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Rogue waves of the sixth-order nonlinear Schrödinger equation on a periodic background

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Abstract

In this paper, we construct the rogue wave solutions of the sixth-order nonlinear Schrödinger equation on a background of Jacobian elliptic functions dn and cn by means of the nonlinearization of a spectral problem and Darboux transformation approach. The solutions we find present the dynamic phenomena of higher-order nonlinear wave equations.

Keywords: rogue wave on a periodic background, sixth-order nonlinear Schrödinger equation, Darboux transformation, Jacobian elliptic function

(Some figures may appear in colour only in the online journal)

1. Introduction

The nonlinear Schrödinger (NLS) equation plays a significant role in the field of nonlinear physics. It is a partial differential equation used to describe nonlinear waves and has a strict connection with many nonlinear physics problems, such as nonlinear optics, ion acoustic waves in plasma, and so on.

A 'rogue wave' means a strange wave with extremely large amplitude. It usually occurs in the ocean and comes out of nowhere and disappears without a trace [1], which may lead to a fatal catastrophe. However, at present, people still do not have an effective means of forecasting rogue waves accurately in advance. As a result, the study of rogue waves is necessary and relevant. Recently, some meaningful and important related papers have been published, such as [2, 3].

In 1983, Peregrine [4] found the analytic expression for a rational solution of the NLS equation localized both in time and in space that is now known as the Peregrine breather. As was shown in 2009, this solution represented the first-order rogue wave of the NLS equation. Rogue waves have been studied in many fields, such as the ocean, optical fibers and Bose–Einstein condensates. Recently, Chen *et al* [5] studied rogue waves on a periodic background in 2018. Such rogue waves are a kind of wave formed on the periodic background of the Jacobian elliptic functions dn and cn. By means of the nonlinearization of a spectral problem [6] and the Darboux transformation approach [7–12], periodic standing waves of various equations have been investigated, such as the mKdV equation [13], the NLS equation [5, 14–16], the fifth-order Ito equation [17], the sine-Gordon equation [18] and the Hirota equation [19].

Recently, studies related to higher-order rogue wave solutions of the NLS equation have been developing rapidly. For example, Zhang *et al* [20] constructed rogue wave solutions on the periodic background for the fourth-order NLS equation. Yue *et al* [21] investigated modulation instability, rogue waves and spectral analysis for the sixth-order NLS equation. In addition, many researchers have also considered the higher-order Schrödinger equations and their applications. For instance, Brocchi *et al* [22] investigated a class of sharp Fourier extension inequalities for fractional and higher-order Schrödinger equations. Duan *et al* [23] studied the unique continuation properties of the higher-order NLS equations and proposed exponential decay weighted estimates as well as an L^p-type Carleman estimate based on the Littlewood–Paley theory.

In the past few years, there have been abundant studies with profound significance and advances in the higher-order NLS hierarchy. In 2015, Kedziora *et al* [24] presented an infinite NLS equation hierarchy of integrable equations. In 2016, Ankiewicz *et al* [25] studied the infinite integrable NLS equation hierarchy and put forward the generalized Lax pair and various solutions.

The infinite integrable NLS equation takes the following form:

$$\begin{aligned}&\mathrm{i}q_t + \delta_2\Gamma_2(q) - \mathrm{i}\delta_3\Gamma_3(q) + \delta_4\Gamma_4(q)\\&-\mathrm{i}\delta_5\Gamma_5(q) + \delta_6\Gamma_6(q) - \mathrm{i}\delta_7\Gamma_7(q)+\cdots = 0,\end{aligned} \tag{1}$$

with

$$\begin{aligned}&\Gamma_2(q) = q_{xx} + 2q|q|^2,\\&\Gamma_3(q) = q_{xxx} + 6|q|^2q_x,\\&\Gamma_4(q) = q_{xxxx} + 6q^*q_x^2 + 4q|q_x|^2\\&+8|q|^2q_{xx} + 2q^2q_{xx}^* + 6|q|^4q,\\&\Gamma_5(q) = q_{xxxxx} + 10|q|^2q_{xxx} + 30|q|^4q_x + 10qq_xq_{xx}^*\\&+10qq_x^*q_{xx} + 20q^*q_xq_{xx} + 10q_x^2q_x^*,\\&\Gamma_6(q) = q_{xxxxxx} + q^2[60|q_x|^2q^* + 50q_{xx}(q^*)^2\\&+2q_{xxxx}^*] + q[12q^*q_{xxxx} + 18q_x^*q_{xxx}\\&+8q_xq_{xxx}^* + 70(q^*)^2q_x^2 + 22|q_{xx}|^2]\\&+10q_x[3q^*q_{xxx} + 5q_x^*q_{xx} + 2q_xq_{xx}^*]\\&+10q^3[2q^*q_{xx}^* + (q_x^*)^2] + 20q^*q_{xx}^2 + 20q|q|^6,\\&\Gamma_7(q) = q_{xxxxxxx} + 70q_{xx}^2q_x^* + 112q_x|q_{xx}|^2\\&+98|q_x|^2q_{xxx} + 28q_x^2q_{xxx}^* + 70q^2\{q_x[2q^*q_{xx}^*\\&+(q_x^*)^2] + q^*(2q_{xx}q_x^* + q_{xxx}q^*)\}\\&+14q[q^*(20|q_x|^2q_x + q_{xxxxx}) + 3q_{xxx}q_{xx}^*\\&+2q_{xx}q_{xxx}^* + 2q_{xxxx}q_x^* + q_x + q_{xxxx}^*\\&+20q_xq_{xx}(q^*)^2] + 140|q|^6q_x + 70q_x^3(q^*)^2\\&+14q^*(5q_{xx}q_{xxx} + 3q_xq_{xxxx}),\end{aligned} \tag{2}$$

where $q = q(x, t)$, x is the spatial coordinate variable, t is the scaled time coordinate variable, the function $|q| = |q(x, t)|$ denotes the envelope of the wave, the asterisk * denotes the complex conjugate, $\delta_i (i = 2, 3, 4, \cdots, \infty)$ represents the i-order real coefficients, $\Gamma_3(q)$ is the Hirota operator [26], $\Gamma_4(q)$ is the fourth-order Lakshmanan–Porsezian–Daniel (LPD) operator [27], $\Gamma_5(q)$ is the quintic operator [28], and $\Gamma_6(q)$ and $\Gamma_7(q)$ are the sextic and heptic operators, respectively. As for the parameter δ_2, many papers [29, 30] have chosen $\delta_2 = \frac{1}{2}$ for convenience. As a result, here we also take $\delta_2 = \frac{1}{2}$.

