inequality, let us introduce the following lemma proved in [7].

Lemma 1. *If $|\psi\rangle = \frac{K_i|\phi\rangle}{\sqrt{\text{Tr}[K_i|\phi\rangle\langle\phi|K_i^\dagger]}}$, where $\{K_i\}$ is a set of incoherent-preserving Kraus operators, then $C_0(|\psi\rangle\langle\psi|) \leq C_0(|\phi\rangle\langle\phi|)$.*

Clearly, lemma 1 also holds when $\{K_i\}$ is a block-incoherent operation.

Let $\{p_i, |\psi_i\rangle\}$ be the decomposition such that

$$C_0(\rho, \mathbf{P}) = \max_i \log_2 M(|\psi_i\rangle). \quad (59)$$

Let Λ_{BI} be any block-incoherent operation with $\Lambda_{\text{BI}}(\rho) = \sum_n K_n \rho K_n^\dagger$. For a given state $\rho = \sum_{i,p_i} p_i |\psi_i\rangle\langle\psi_i|$, then the post-measurement state of the n th outcome is

$$|\psi_i^n\rangle = \frac{K_n|\psi_i\rangle}{\sqrt{\text{Tr}[K_n|\psi_i\rangle\langle\psi_i|K_n^\dagger]}}. \quad (60)$$

Thus, we can get an ensemble $\{p(i|n), |\psi_i^n\rangle\}$, where the probability is

$$p(i|n) = p_i \frac{\text{Tr}[K_n|\psi_i\rangle\langle\psi_i|K_n^\dagger]}{\text{Tr}[K_n \rho K_n^\dagger]}, \quad (61)$$

and the corresponding density operator ρ_n of the n th outcome is

$$\rho_n = \frac{\sum_i p_i K_n |\psi_i\rangle\langle\psi_i| K_n^\dagger}{\text{Tr}[K_n \rho K_n^\dagger]}. \quad (62)$$

From lemma 1, we can know that for the minimum ensemble decomposition, there is $C_0(\rho_n, \mathbf{P}) \leq C_0(\rho, \mathbf{P})$, and then

$$C_0(\sum_n K_n \rho K_n^\dagger, \mathbf{P}) \leq C_0(\rho, \mathbf{P}).$$

This implies that $C_0(\rho, \mathbf{P})$ satisfies (B2).

Now we discuss the relationship between $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$ and $C_0^\epsilon(\rho, \mathbf{P})$.

Theorem 2. *For $\epsilon > 0$, the value of the one-shot block-coherence cost under MBI is equal to $C_0^\epsilon(\rho, \mathbf{P})$, namely,*

$$C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = C_0^\epsilon(\rho, \mathbf{P}). \quad (63)$$

Proof. We begin with the lower bound on $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$. Let $\log_2 N = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$, then there exists an operation Λ_{MBI} such that $F[\Lambda_{\text{MBI}}(\psi_N), \rho] \geq 1 - \epsilon$. Thus, we obtain

$$\begin{aligned} C_0^\epsilon(\rho, \mathbf{P}) &\leq C_0(\Lambda_{\text{MBI}}(\psi_N), \mathbf{P}) \\ &\leq C_0(\psi_N, \mathbf{P}) = \log_2 N = C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}). \end{aligned} \quad (64)$$

For the upper bound on $C_{\text{MBI}}^\epsilon(\rho, \mathbf{P})$, we select the state ρ' reaching the minimum such that $C_0^\epsilon(\rho, \mathbf{P}) = C_0(\rho', \mathbf{P})$. Let

$C_0(\rho', \mathbf{P}) = \log_2 N'$, as the MBI operation is constructed in the deterministic coherence dilution process discussed above, there is a Λ_{MBI} such that $F[\Lambda_{\text{MBI}}(\psi_{N'}), \rho] = F[\rho', \rho] \geq 1 - \epsilon$; thus,

$$\begin{aligned} C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) &\leq C_{\text{MBI}}(\rho', \mathbf{P}) = \log_2 N' \\ &= C_0(\rho', \mathbf{P}) = C_0^\epsilon(\rho, \mathbf{P}). \end{aligned} \quad (65)$$

Therefore, we get

$$C_{\text{MBI}}^\epsilon(\rho, \mathbf{P}) = C_0^\epsilon(\rho, \mathbf{P}), \quad (66)$$

as desired.

