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Friedberg-Lee neutrino model with μ–τ reflection symmetry

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Abstract

In this letter, we make an attempt to embed the μ–τ reflection symmetry (which predicts maximal atmospherical mixing angle and Dirac CP phase) in the Friedberg-Lee neutrino model (which employs a translational flavor symmetry and keeps one neutrino mass vanishing) and study the consequences of such a combination.

Keywords: neutrino masses, neutrino mixing, Friedberg-Lee symmetry, mu-tau reflection symmetry

1. Introduction

As we know, the observations of the phenomena of neutrino oscillations indicate that neutrinos are massive and three lepton flavors are mixed [1]. In the basis where the mass eigenstates of three charged leptons coincide with their flavor eigenstates, the lepton flavor mixing matrix U is identical with the unitary matrix for diagonalizing the neutrino mass matrix M_ν[2]:

$$U^\dagger M_\nu U^* = D_\nu = \mathrm{diag}(m_1, m_2, m_3), \tag{1}$$

with m_i being three-light neutrino masses. In the standard form, U is parameterized in terms of three lepton flavor mixing angles θ_{ij} (for $ij = 12, 13, 23$), the Dirac CP phase δ, and two Majorana CP phases ρ and σ:

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \times \begin{pmatrix} \mathrm{e}^{\mathrm{i}\rho} & & \\ & \mathrm{e}^{\mathrm{i}\sigma} & \\ & & 1 \end{pmatrix}, \tag{2}$$

where the symbols $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$ have been employed.

On the experimental side, the neutrino mass squared differences $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ and three lepton flavor mixing angles have been measured with good degrees of accuracy. And there has also been a preliminary result for δ. For the convenience of the reader, the global-fit results for these parameters [2, 3] are presented here in table 1. However, the sign of Δm_{31}^2 remains unknown, leaving us with two possible neutrino mass orderings: the normal ordering (NO) $m_1 < m_2 < m_3$ and inverted ordering (IO) $m_3 < m_1 < m_2$. And there has not been any lower constraint on the smallest neutrino mass, nor any constraint on the Majorana CP phases, whose values can only be inferred from some non-oscillatory processes (e.g. the neutrinoless double beta decays [4]) or cosmological observations.

From table 1 we see that θ_{12} and θ_{23} are close to some special values: $\sin^2\theta_{12} \sim 1/3$ and $\sin^2\theta_{23} \sim 1/2$ (i.e. $\theta_{23} \sim \pi/4$), and θ_{13} is not far from zero. As is well known, the μ–τ symmetry (under which M_ν keeps invariant with respect to the $\nu_\mu \leftrightarrow \nu_\tau$ transformation [5, 6]) can naturally accommodate $\theta_{23} = \pi/4$ and $\theta_{13} = 0$. After the observation of a relatively large θ_{13} (compared to zero) and a preliminary hint for $\delta \sim -\pi/2$ [7], its variant—the μ–τ reflection symmetry [6, 8] has attracted particular attention. This symmetry is defined in a way that M_ν keeps invariant with respect to the following transformations of three neutrino fields

$$\nu_e \leftrightarrow \nu_e^c, \quad \nu_\mu \leftrightarrow \nu_\tau^c, \quad \nu_\tau \leftrightarrow \nu_\mu^c, \tag{3}$$

where the superscript c denotes the charge conjugation of the relevant neutrino fields. To be explicit, M_ν takes a form as

$$M_\nu = \begin{pmatrix} A & B & B^* \\ B & C & D \\ B^* & D & C^* \end{pmatrix}, \tag{4}$$

[1] Co-first author.

[2] In this letter we consider the scenario of neutrinos being Majorana particles where M_ν is symmetric.

Table 1. The best-fit values, 1σ errors and 3σ ranges of six neutrino oscillation parameters extracted from a global analysis of the existing neutrino oscillation data [2].