Inspired by equation (1), we consider the following sixth-order NLS equation.

$$\begin{aligned}&-q_t^* + \frac{1}{2}\Gamma'_2(q) - \mathrm{i}\alpha\Gamma'_3(q) + \beta\Gamma'_4(q)\\&-\mathrm{i}\gamma\Gamma'_5(q) + \delta\Gamma'_6(q) = 0,\end{aligned} \tag{3}$$

where

$$\begin{aligned}&\Gamma'_2(q) = \mathrm{i}q_{xx}^* + 2\mathrm{i}q^*|q|^2,\\&\Gamma'_3(q) = \mathrm{i}q_{xxx}^* + 6\mathrm{i}|q|^2q_x^*,\\&\Gamma'_4(q) = \mathrm{i}q_{xxxx}^* + 6\mathrm{i}q(q_x^*)^2 + 4\mathrm{i}q^*|q_x|^2\\&+8\mathrm{i}|q|^2q_{xx}^* + 2\mathrm{i}(q^*)^2q_{xx} + 6\mathrm{i}|q|^4q^*,\\&\Gamma'_5(q) = \mathrm{i}q_{xxxxx}^* + 10\mathrm{i}|q|^2q_{xxx}^* + 30\mathrm{i}|q|^4q_x^* + 10\mathrm{i}q^*q_x^*q_{xx}\\&+10\mathrm{i}q^*q_xq_{xx}^* + 20\mathrm{i}qq_x^*q_{xx}^* + 10\mathrm{i}(q^*)_x^2q_x,\\&\Gamma'_6(q) = \mathrm{i}q_{xxxxxx}^* + \mathrm{i}(q^*)^2[60|q_x|^2q + 50q^2q_{xx}^*\\&+2q_{xxxx}] + q^*[12\mathrm{i}qq_{xxxx}^* + 18\mathrm{i}q_xq_{xxx}^*\\&+8\mathrm{i}q_x^*q_{xxx} + 70\mathrm{i}q^2(q_x^*)^2 + 22|q_{xx}|^2]\\&+10\mathrm{i}q_x^*[3qq_{xxx}^* + 5q_xq_{xx}^* + 2q_x^*q_{xx}]\\&+10\mathrm{i}(q^*)^3[2qq_{xx} + (q_x)^2]\\&+20\mathrm{i}q(q^*)_{xx}^2 + 20\mathrm{i}q^*|q|^6.\end{aligned}$$

Equation (3) has the Lax pair

$$\begin{aligned}&\Phi_x = U\Phi,\ U = \begin{pmatrix}\lambda & q\\ -q^* & -\lambda\end{pmatrix},\\&\Phi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix},\end{aligned} \tag{5}$$

$$\Phi_t = V\Phi,\ V = \sum_{c=0}^{6}\lambda^c\begin{pmatrix}A_c & B_c^*\\ B_c & -A_c\end{pmatrix}, \tag{6}$$

with

$$\begin{aligned}&A_0 = -\frac{1}{2}|q|^2 - 3\beta|q|^4 - 10\delta|q|^6 + \beta|q_x|^2\\&-\delta|q_{xx}|^2 - 5\delta(q^2(q_x^*)^2 + (q^*)^2q_x^2)\\&-(10\delta|q|^2 + \beta)(qq_{xx}^* + q^*q_{xx}) + \delta(q_xq_{xxx}^*\\&+q_x^*q_{xxx} - qq_{xxxx}^* - q^*q_{xxxx})\\&-(\mathrm{i}\alpha + 6\mathrm{i}\gamma|q|^2)(q^*q_x - q_x^*q)\\&-\mathrm{i}\gamma(q_xq_{xx}^* - q_x^*q_{xx} + q^*q_{xxx} - qq_{xxx}^*),\\&A_1 = 2\alpha|q|^2 + 6\gamma|q|^4 - 2\gamma|q_x|^2\\&+2\gamma(qq_{xx}^* + q^*q_{xx}) - (2\mathrm{i}\beta + 12\mathrm{i}\delta|q|^2)\\&\times(q^*q_x - qq_x^*) - 2\mathrm{i}\delta(q_xq_{xx}^*\\&-q_x^*q_{xx} + q^*q_{xxx} - qq_{xxx}^*),\\&A_2 = 1 + 4\beta|q|^2 + 12\delta|q|^4 + 4\delta(qq_{xx}^*\\&+q^*q_{xx} - |q_x|^2) - 4\mathrm{i}\gamma(qq_x^* - q^*q_x),\\&A_3 = -4\alpha - 8\gamma|q|^2 - 8\mathrm{i}\delta(qq_x^* - q^*q_x),\\&A_4 = -8\beta - 16\delta|q|^2,\\&A_5 = 16\gamma,\quad A_6 = 32\delta,\\&B_0 = 2\mathrm{i}\alpha q^*|q|^2 + 6\mathrm{i}\gamma q^*|q|^4 + 4\mathrm{i}\gamma q^*|q_x|^2\\&+6\mathrm{i}\gamma q(q_x^*)^2 + 2\mathrm{i}\gamma(q^*)^2q_{xx} + \mathrm{i}\alpha q_{xx}^*\\&+8\mathrm{i}\gamma|q|^2q_{xx}^* + \mathrm{i}\gamma q_{xxxx}^* - \frac{1}{2}q_x^*\\&-\beta q_{xxx}^* - \delta q_{xxxxx}^* - 6\beta|q|^2q_x^*\\&-10\delta(q^*q_xq_{xx}^* + q^*q_x^*q_{xx} + |q|^2q_{xxx}^* + |q_x|^2q_x^*)\\&-20\delta qq_x^*q_{xx}^* - 30\delta|q|^4q_x^*,\\&B_1 = \mathrm{i}q^* + 4\mathrm{i}\beta q^*|q|^2 + 12\mathrm{i}\delta q^*|q|^4 + 8\mathrm{i}\delta q^*|q_x|^2\\&+12\mathrm{i}\delta q(q_x^*)^2 + 4\mathrm{i}\delta(q^*)^2q_{xx}\\&+2\mathrm{i}\beta q_{xx}^* + 16\mathrm{i}\delta|q|^2q_{xx}^* + 2\mathrm{i}\delta q_{xxxx}^*\\&+2\alpha q_x^* + 12\gamma|q|^2q_x^* + 2\gamma q_{xxx}^*,\\&B_2 = -4\mathrm{i}\alpha q^* - 8\mathrm{i}\gamma q^*|q|^2 - 4\mathrm{i}\alpha q_{xx}^*\\&+4\beta q_x^* + 24\delta|q|^2q_x^* + 4\delta q_{xxx}^*,\\&B_3 = -8\mathrm{i}\beta q^* - 16\mathrm{i}\delta q^*|q|^2 - 8\mathrm{i}\delta q_{xx}^*\\&-8\gamma q_x^*,\quad B_4 = 16\mathrm{i}\gamma q^* - 16\delta q_x^*,\\&B_5 = 32\mathrm{i}\delta q^*,\quad B_6 = 0.\end{aligned} \tag{7}$$