4. POVM-based coherence measures

For a POVM $\mathbf{E} = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ on a d -dimensional Hilbert space \mathcal{H} [21, 22, 24], a canonical Naimark extension projective measurement $\mathbf{P} = \{P_i\}_{i=1}^n$ of \mathbf{E} is described by a unitary matrix V on Naimark space \mathcal{H}' as

$$P_i = V^\dagger \bar{P}_i V, \quad (67)$$

where

$$V = \sum_{i,j=1}^n A_{ij} \otimes |i\rangle\langle j|, \quad (68)$$

with $\{A_{ij}\}_{i,j=1}^n$ satisfying the conditions [22]

$$\sum_{i=1}^n A_{ij}^\dagger A_{ik} = \delta_{jk} I_d, \quad \sum_{k=1}^n A_{ik}^\dagger A_{jk} = \delta_{ij} I_d, \quad A_{i1} = A_i, \quad (69)$$

and

$$\bar{\mathbf{P}} = \{\bar{P}_i = I_d \otimes |i\rangle\langle i|\}_{i=1}^n. \quad (70)$$

Let $\mathcal{C}(\rho', \bar{\mathbf{P}})$ be a unitary invariant block-coherence measure, that is,

$$\mathcal{C}(\rho', \bar{\mathbf{P}}) = \mathcal{C}(U\rho'U^\dagger, U\bar{\mathbf{P}}U^\dagger), \quad (71)$$

for any unitary transformation U on the Hilbert space [21]. The POVM-based coherence measure $\mathcal{C}(\rho, \mathbf{E})$ of ρ under POVM \mathbf{E} is defined [21] as

$$\mathcal{C}(\rho, \mathbf{E}) = \mathcal{C}(\varepsilon(\rho), \bar{\mathbf{P}}) = \mathcal{C}(\rho \otimes |1\rangle\langle 1|, \mathbf{P}), \quad (72)$$

where

$$\varepsilon(\rho) = \sum_{i,j=1}^n A_i \rho A_j^\dagger \otimes |i\rangle\langle j|, \quad (73)$$

is a state on the embedded state Hilbert space \mathcal{H}_ε .

From the conclusions in references [21, 22, 24], we know that the quantity $\mathcal{C}(\rho, \mathbf{E})$ is a POVM-based coherence measure satisfying the conditions (P1), ..., (P4).

Next, we discuss a concrete POVM-based coherence measure.

Proposition 3. *Let $\mathbf{E} = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ be a POVM on the Hilbert space \mathcal{H} , the quantity based on the max-relative entropy*

$$\begin{aligned} C_{\max}(\rho, \mathbf{E}) &= C_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}) \\ &= \min_{\sigma \in \mathcal{I}_B(\mathcal{H}_\varepsilon)} \log_2 \min\{\lambda | \varepsilon(\rho) \leq \lambda \sigma\} \end{aligned} \quad (74)$$

is a block-coherence monotone and quasi-convex. Here, $\mathcal{I}_B(\mathcal{H}_\varepsilon)$ is the set of block-incoherent states in the Hilbert space \mathcal{H}_ε .

Proof. We first prove that $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ is invariant under unitary transformation. Note that

$$\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}) = \min_{\sigma \in \mathcal{S}(\mathcal{H}_\varepsilon)} \log_2 \min \{ \lambda | \varepsilon(\rho) \leq \lambda \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i \}, \quad (75)$$

where σ is an arbitrary density operator on the state set $\mathcal{S}(\mathcal{H}_\varepsilon)$.