	Normal ordering		Inverted ordering	
	bf $\pm$ 1σ	3σ range	bf $\pm$ 1σ	3σ range
$\sin^2\theta_{12}$	$0.318^{+0.016}_{-0.016}$	$0.271 \to 0.370$	$0.318^{+0.016}_{-0.016}$	$0.271 \to 0.370$
$\sin^2\theta_{23}$	$0.566^{+0.016}_{-0.022}$	$0.441 \to 0.609$	$0.566^{+0.018}_{-0.023}$	$0.446 \to 0.609$
$\sin^2\theta_{13}$	$0.02225^{+0.00055}_{-0.00078}$	$0.02015 \to 0.024\,17$	$0.02250^{+0.00056}_{-0.00076}$	$0.02039 \to 0.024\,41$
δ/π	$1.20^{+0.23}_{-0.14}$	$0.80 \to 2.00$	$1.54^{+0.13}_{-0.13}$	$1.14 \to 1.90$
$\Delta m^2_{21}/(10^{-5}\,\mathrm{eV}^2)$	$7.50^{+0.22}_{-0.20}$	$6.94 \to 8.14$	$7.50^{+0.22}_{-0.20}$	$6.94 \to 8.14$
$\lvert\Delta m^2_{31}\rvert/(10^{-3}\,\mathrm{eV}^2)$	$2.56^{+0.03}_{-0.04}$	$2.46 \to 2.65$	$2.46^{+0.03}_{-0.03}$	$2.37 \to 2.55$

with A and D being real. Such an M_ν leads to the following interesting predictions for the lepton flavor mixing parameters

$$\theta_{23} = \frac{\pi}{4}, \quad \delta = \pm\frac{\pi}{2}, \quad \rho, \sigma = 0 \text{ or } \frac{\pi}{2}. \tag{5}$$

Throughout this letter, we will take $\delta = -\pi/2$ (which is more favored by the experimental results) out of $\pm\pi/2$.

Considering that the μ–τ reflection symmetry has no predictive power for the neutrino masses, in this letter we make an attempt to embed it in the interesting Friedberg-Lee (FL) neutrino model which employs a translational flavor symmetry and keeps one neutrino mass vanishing ($m_1 = 0$ or $m_3 = 0$ in the NO or IO case) and study the consequences of such a combination. In this scenario, both the lepton flavor mixing parameters and neutrino masses will be constrained by the flavor symmetries.

2. FL neutrino model with μ–τ reflection symmetry

Let us first briefly recapitulate the salient features of the FL neutrino model [9–11]. Under the FL symmetry, the lagrangian relevant for the neutrino masses is given by

$$\begin{aligned}\mathcal{L}_{\rm mass} = {} & a(\eta_\tau^*\bar{\nu}_\mu - \eta_\mu^*\bar{\nu}_\tau)(\eta_\tau^*\nu_\mu^c - \eta_\mu^*\nu_\tau^c) \\ & + b(\eta_\mu^*\bar{\nu}_e - \eta_e^*\bar{\nu}_\mu)(\eta_\mu^*\nu_e^c - \eta_e^*\nu_\mu^c) \\ & + c(\eta_\tau^*\bar{\nu}_e - \eta_e^*\bar{\nu}_\tau)(\eta_\tau^*\nu_e^c - \eta_e^*\nu_\tau^c) + \text{h.c.},\end{aligned} \tag{6}$$

where a, b and c are mass parameters and $\eta_{e,\mu,\tau}$ are coefficients. One can easily see that $\mathcal{L}_{\rm mass}$ keeps invariant with respect to the following translational transformations of three neutrino fields

$$\nu_e \to \nu_e + \eta_e\xi, \quad \nu_\mu \to \nu_\mu + \eta_\mu\xi, \quad \nu_\tau \to \nu_\tau + \eta_\tau\xi, \tag{7}$$

where ξ is a space-time independent element of the Grassmann algebra and anti-commutates with the neutrino fields. It is worth noting that the following neutrino mass terms also keep invariant with respect to the translational transformations of three neutrino fields in equation (7)[3]:

$$(\eta_\tau^*\bar{\nu}_\mu - \eta_\mu^*\bar{\nu}_\tau)(\eta_\mu^*\nu_e^c - \eta_e^*\nu_\mu^c). \tag{8}$$

However, considering that these terms are not present in the original FL model and otherwise they would introduce additional parameters, we will not include them in our analysis.