The compatibility condition $U_t - V_x + [U, V] = 0$ of Lax pair (5)–(6) gives rise to equation (3).

To our knowledge, rogue wave solutions of equation (3) on a periodic background have not been constructed. Hence, in the rest of the paper, our purpose is to construct the rogue wave solutions of equation (3) on a periodic background through the nonlinearization of a spectral problem and the Darboux transformation approach. The structure of this paper is given as follows. In section 2, we deduce two families of periodic solutions called dn and cn for equation (3). In section 3, we nonlinearize the Lax pair of equation (3). In section 4, we obtain the periodic and non-periodic wave solutions of equation (3). In section 5, we use Darboux transformation to construct the rogue wave solutions of equation (3) on the dn-periodic background and cn-periodic background. In section 6, some conclusions are given.

2. Two families of periodic solutions of equation (3)

To obtain the periodic wave solutions of equation (3), we take

$$q(x, t) = Q(x)e^{ict}, \tag{8}$$

where $Q(x)$ is the real periodic function to be determined and c is a real constant representing the speed of waves. In addition, we can easily find that $|q|^2 = qq^* = Q^2$.

Substituting (8) into equation (3), we obtain a sixth-order nonlinear ordinary differential equation, for which it is difficult to find the exact solutions. However, by means of the Jacobi elliptic function expansion method, we can turn the sixth-order nonlinear ordinary differential equation into a first-order nonlinear ordinary differential equation to obtain two families of periodic solutions of equation (3) expressed by the Jacobi elliptic functions dn and cn as

$$\begin{aligned}&Q(x) = \mathrm{dn}(x; k),\quad c = 1 - \frac{1}{2}k^2 + 6\beta - 6k^2\beta\\&+k^4\beta + 20\delta - 30k^2\delta + 12k^4\delta - k^6\delta,\\&(2 - k^2)\alpha + (6 - 6k^2 + k^4)\gamma = 0,\end{aligned} \tag{9}$$

and

$$\begin{aligned}&Q(x) = k\,\mathrm{cn}(x; k),\quad c = -\frac{1}{2} + k^2 + \beta - 6k^2\beta\\&+6k^4\beta - \delta + 12k^2\delta - 30k^4\delta + 20k^6\delta,\\&-\alpha + 2k^2\alpha + \gamma - 6k^2\gamma + 6k^4\gamma = 0,\end{aligned} \tag{10}$$

where $k \in (0, 1)$ is elliptic modulus and (9)–(10) satisfy the following two elliptic equations:

$$\begin{aligned}&Q_{xx} = -2Q^3 + a_0Q,\\&Q_x^2 = -Q^4 + a_0Q^2 + a_1,\end{aligned} \tag{11}$$

where a_0 and a_1 are two real constants.

In the case of the dn-function solution, we have $a_0 = 2 - k^2$ and $a_1 = k^2 - 1$. In the case of the cn-function solution, we have $a_0 = 2k^2 - 1$ and $a_1 = k^2(1 - k^2)$ on the other side.

3. Nonlinearization of the Lax pair

We introduce the following Bargmann constraint [31, 32]:

$$q(x, t) = \varphi_1^2 + \varphi_2^{2*}, \tag{12}$$

where $\Phi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ is a non-zero solution of the Lax pair (9)–(10) with $\lambda = \lambda_1$.

Substituting (12) into (5) yields a finite-dimensional Hamiltonian system

$$\frac{d\varphi_1}{dx} = \frac{\partial H}{\partial \varphi_2},\quad \frac{d\varphi_2}{dx} = -\frac{\partial H}{\partial \varphi_1}, \tag{13}$$

where

$$\begin{aligned}H = {}&\lambda_1\varphi_1\varphi_2 + \lambda_1^*\varphi_1^*\varphi_2^*\\&+\frac{1}{2}(\varphi_1^2 + \varphi_2^{2*})(\varphi_1^{2*} + \varphi_2^2).\end{aligned} \tag{14}$$

Taking two conserved integrals of (13) as

$$G_0 = i(\varphi_1\varphi_2 - \varphi_1^*\varphi_2^*), \tag{15}$$

$$G_1 = \lambda_1\varphi_1\varphi_2 + \lambda_1^*\varphi_1^*\varphi_2^* + \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2)^2, \tag{16}$$

where G_0 and G_1 are constants of variable x and there exists a relationship between H, G_0 and G_1 as $H = G_1 - \frac{1}{2}G_0^2$.