For any unitary transformation U on \mathcal{H}_ε , we derive

$$\begin{aligned} & \mathcal{C}_{\max}(U\varepsilon(\rho)U^\dagger, U\bar{\mathbf{P}}U^\dagger) \\ &= \min_{\sigma \in \mathcal{S}(\mathcal{H}_\varepsilon)} \log_2 \min \{ \lambda | U\varepsilon(\rho)U^\dagger \leq \lambda U \sum_{i=1}^n \bar{P}_i U^\dagger \sigma U \bar{P}_i U^\dagger \} \\ &= \min_{\sigma \in \mathcal{S}(\mathcal{H}_\varepsilon)} \log_2 \min \{ \lambda | \varepsilon(\rho) \leq \lambda \sum_{i=1}^n \bar{P}_i U^\dagger \sigma U \bar{P}_i \} \\ &= \min_{\sigma \in \mathcal{S}(\mathcal{H}_\varepsilon)} \log_2 \min \{ \lambda | \varepsilon(\rho) \leq \lambda \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i \} \\ &= \mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}). \end{aligned} \quad (76)$$

Then, we show that $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ is a block-coherence monotone.

Firstly, we prove that $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}) \geq 0$, with equality if and only if $\varepsilon(\rho) = \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i$, i.e., $\varepsilon(\rho)$ is the block-incoherent state on \mathcal{H}_ε .

By the definition, one has

$$\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}) = \min_{\sigma \in \mathcal{S}(\mathcal{H})} \log_2 \min \{ \lambda | \varepsilon(\rho) \leq \lambda \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i \}. \quad (77)$$

Since $\varepsilon(\rho) \leq \lambda \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i$, there is $\text{Tr}(\lambda \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i - \varepsilon(\rho)) \geq 0$. So $\lambda \geq 1$ holds. Hence,

$$\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}}) \geq 0. \quad (78)$$

According to the properties of maximum relative entropy, the equality holds if and only if $\varepsilon(\rho) = \sum_{i=1}^n \bar{P}_i \sigma \bar{P}_i$; thus, $\varepsilon(\rho)$ is a block-incoherent state on \mathcal{H}_ε . This implies that $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ satisfies (B1).

The monotonicity of $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ follows directly from the properties of the max-relative entropy. Hence, $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ satisfies (B2).

It is easy to show that $\mathcal{C}_{\max}(\varepsilon(\rho), \bar{\mathbf{P}})$ is also quasi-convex, i.e.,

$$\mathcal{C}_{\max}(\sum_i p_{\varepsilon_i} \varepsilon_i(\rho), \bar{\mathbf{P}}) \leq \max_i \mathcal{C}_{\max}(\varepsilon_i(\rho), \bar{\mathbf{P}}), \quad (79)$$

where $p_{\varepsilon_i} = \text{Tr}(K_i' \varepsilon(\rho) (K_i')^\dagger)$, $\varepsilon_i(\rho) = \frac{K_i' \varepsilon(\rho) (K_i')^\dagger}{p_{\varepsilon_i}}$, and $\{K_i'\}$ is the set of the Kraus operations.

Therefore, the quantity $\mathcal{C}_{\max}(\rho, \mathbf{E})$ is a block-coherence monotone and quasi-convex.

Now, we define the one-shot block-coherence cost under the maximally POVM-incoherent operations.

Definition 4. Let $\mathbf{E} = \{E_i = A_i^\dagger A_i\}_{i=1}^n$ be a POVM on the d -dimensional Hilbert space \mathcal{H} , and $\mathbf{P} = \{P_i = V^\dagger \mathbb{I} \otimes |i\rangle\langle i| V\}_{i=1}^n$ be a canonical Naimark extension of $\mathbf{E} = \{E_i\}_{i=1}^n$. We use \mathcal{O} to denote the set of the

maximally POVM-incoherent operations. For a state ρ and $\varepsilon \geq 0$, the one-shot block-coherence cost under \mathcal{O} is defined as