As a result of the FL symmetry, M_ν takes a form as

$$M_\nu = \begin{pmatrix} b\eta_\mu^{*2} + c\eta_\tau^{*2} & -b\eta_e^*\eta_\mu^* & -c\eta_e^*\eta_\tau^* \\ -b\eta_e^*\eta_\mu^* & a\eta_\tau^{*2} + b\eta_e^{*2} & -a\eta_\mu^*\eta_\tau^* \\ -c\eta_e^*\eta_\tau^* & -a\eta_\mu^*\eta_\tau^* & a\eta_\mu^{*2} + c\eta_e^{*2} \end{pmatrix}. \tag{9}$$

It is direct to verify that the relation $M_\nu(\eta_e^*, \eta_\mu^*, \eta_\tau^*)^T = 0$ holds. This means that one eigenvalue of M_ν (i.e. one neutrino mass) is vanishing and the eigenvector corresponding to it is proportional to $(\eta_e^*, \eta_\mu^*, \eta_\tau^*)^T$. Therefore, in the NO (or IO) case with $m_1 = 0$ (or $m_3 = 0$), the first (or third) column of U will be proportional to $(\eta_e^*, \eta_\mu^*, \eta_\tau^*)^{\rm T}$.

Now, we are ready to study the embedding of the μ–τ reflection symmetry in the FL neutrino model. It is found that in order for M_ν in equation (9) to conform to the pattern described in equation (4) the following relations must hold

$$\begin{aligned} & \phi_\mu + \phi_\tau = \phi_a, \quad \phi_b + \phi_c = 2\phi_e + \phi_a, \\ & |b||\eta_\mu| = |c||\eta_\tau|, \\ & |\eta_\mu|\sin(\phi_b - 2\phi_\mu) + |\eta_\tau| \\ & \times\sin(\phi_c - 2\phi_\tau) = 0, \\ & |a|(|\eta_\mu|^2 - |\eta_\tau|^2){\rm e}^{{\rm i}(\phi_\mu - \phi_\tau)} = |\eta_e|^2 \\ & \times[|b|{\rm e}^{{\rm i}(\phi_a - \phi_c)} - |c|{\rm e}^{{\rm i}(\phi_b - \phi_a)}], \end{aligned} \tag{10}$$

with $\phi_\alpha \equiv \arg(\eta_\alpha)$ and $\phi_a = \arg(a)$ (and similarly for ϕ_b and ϕ_c). In the present paper, we will study the following instructive solution of equation (10):

$$b = c^*, \quad \eta_\mu = \eta_\tau^*, \quad \phi_e = \phi_a = 0. \tag{11}$$

For the sake of simplicity, without loss of generality, hereafter we will take $\eta_e = 1$. Therefore, in the FL neutrino model with

[3] We thank the anonymous referee for pointing out this point.

the μ–τ reflection symmetry, M_ν can be expressed as

$$M_\nu = \begin{pmatrix} 2\,\mathrm{Re}(b\eta_\mu^{*2}) & -b\eta_\mu^* & -b^*\eta_\mu \\ -b\eta_\mu^* & a\eta_\mu^2 + b & -a|\eta_\mu|^2 \\ -b^*\eta_\mu & -a|\eta_\mu|^2 & a\eta_\mu^{*2} + b^* \end{pmatrix}, \tag{12}$$

with a being real.

By diagonalizing the above M_ν according to equation (1), one can derive the expressions of the resulting lepton flavor mixing angles and neutrino masses in terms of the model parameters a, b and η_μ. Before doing that, we first transform M_ν to the following form

$$M_\nu' = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} M_\nu \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{Re}(b\eta_\mu^{*2}) & -b\eta_\mu^* & b^*\eta_\mu \\ -b\eta_\mu^* & a\eta_\mu^2 + b & a|\eta_\mu|^2 \\ b^*\eta_\mu & a|\eta_\mu|^2 & a\eta_\mu^{*2} + b^* \end{pmatrix}, \tag{13}$$

via a redefinition of the phases of three neutrino fields (which are unphysical). Then, diagonalization of M_ν' leads us to