Equations (12) and (15) yield

$$\lambda_1\varphi_1^2 - \lambda_1^*\varphi_1^{2*} = \frac{1}{2}q_x + iqG_0. \tag{17}$$

Some other constraints with $\lambda_1 = \rho + i\mu$ can be referred from [5] as

$$\begin{aligned}&q_xq^* - qq_x^* = 2i(2\mu - G_0)|q|^2\\&+2iG_0(G_0^2 + 2G_0\mu - 2G_1),\end{aligned} \tag{18}$$

$$\begin{aligned}&q_{xx} + 2|q|^2q - 4(\rho^2 + \mu^2 + G_1\\&-\frac{1}{2}G_0^2 - 2\mu G_0)q = 2i(2\mu - G_0)q_x,\end{aligned} \tag{19}$$

$$\begin{aligned}|q_x|^2 = {}&-|q|^4 + 4(\rho^2 + \mu^2 + G_1\\&-\frac{1}{2}G_0^2 - 2\mu G_0)|q|^2 + 4\rho^2G_0^2\\&-(G_0^2 + 2\mu G_0 - 2G_1)(5G_0^2 - 2\mu G_0 - 2G_1),\end{aligned}$$

where ρ and μ are the real and imaginary parts of λ_1 respectively.

Substituting (8) into (18), it is easy to notice that the left-hand side of (18) is zero, which yields

$$G_0 = 2\mu,\quad \mu(G_1 - 4\mu^2) = 0. \tag{21}$$

Substituting (8) into (19)–(20) and comparing them with the two equations in (11), we have

$$\begin{aligned}&a_0 = 4(\rho^2 - 5\mu^2 + G_1),\\&a_1 = -4(G_1^2 - 12G_1\mu^2 + 32\mu^4 - 4\mu^2\rho^2).\end{aligned} \tag{22}$$

Considering the second equation in (21), there exist two cases, either $\mu = 0$ or $\mu \neq 0$ and $G_1 = 4\mu^2$.

Case 1. For $\mu = 0$, relations (21) yield $G_0 = 0$. Then, relations (22) yield

$$a_0 = 4(\rho^2 + G_1), \quad a_1 = -4G_1^2. \tag{23}$$

As $a_1 = -4G_1^2 < 0$, we choose the dn-function solution in (9) and have

$$G_1 = \pm\frac{1}{2}\sqrt{1 - k^2}, \quad \lambda_1^2 = \frac{1}{4}(2 - k^2 \pm 2\sqrt{1 - k^2}). \tag{24}$$

Then, we obtain the expression for eigenvalue λ_1 with two real eigenvalues in the right half-plane:

$$\lambda_1 = \rho = \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 - k^2}), \tag{25}$$

and other two $-\lambda_\pm$ in the left half-plane.

Case 2. For $\mu \neq 0$, relations (21) yield $G_0 = 2\mu$ and $G_1 = 4\mu^2$. Then, relations (22) yield

$$a_0 = 4(\rho^2 - \mu^2), \quad a_1 = 16\mu^2\rho^2. \tag{26}$$

As $a_1 = 16\mu^2\rho^2 > 0$, we choose the cn-function solution in (10) and have

$$\lambda_1^2 = \frac{1}{4}(2k^2 - 1 \pm 2\mathrm{i}k\sqrt{1 - k^2}). \tag{27}$$

There also exist two pairs of eigenvalues with $\lambda_1 = +\lambda_\pm$ and $\lambda_1 = -\lambda_\pm$, where

$$\lambda_\pm = \frac{1}{2}(k \pm \mathrm{i}\sqrt{1 - k^2}). \tag{28}$$

4. Periodic and non-periodic solutions of the Lax pair

According to (12), (17) and (21), we have

$$\varphi_1^2 = \frac{2\lambda_1 q + q_x}{2(\lambda_1 + \lambda_1^*)}, \quad \varphi_2^{2*} = \frac{2\lambda_1^* q - q_x}{2(\lambda_1 + \lambda_1^*)}. \tag{29}$$

Since $q(x, t) = Q(x)\mathrm{e}^{\mathrm{i}ct}$, we take

$$\varphi_1(x, t) = \Phi_1(x)\mathrm{e}^{\frac{1}{2}\mathrm{i}ct}, \quad \varphi_2(x, t) = \Phi_2(x)\mathrm{e}^{-\frac{1}{2}\mathrm{i}ct}. \tag{30}$$

Substituting (8) and (30) into (29) yields

$$\Phi_1^2(x) = \frac{2\lambda_1 Q + Q_x}{2(\lambda_1 + \lambda_1^*)}, \quad \Phi_2^{2*}(x) = \frac{2\lambda_1^* Q - Q_x}{2(\lambda_1 + \lambda_1^*)}. \tag{31}$$

Based on (31), we have

$$\Phi_1^2 + \Phi_2^2 = \frac{2\lambda_1 Q}{(\lambda_1 + \lambda_1^*)}, \quad \Phi_1^2 - \Phi_2^2 = \frac{Q_x}{(\lambda_1 + \lambda_1^*)}. \tag{32}$$

Substituting (30) into (14) and (15) yields $\lambda_1\Phi_1\Phi_2 + \lambda_1^*\Phi_1^*\Phi_2^* - H + \frac{1}{2}Q^2 = 0$ and $\Phi_1^*\Phi_2^* = \Phi_1\Phi_2 + \mathrm{i}G_0$. Thus, we obtain

$$\Phi_1\Phi_2 = -\frac{Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*}{2(\lambda_1 + \lambda_1^*)}. \tag{33}$$

For the dn-function solution $Q(x) = \mathrm{dn}(x; k)$ in (13), we already know that $G_0 = 0$, $H = G_1 = \pm\frac{1}{2}\sqrt{1 - k^2}$ and choose $\lambda_1 = \rho = \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 - k^2})$. Then, we rewrite (32) and (33) as

$$\Phi_1\Phi_2 = -\frac{1}{4\rho}(Q^2 + \sqrt{1 - k^2}) = -\frac{\mathrm{dn}^2(x; k) + \sqrt{1 - k^2}}{2(1 \pm \sqrt{1 - k^2})},$$
$$\Phi_1^2 + \Phi_2^2 = Q = \mathrm{dn}(x; k),$$
$$\Phi_1^2 - \Phi_2^2 = \frac{1}{2\rho}Q_x = -\frac{k^2\,\mathrm{sn}(x; k)\mathrm{cn}(x; k)}{1 \pm \sqrt{1 - k^2}}.$$