$$\mathcal{C}_{\mathcal{O}}^\varepsilon(\rho, \mathbf{E}) = \min_{\Lambda \in \mathcal{O}} \times \{ \log_2 N' | F[\Lambda_{\mathcal{O}}(\psi_{N'}), \rho \otimes |1\rangle\langle 1|] \geq 1 - \varepsilon \}, \quad (80)$$

where $F(\varrho, \varsigma) = (\text{Tr}[\sqrt{\sqrt{\varrho}\varsigma\sqrt{\varrho}}])^2$ is the fidelity between two quantum states ϱ and ς , and

$$|\psi_{N'}\rangle = \frac{1}{\sqrt{N'}} \sum_{i=1}^{N'} |i\rangle, \quad (81)$$

is an maximally block-coherent state in the extended Hilbert space \mathcal{H}' . Here, $|i\rangle$ is a state in the space $P_i \mathcal{H}'$, and $\Lambda_{\mathcal{O}}$ is a block-incoherent operation in the extended Hilbert space, corresponding to the POVM-incoherent operation Λ .

Similar to theorem 1, for the one-shot block-coherence cost under the maximally POVM-incoherent operations, the following is true.

Theorem 3. For quantum state ρ and $\varepsilon > 0$, we have

$$\mathcal{C}_{\max}^\varepsilon(\rho, \mathbf{E}) \leq \mathcal{C}_{\mathcal{O}}^\varepsilon(\rho, \mathbf{E}) \leq \mathcal{C}_{\max}^\varepsilon(\rho, \mathbf{E}) + 1. \quad (82)$$

5. Conclusion

In the resource theory of block-coherence, we have presented a block-coherence measure $\mathcal{C}_{\max}(\rho, \mathbf{P})$ based on maximum relative entropy, and shown that it is a coherence monotone and quasi-convex under the MBI operations. We have proposed the one-shot block-coherence cost under the MBI operations and found the relationship between the coherence measure $\mathcal{C}_{\max}(\rho, \mathbf{P})$ and the one-shot block-coherence cost. We have described the deterministic coherence dilution process by constructing block-incoherent operations based on the resource theory of block-coherence. We also introduced the coherence measure $\mathcal{C}_0(\rho, \mathbf{P})$ based on coherent rank, and obtained the relationship with the one-shot block-coherence cost. Based on the POVM coherence resource theory, we proposed a POVM-based coherence measure by using the known scheme of building POVM-based coherence measures from block-coherence measures, and the one-shot block-coherence cost under the maximally POVM-incoherent operations. The relationship between the POVM-based coherence measure and the one-shot block-coherence cost under the maximally POVM-incoherent operations has been analyzed.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 12071110, the Hebei Natural Science Foundation of China under Grant No. A2020205014, and the Science and Technology Project of

Hebei Education Department under Grant Nos. ZD2020167 and ZD2021066.