$$\begin{aligned} \tan\theta_{13} &= -\frac{\mathrm{Im}(b) + a\,\mathrm{Im}(\eta_\mu^2)}{\sqrt{2}\,\mathrm{Re}(b\eta_\mu^*)}, \\ \tan 2\theta_{13} &= -\frac{2\sqrt{2}\,\mathrm{Im}(b\eta_\mu^*)}{\mathrm{Re}(b) + a\,\mathrm{Re}(\eta_\mu^2) + a|\eta_\mu|^2 + 2\,\mathrm{Re}(b\eta_\mu^{*2})}, \\ \tan 2\theta_{12} &= \frac{2B}{C - A}, \\ m_1 &= |c_{12}^2 A + s_{12}^2 C - 2c_{12}s_{12}B|, \\ m_2 &= |s_{12}^2 A + c_{12}^2 C + 2c_{12}s_{12}B|, \\ m_3 &= |c_{13}^2[\mathrm{Re}(b) + a\,\mathrm{Re}(\eta_\mu^2) + a|\eta_\mu|^2] \\ &\quad - 2s_{13}^2\,\mathrm{Re}(b\eta_\mu^{*2}) - 2\sqrt{2}\,c_{13}s_{13}\,\mathrm{Im}(b\eta_\mu^*)|, \end{aligned} \tag{14}$$

with

$$\begin{aligned} A &= 2c_{13}^2\,\mathrm{Re}(b\eta_\mu^{*2}) - s_{13}^2[\mathrm{Re}(b) + a\,\mathrm{Re}(\eta_\mu^2) \\ &\quad + a|\eta_\mu|^2] - 2\sqrt{2}\,c_{13}s_{13}\,\mathrm{Im}(b\eta_\mu^*), \\ B &= -\sqrt{2}\,c_{13}\,\mathrm{Re}(b\eta_\mu^*) + s_{13}[\mathrm{Im}(b) + a\,\mathrm{Im}(\eta_\mu^2)], \\ C &= \mathrm{Re}(b) + a\,\mathrm{Re}(\eta_\mu^2) - a|\eta_\mu|^2. \end{aligned} \tag{15}$$

Conversely, one can derive the expressions of the model parameters in terms of the lepton flavor mixing angles and neutrino masses by comparing M_ν' in equation (13) and M_ν obtained via the reconstruction relation $M_\nu = UD_\nu U^{\mathrm{T}}$:

$$\begin{aligned} (M_\nu)_{ee} &= m_1\eta_\rho c_{12}^2 c_{13}^2 + m_2\eta_\sigma s_{12}^2 c_{13}^2 - m_3 s_{13}^2, \\ (M_\nu)_{e\mu} &= -\frac{1}{\sqrt{2}}m_1\eta_\rho c_{12}c_{13}(s_{12} - \mathrm{i}c_{12}s_{13}) \\ &\quad + \frac{1}{\sqrt{2}}m_2\eta_\sigma s_{12}c_{13}(c_{12} + \mathrm{i}s_{12}s_{13}) + \frac{1}{\sqrt{2}}\mathrm{i}m_3 c_{13}s_{13}, \\ (M_\nu)_{\mu\mu} &= \frac{1}{2}m_1\eta_\rho(s_{12} - \mathrm{i}c_{12}s_{13})^2 \\ &\quad + \frac{1}{2}m_2\eta_\sigma(c_{12} + \mathrm{i}s_{12}s_{13})^2 + \frac{1}{2}m_3 c_{13}^2, \\ (M_\nu)_{\mu\tau} &= -\frac{1}{2}m_1\eta_\rho(s_{12}^2 + c_{12}^2 s_{13}^2) \\ &\quad - \frac{1}{2}m_2\eta_\sigma(c_{12}^2 + s_{12}^2 s_{13}^2) + \frac{1}{2}m_3 c_{13}^2, \end{aligned} \tag{16}$$

with $\eta_\rho = 1$ or -1 for $\rho = 0$ or $\pi/2$ (and similarly for η_σ), $(M_\nu)_{e\tau} = -(M_\nu)_{e\mu}^*$, and $(M_\nu)_{\tau\tau} = (M_\nu)_{\mu\mu}^*$. With the help of these expressions, by inputting the experimental results of the lepton flavor mixing angles and neutrino mass squared differences, we can calculate the values of $(M_\nu)_{\alpha\beta}$ and subsequently the values of the model parameters.