Based on (31) with $\lambda_1 = \lambda_1^* = \rho$, we know that Φ_1^2 and Φ_2^2 are real. In addition, $\Phi_1^2 + \Phi_2^2 = Q = \mathrm{dn}(x; k) > 0$, which yields that Φ_1 and Φ_2 are real. Thus, we rewrite the first equation of (34) as

$$\Phi_1\Phi_2^* = -\frac{1}{4\rho}(Q^2 + \sqrt{1 - k^2}) = -\frac{\mathrm{dn}^2(x; k) + \sqrt{1 - k^2}}{2(1 \pm \sqrt{1 - k^2})}. \tag{35}$$

For the cn-function solution $Q(x) = k\,\mathrm{cn}(x; k)$ in (10), we already know that $G_0 = 2\mu$, $G_1 = 4\mu^2$, $H = 2\mu^2$ and choose $\lambda_1 = \lambda_\pm = \frac{1}{2}(k \pm \mathrm{i}\sqrt{1 - k^2})$. Then, we rewrite (33) as

$$\Phi_1\Phi_2 = -\frac{1}{2k}(Q^2 + \mathrm{i}k\sqrt{1 - k^2}). \tag{36}$$

Equations (16) and (33) yield $(|\Phi_1|^2 + |\Phi_2|^2)^2 = 1 - k^2 + |u|^2$. If we consider the positive square root, we have

$$|\Phi_1|^2 + |\Phi_2|^2 = \mathrm{dn}(x; k). \tag{37}$$

Noticing that $G_1 = G_0^2$, we have

$$\lambda_1\Phi_1\Phi_2 + \lambda_1^*\Phi_1^*\Phi_2^* + \Phi_1^2\Phi_2^2 + \Phi_1^{2*}\Phi_2^{2*} + \frac{1}{2}(|\Phi_1|^2 - |\Phi_2|^2)^2 = 0. \tag{38}$$

Equations (28), (36) and (38) yield $(|\Phi_1|^2 - |\Phi_2|^2)^2 = |u|^2 - \frac{1}{k^2}|u|^4$. If we consider the negative square root, we have

$$|\Phi_1|^2 - |\Phi_2|^2 = -k\,\mathrm{sn}(x;\,k)\mathrm{cn}(x;\,k). \tag{39}$$

Solving equations (37) and (39) yields

$$|\Phi_1|^2 = \frac{1}{2}[\mathrm{dn}(x;\,k) - k\,\mathrm{sn}(x;\,k)\mathrm{cn}(x;\,k)],$$
$$|\Phi_2|^2 = \frac{1}{2}[\mathrm{dn}(x;\,k) + k\,\mathrm{sn}(x;\,k)\mathrm{cn}(x;\,k)]. \tag{40}$$

Based on (28) and (31), we have

$$\Phi_1^2\Phi_2^{2*} = \frac{1}{4}[\mathrm{cn}^2(x;\,k) - \mathrm{sn}^2(x;\,k)\mathrm{dn}^2(x;\,k) + 2\mathrm{i}\sqrt{1-k^2}\,\mathrm{sn}(x;\,k)\mathrm{cn}(x;\,k)\mathrm{dn}(x;\,k)], \tag{41}$$

which yields

$$\Phi_1\Phi_2^* = -\frac{1}{2}[\mathrm{cn}(x;\,k)\mathrm{dn}(x;\,k) + \mathrm{i}\sqrt{1-k^2}\,\mathrm{sn}(x;\,k)]. \tag{42}$$

Let us make an assumption that $(\varphi_1, \varphi_2)^{\mathrm{T}}$ is the periodic solution of the Lax pair (5)–(6) with $\lambda = \lambda_1$, and $(\psi_1, \psi_2)^{\mathrm{T}}$ is the second linearly independent solution of the Lax pair (5)–(6) with the same $\lambda = \lambda_1$, where ψ_1 and ψ_2 are non-periodic solutions and have the forms

$$\psi_1 = \frac{\theta - 1}{\varphi_2}, \quad \psi_2 = \frac{\theta + 1}{\varphi_1}, \tag{43}$$

where $\theta = \theta(x, t)$ is a function to be determined.

Using (43) and (5), we have

$$\theta_x = \theta\frac{q\varphi_2^2 - q^*\varphi_1^2}{\varphi_1\varphi_2} + \frac{q\varphi_2^2 + q^*\varphi_1^2}{\varphi_1\varphi_2}. \tag{44}$$

Using (8) and (30), we rewrite (44) as

$$\theta_x = \theta Q\frac{\Phi_2^2 - \Phi_1^2}{\Phi_1\Phi_2} + Q\frac{\Phi_2^2 + \Phi_1^2}{\Phi_1\Phi_2}. \tag{45}$$

Substituting (32) and (33) into (45) yields

$$\left(\frac{\theta}{Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*}\right)_x = \frac{-4\lambda_1 Q^2}{(Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*)^2}. \tag{46}$$

Integrating (46) yields

$$\theta = (Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*) = \left[\int_0^x \frac{-4\lambda_1 Q^2(y)}{(Q^2(y) - 2H + 2\mathrm{i}G_0\lambda_1^*)^2}\mathrm{d}y + \theta_0(t)\right], \tag{47}$$

where $\theta_0(t)$ is an integral constant depending on t to be determined.