References

- [1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
- [2] Baumgratz T, Cramer M and Plenio M B 2014 Quantifying coherence *Phys. Rev. Lett.* **113** 140401
- [3] Chitambar E and Gour G 2016 Comparison of incoherent operations and measures of coherence *Phys. Rev. A* **94** 052336
- [4] Aberg J *Quantifying superposition* arXiv: quant-ph/0612146
- [5] Marvian I and Spekkens R W 2016 How to quantify coherence: Distinguishing speakable and unspeakable notions *Phys. Rev. A* **94** 052324
- [6] Chitambar E and Gour G 2016 Critical examination of incoherent operations and a physically consistent resource theory of quantum coherence *Phys. Rev. Lett.* **117** 030401
- [7] Winter A and Yang D 2016 Operational resource theory of coherence *Phys. Rev. Lett.* **116** 120404
- [8] Yadin B, Ma J, Girolami D, Gu M and Vedral V 2016 Quantum processes which do not use coherence *Phys. Rev. X* **6** 041028
- [9] Yuan X, Zhou H, Cao Z and Ma X 2015 Intrinsic randomness as a measure of quantum coherence *Phys. Rev. A* **92** 022124
- [10] Qi X F, Gao T and Yan F L 2017 Measuring coherence with entanglement concurrence *J. Phys. A: Math. Theor.* **50** 285301
- [11] de Vicente J I and Streltsov A 2017 Genuine quantum coherence *J. Phys. A: Math. Theor.* **50** 045301
- [12] Liu C, Ding Q and Tong D M 2018 Superadditivity of convex roof coherence measures *J. Phys. A: Math. Theor.* **51** 414012
- [13] Mandal S, Narozniak M, Radhakrishnan C, Jiao Z Q, Jin X M and Byrnes T 2020 Characterizing coherence with quantum observables *Phys. Rev. Research* **2** 013157
- [14] Zhang L M, Gao T and Yan F L 2021 Transformations of multilevel coherent states under coherence-preserving operations *Sci. China-Phys. Mech. Astron.* **64** 260313
- [15] Zhao Q, Liu Y, Yuan X, Chitambar E and Winter A 2019 One-shot coherence distillation: Towards completing the picture *IEEE Trans. Inf. Theory* **65** 6441
- [16] Liu C L and Zhou D L 2019 Deterministic coherence distillation *Phys. Rev. Lett.* **123** 070402
- [17] Regula B, Fang K, Wang X and Adesso G 2018 One-shot coherence distillation *Phys. Rev. Lett.* **121** 010401
- [18] Chen S, Zhang X, Zhou Y and Zhao Q 2019 One-shot coherence distillation with catalysts *Phys. Rev. A* **100** 042323
- [19] Zhao Q, Liu Y, Yuan X, Chitambar E and Ma X 2018 One-shot coherence dilution *Phys. Rev. Lett.* **120** 070403
- [20] Lian Y F, Luo Y and Li Y M 2020 Protocol of deterministic coherence distillation and dilution of pure states *Laser Phys. Lett.* **17** 085201
- [21] Bischof F, Kampermann H and Bruß D 2019 Resource theory of coherence based on positive-operator-valued measures *Phys. Rev. Lett.* **123** 110402
- [22] Xu J W, Shao L H and Fei S M 2020 Coherence measures with respect to general quantum measurements *Phys. Rev. A* **102** 012411
- [23] Theurer T, Killoran N, Egloff D and Plenio M B 2017 Resource theory of superposition *Phys. Rev. Lett.* **119** 230401
- [24] Bischof F, Kampermann H and Bruß D 2021 Quantifying coherence with respect to general quantum measurements *Phys. Rev. A* **103** 032429
- [25] Decker T, Janzing D and Rötteler M 2005 Implementation of group-covariant positive operator valued measures by orthogonal measurements *J. Math. Phys.* **46** 012104
- [26] Müller-Lennert M, Dupuis F, Szehr O, Fehr S and Tomamichel M 2013 On quantum Rényi entropies: A new generalization and some properties *J. Math. Phys.* **54** 122203
- [27] Datta N 2009 Min- and max-relative entropies and a new entanglement monotone *IEEE Trans. Inf. Theory* **55** 2816
- [28] Killoran N, Steinhoff F E S and Plenio M B 2016 Converting nonclassicality into entanglement *Phys. Rev. Lett.* **116** 080402
- [29] Bowen G and Datta N 2006 Beyond i.i.d. in quantum information theory *IEEE Int. Sym. Inf. Theory* **2006** 451–5
- [30] Nagaoka H and Hayashi M 2007 An information-spectrum approach to classical and quantum hypothesis testing for simple hypotheses *IEEE Trans. Inf. Theory* **53** 534
- [31] Du S, Bai Z and Guo Y 2015 Conditions for coherence transformations under incoherent operations *Phys. Rev. A* **91** 052120
- [32] Torun G and Yildiz A 2018 Deterministic transformations of coherent states under incoherent operations *Phys. Rev. A* **97** 052331
- [33] Bhatia R 1997 *Matrix Analysis* (New York: Springer)
- [34] Zhu H, Ma Z, Cao Z, Fei S M and Vedral V 2017 Operational one-to-one mapping between coherence and entanglement measures *Phys. Rev. A* **96** 032316