Let us first consider the NO case (i.e. $m_1 = 0$). For $\sigma = 0$, allowing the lepton flavor mixing angles and neutrino mass squared differences to vary in their 3σ ranges, the ranges of $(M_\nu)_{\alpha\beta}$ are found to be

$$\begin{aligned} &1.0 \leqslant (M_\nu)_{ee}/\mathrm{meV} \leqslant 2.2, \\ &21 \leqslant (M_\nu)_{\mu\tau}/\mathrm{meV} \leqslant 23, \\ &5.8 \leqslant |(M_\nu)_{e\mu}|/\mathrm{meV} \leqslant 6.7, \\ &0.33 \leqslant \arg[(M_\nu)_{e\mu}]/\pi \leqslant 0.37, \\ &27 \leqslant |(M_\nu)_{\mu\mu}|/\mathrm{meV} \leqslant 28, \\ &0.006 \leqslant \arg[(M_\nu)_{\mu\mu}]/\pi \leqslant 0.008. \end{aligned} \tag{17}$$

We see that there exist large hierarchies among $(M_\nu)_{\alpha\beta}$: $|(M_\nu)_{\mu\mu}| \sim |(M_\nu)_{\mu\tau}| \gg |(M_\nu)_{e\mu}| \gg |(M_\nu)_{ee}|$. And $(M_\nu)_{\mu\mu}$ is nearly purely real. These results can be understood from equation (16) by taking $m_1 = 0$. Subsequently, the ranges of the model parameters are obtained as

$$\begin{aligned} &69 \leqslant a/\mathrm{meV} \leqslant 110, \\ &11 \leqslant |b|/\mathrm{meV} \leqslant 14, \\ &0.27 \leqslant \phi_b/\pi \leqslant 0.29, \\ &0.45 \leqslant |\eta_\mu| \leqslant 0.56, \\ &0.92 \leqslant \phi_\mu/\pi \leqslant 0.94. \end{aligned} \tag{18}$$

One can see that the relations $|a| \gg |b|$ and $\phi_b \sim \arg[(M_\nu)_{e\mu}]$ hold. And η_μ is nearly purely real. These results can be understood with the help of equations (13), (17). It is interesting to note that $|\eta_\mu|$ can take the special value of 1/2 that corresponds to the so-called TM1 mixing [12]: after a proper redefinition of the phases of three neutrino fields, the first column of U will be proportional to $(2,1,1)^{\mathrm{T}}$ as predicted by the TM1 mixing.

For $\sigma = \pi/2$, the ranges of $(M_\nu)_{\alpha\beta}$ turn out to be

$$\begin{aligned}
&-4.5 \leqslant (M_\nu)_{ee}/\text{meV} \leqslant -3.2,\\
&27 \leqslant (M_\nu)_{\mu\tau}/\text{meV} \leqslant 28,\\
&5.4 \leqslant |(M_\nu)_{e\mu}|/\text{meV} \leqslant 6.0,\\
&0.64 \leqslant \arg[(M_\nu)_{e\mu}]/\pi \leqslant 0.69,\\
&21 \leqslant |(M_\nu)_{\mu\mu}|/\text{meV} \leqslant 23,\\
&-0.010 \leqslant \arg[(M_\nu)_{\mu\mu}]/\pi \leqslant -0.008.
\end{aligned} \tag{19}$$

In this case, $|(M_\nu)_{ee}|$ becomes comparable to $|(M_\nu)_{e\mu}|$. Subsequently, the ranges of a and b are obtained as

$$\begin{aligned}
&87 \leqslant a/\text{meV} \leqslant 142,\\
&9.8 \leqslant |b|/\text{meV} \leqslant 13,\\
&0.57 \leqslant \phi_b/\pi \leqslant 0.62.
\end{aligned} \tag{20}$$

On the other hand, the results of η_μ are completely the same as in equation (18). This is because the change of σ from 0 to $\pi/2$ does not affect the first column of U which is in turn proportional to $(1, \eta_\mu^*, \eta_\mu)^{\rm T}$ in the case under consideration.