Notice that (43) and (6) yield a rather complex expression for θ_t. However, with the help of the Jacobi elliptic function expansion method and using (30), (32), (33) and (45), we have

$$\theta_t = (Q^2(y) - 2H + 2\mathrm{i}G_0\lambda_1^*)\Delta_0 + \Delta_1, \tag{48}$$

where

$$\Delta_0 = \mathrm{i}[1 + 2a_0^2\delta - 4a_1\delta + 4\mathrm{i}\alpha\lambda_1 + 8\beta\lambda_1^2 + 16\mathrm{i}\gamma\lambda_1^3 + 32\delta\lambda_1^4 + 2a_0(\beta + 2\mathrm{i}\gamma\lambda_1 + 4\delta\lambda_1^2)],$$
$$\Delta_1 = (a_0\alpha + a_0^2\gamma - 2a_1\gamma)\theta_x. \tag{49}$$

For the rogue waves on the dn-periodic background, we already know that $a_0 = 2 - k^2$ and $a_1 = k^2 - 1$. Using the third equation of (9), it emerges that $\Delta_1 = 0$. For the rogue waves on the cn-periodic background, we already know that $a_0 = 2k^2 - 1$ and $a_1 = k^2(1 - k^2)$. Using the third equation of (10), it also emerges that $\Delta_1 = 0$. Therefore, on the background of Jacobian elliptic functions dn and cn, we finally rewrite (48) as

$$\theta_t = (Q^2(y) - 2H + 2\mathrm{i}G_0\lambda_1^*)\Delta_0. \tag{50}$$

Taking a derivative of (47) with respect to t, we have

$$\theta_t = (Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*)\theta_{0,t}(t). \tag{51}$$

Comparing (50) and (51) yields $\theta_{0,t}(t) = \Delta_0$. Integrating $\theta_{0,t}(t)$, we have

$$\theta_0(t) = \mathrm{i}[1 + 2a_0^2\delta - 4a_1\delta + 4\mathrm{i}\alpha\lambda_1 + 8\beta\lambda_1^2 + 16\mathrm{i}\gamma\lambda_1^3 + 32\delta\lambda_1^4 + 2a_0(\beta + 2\mathrm{i}\gamma\lambda_1 + 4\delta\lambda_1^2)]t + \vartheta, \tag{52}$$

where ϑ is a constant.

Substituting (52) into (47), we finally arrive at the expression of θ:

$$\theta = (Q^2 - 2H + 2\mathrm{i}G_0\lambda_1^*) \times \left[\int_0^x \frac{-4\lambda_1 Q^2(y)}{(Q^2(y) - 2H + 2\mathrm{i}G_0\lambda_1^*)^2}\mathrm{d}y + \Delta_0 t + \vartheta\right], \tag{53}$$

where

$$\Delta_0 = \mathrm{i}[1 + 2a_0^2\delta - 4a_1\delta + 4\mathrm{i}\alpha\lambda_1 + 8\beta\lambda_1^2 + 16\mathrm{i}\gamma\lambda_1^3 + 32\delta\lambda_1^4 + 2a_0(\beta + 2\mathrm{i}\gamma\lambda_1 + 4\delta\lambda_1^2)]. \tag{54}$$

5. Rogue waves on the periodic background

5.1. Darboux transformation

According to [33], the elementary Darboux transformation of equation (3) can be redefined as

$$q[1] = q + \frac{2(\lambda_1 + \lambda_1^*)\phi_{11}\phi_{21}^*}{|\phi_{11}|^2 + |\phi_{21}|^2}, \tag{55}$$

where $(\phi_{11}, \phi_{21})^{\mathrm{T}}$ is a non-zero solution of the Lax pair (5)–(6) with $\lambda = \lambda_1$.

5.2. Rogue waves on the dn-periodic background

In order to obtain the rogue wave solution of equation (3) on the dn-periodic background, we apply the one-fold Darboux transformation (55) to the Jacobian elliptic function dn, set the seed solution $q = Q(x)e^{\mathrm{i}ct}$ and choose the real eigenvalue $\lambda_1 = \rho = \lambda_+ = \frac{1}{2}(1 + \sqrt{1-k^2})$ in (25). Substituting $(\phi_{11}, \phi_{21})^{\mathrm{T}} = (\psi_1, \psi_2)^{\mathrm{T}}$ defined by (43) into (53) and using

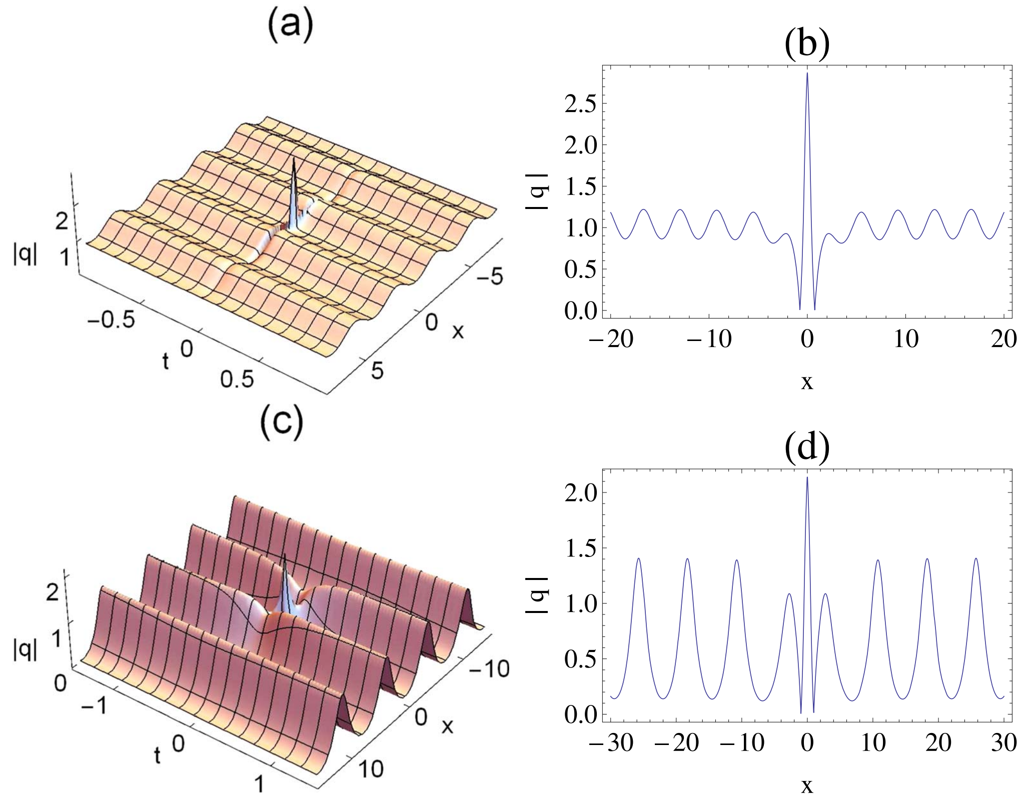

Figure 1. Rogue waves on the dn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 0.5$. (a) Three-dimensional plot, (b) transverse plot at $t = 0$. Rogue waves on the dn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 0.99$. (c) Three-dimensional plot, (d) transverse plot at $t = 0$.

(34), (35), we obtain the rogue wave solution of equation (3) on the dn-periodic background as

$$q_{\mathrm{dn-rogue}} = \mathrm{e}^{\mathrm{i}ct}\left[\mathrm{dn}(x;\, k) + \frac{(1 - 2\mathrm{i}\mathrm{Im}\theta_{\mathrm{dn}} - |\theta_{\mathrm{dn}}|^2)(\mathrm{dn}^2(x;k) + \sqrt{1-k^2})}{(1 + |\theta_{\mathrm{dn}}|^2)\,\mathrm{dn}(x;k) + 2(1 - \sqrt{1-k^2})\,\mathrm{Re}\,\theta_{\mathrm{dn}}\,\mathrm{sn}(x;k)\mathrm{cn}(x;k)}\right], \tag{56}$$

with

$$\theta_{\mathrm{dn}} = (\mathrm{dn}^2(x;\, k) + \sqrt{1-k^2})\left[-2(1 + \sqrt{1-k^2}) \times \int_0^x \frac{\mathrm{dn}^2(y)}{(\mathrm{dn}^2(y) + \sqrt{1-k^2})^2}\mathrm{d}y + \Delta_0 t + \vartheta\right], \tag{57}$$

where Δ_0 is defined in (54) and ϑ is a constant.

The rogue waves on the dn-periodic background (56) are presented in figures 1 (a), (b) with $k = 0.5$ and (c), (d) $k = 0.99$. As we can see in figure 1, the central part corresponds to a rogue wave and the background is periodic to x.

In particular, when $k = 0$ or $k = 1$, we obtain the degenerate solutions and present them in figure 2.

5.3. Rogue waves on the cn-periodic background

In order to obtain the rogue wave solution of equation (3) on the cn-periodic background, we apply the one-fold Darboux transformation (55) to the Jacobian elliptic function cn, set the seed solution $q = Q(x)e^{\mathrm{i}\,ct}$ and choose the complex eigenvalue $\lambda_1 = \lambda_\pm = \frac{1}{2}(k \pm \mathrm{i}\sqrt{1-k^2})$ in (28). Substituting $(\phi_{11},\, \phi_{21})^{\mathrm{T}} = (\psi_1,\, \psi_2)^{\mathrm{T}}$ defined by (43) into (53) and using (37), (39) and (42), we obtain the rogue wave solution of equation (3) on the cn-periodic background as

$$q_{\mathrm{cn-rogue}} = \mathrm{e}^{\mathrm{i}ct}\left[k\mathrm{cn}(x;\, k) + \frac{k(1 - 2\mathrm{i}\mathrm{Im}\theta_{\mathrm{cn}} - |\theta_{\mathrm{cn}}|^2)(\mathrm{cn}(x;\, k)\mathrm{dn}(x;\, k) + \mathrm{i}\sqrt{1-k^2}\,\mathrm{sn}(x;\, k))}{(1 + |\theta_{\mathrm{cn}}|^2)\mathrm{dn}(x;\, k) + 2k\mathrm{Re}\theta_{\mathrm{cn}}\mathrm{sn}(x;\, k)\mathrm{cn}(x;\, k)}\right], \tag{58}$$

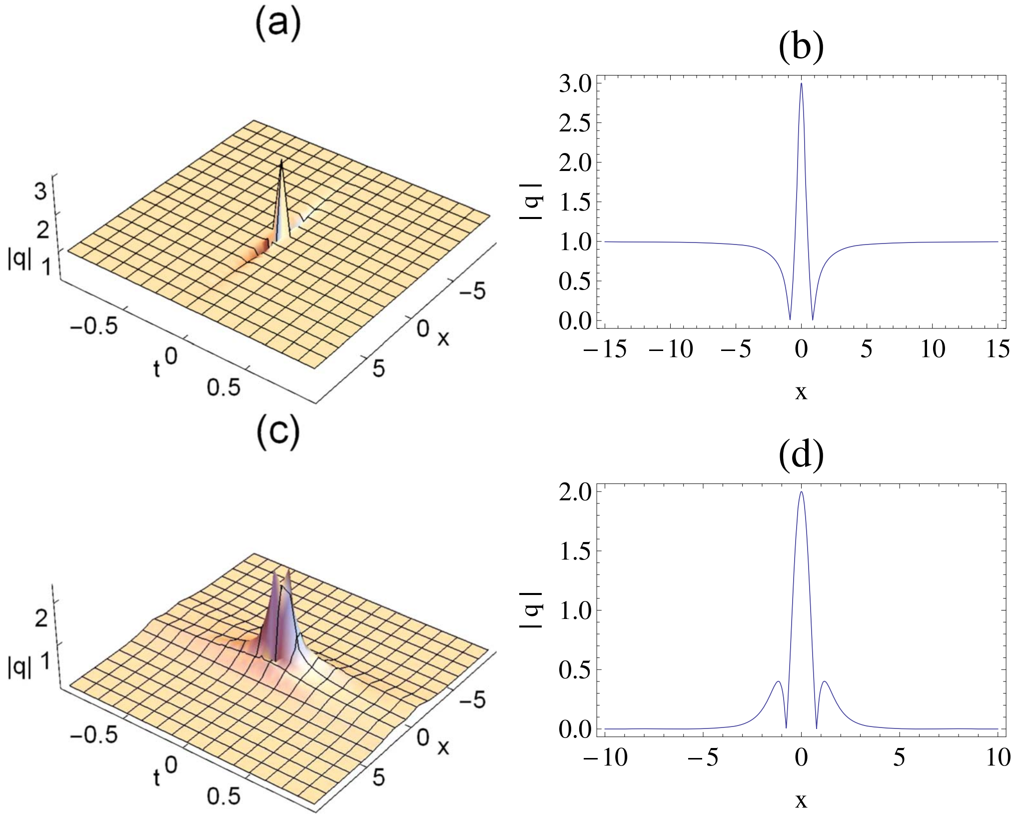

Figure 2. Degenerate rogue waves on the dn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 0$. (a) Three-dimensional plot, (b) transverse plot at $t = 0$. Degenerate rogue waves on the dn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 1$. (c) Three-dimensional plot, (d) transverse plot at $t = 0$.

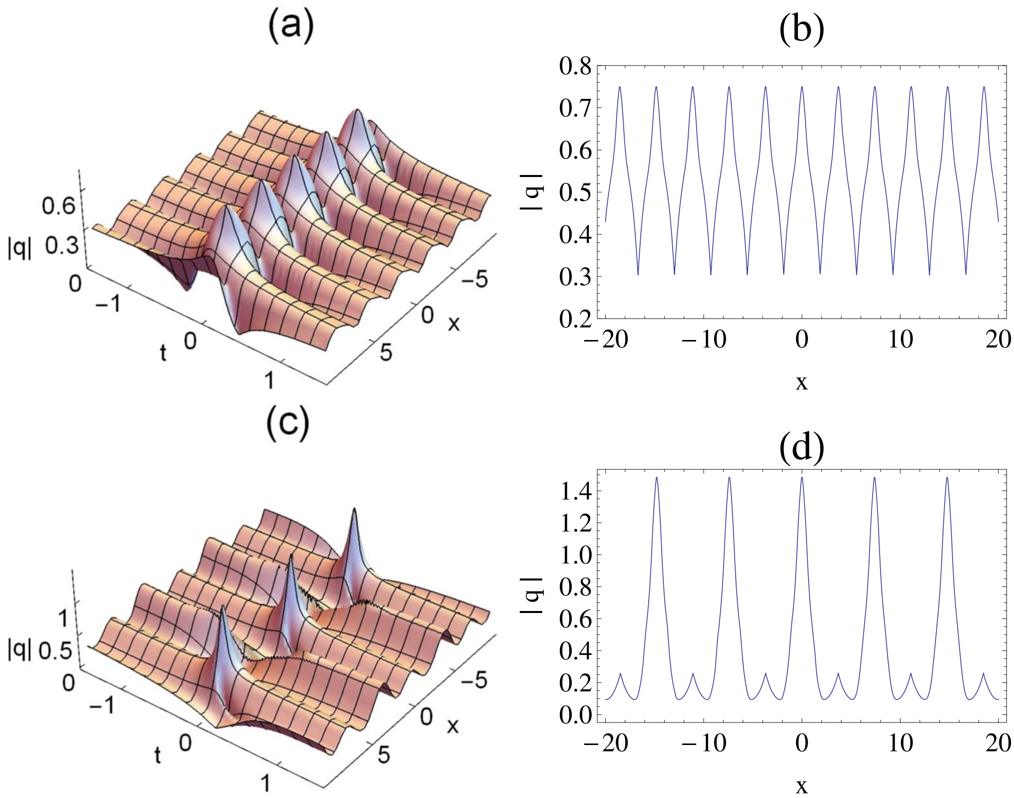

Figure 3. Rogue waves on the cn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 0.5$. (a) Three-dimensional plot, (b) transverse plot at $t = 0$. Rogue waves on the cn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 0.99$. (c) Three-dimensional plot, (d) transverse plot at $t = 0$.

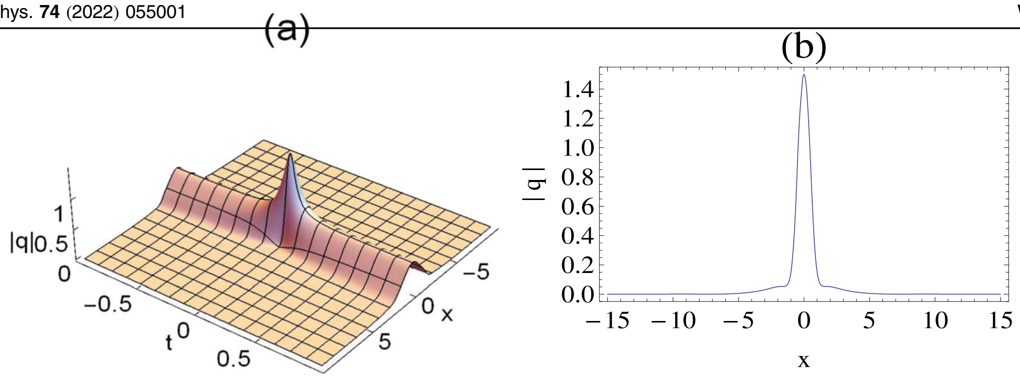

Figure 4. Degenerate rogue waves on the cn-periodic background with $\beta = 1$, $\gamma = 3$, $\delta = 3$, $\vartheta = 0$, $k = 1$. (a) Three-dimensional plot, (b) transverse plot at $t = 0$.

with

$$\theta_{\rm cn} = (k^2{\rm cn}^2(x;\,k) + {\rm i}k\sqrt{1-k^2})\left[-2(k + {\rm i}\sqrt{1-k^2}) \times \int_0^x \frac{k^2{\rm cn}^2(y)}{(k^2{\rm cn}^2(y) + {\rm i}k\sqrt{1-k^2})^2}{\rm d}y + \Delta_0 t + \vartheta\right]. \tag{59}$$

The rogue waves on the cn-periodic background (58) are presented in figures 3 (a), (b) with $k = 0.5$ and (c), (d) $k = 0.99$.

In particular, when $k = 1$, we obtain the degenerate solution and present it in figure 4.

6. Conclusions

In this paper, the rogue waves of the sixth-order NLS equation on the periodic background of Jacobian elliptic functions dn and cn are constructed by means of non-linearization of a spectral problem and Darboux transformation approach. If we compare the results in this paper to the results of some other well-known NLS equations such as [5, 14–16], we find that all the above papers have similar expressions for solutions like (56), (58) and similar plots for solutions, which confirms the correctness of our results. However, our current study is just dependent upon the spectral problem in the AKNS system. In the future, we expect to construct rogue waves on a periodic background based on other spectral problems and we hope our results can provide some inspiration in the study of rogue wave phenomena on periodic backgrounds in the field of nonlinear physics.

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Conflict of interest

The authors declare that they have no conflict of interest.

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