We then consider the IO case (i.e. $m_3 = 0$). In this case, only the difference between ρ and σ is of physical meaning. For $\sigma - \rho = 0$, the ranges of $(M_\nu)_{\alpha\beta}$ are found to be

$$\begin{aligned}
&48 \leqslant (M_\nu)_{ee}/\text{meV} \leqslant 50,\\
&-26 \leqslant (M_\nu)_{\mu\tau}/\text{meV} \leqslant -25,\\
&4.9 \leqslant |(M_\nu)_{e\mu}|/\text{meV} \leqslant 5.5,\\
&0.48 \leqslant \arg[(M_\nu)_{e\mu}]/\pi \leqslant 0.49,\\
&24 \leqslant |(M_\nu)_{\mu\mu}|/\text{meV} \leqslant 25,\\
&0.0006 \leqslant \arg[(M_\nu)_{\mu\mu}]/\pi \leqslant 0.0008.
\end{aligned} \tag{21}$$

We see that the relation $|(M_\nu)_{ee}| \sim 2|(M_\nu)_{\mu\mu}| \sim 2|(M_\nu)_{\mu\tau}| \gg |(M_\nu)_{e\mu}|$ holds. And $(M_\nu)_{e\mu}$ is nearly purely imaginary, while $(M_\nu)_{\mu\mu}$ is nearly purely real. These results can be understood from equation (16) by taking $m_3 = 0$. Subsequently, the ranges of the model parameters are obtained as

$$\begin{aligned}
&-1.3 \leqslant a/\text{meV} \leqslant -1.0,\\
&1.0 \leqslant |b|/\text{meV} \leqslant 1.2,\\
&0.98 \leqslant \phi_b/\pi \leqslant 0.99,\\
&4.5 \leqslant |\eta_\mu| \leqslant 4.9,\\
&\phi_\mu/\pi = -\frac{1}{2}.
\end{aligned} \tag{22}$$

It is found that the relation $|a| \sim |b|$ holds. And b is nearly purely real. In particular, η_μ is purely imaginary. These results can be understood with the help of equations (13), (21).

For $\sigma - \rho = \pi/2$, the ranges of $(M_\nu)_{\alpha\beta}$ become

$$\begin{aligned}
&12 \leqslant (M_\nu)_{ee}/\text{meV} \leqslant 22,\\
&6.4 \leqslant (M_\nu)_{\mu\tau}/\text{meV} \leqslant 12,\\
&31 \leqslant |(M_\nu)_{e\mu}|/\text{meV} \leqslant 34,\\
&0.97 \leqslant \arg[(M_\nu)_{e\mu}]/\pi \leqslant 0.99,\\
&9.6 \leqslant |(M_\nu)_{\mu\mu}|/\text{meV} \leqslant 14,\\
&-0.84 \leqslant \arg[(M_\nu)_{\mu\mu}]/\pi \leqslant -0.73.
\end{aligned} \tag{23}$$

In this case, the relation $|(M_\nu)_{e\mu}| \gg |(M_\nu)_{ee}| \sim |(M_\nu)_{\mu\mu}| \sim |(M_\nu)_{\mu\tau}|$ holds. And $(M_\nu)_{e\mu}$ is nearly purely real. Subsequently, the ranges of a and b are obtained as

$$\begin{aligned}
&0.3 \leqslant a/\text{meV} \leqslant 0.6,\\
&6.3 \leqslant |b|/\text{meV} \leqslant 7.7,\\
&-0.53 \leqslant \phi_b/\pi \leqslant -0.51.
\end{aligned} \tag{24}$$

It turns out that $|b|$ is much larger than $|a|$. And b is nearly purely imaginary. For the same reason as mentioned below equation (20), the results of η_μ are completely the same as in equation (22).

3. Summary

To summarize, in this letter we have made an attempt to embed the μ–τ reflection symmetry which predicts maximal atmospherical mixing angle and Dirac CP phase in the FL neutrino model which employs a translational flavor symmetry and keeps one neutrino mass vanishing ($m_1 = 0$ or $m_3 = 0$ in the NO or IO case). We have first formulated such a combination (see M_ν in equation (12)) and then studied its consequences. Such a scenario is highly restrictive and predictive: M_ν only contains three effective parameters (complex parameters b and η_μ, real parameter a); both the lepton flavor mixing parameters (see equation (5)) and neutrino masses are constrained by the flavor symmetries.

We have derived the expressions of the resulting lepton flavor mixing angles and neutrino masses in terms of the model parameters by diagonalizing M_ν' in equation (13), while the expressions of the latter in terms of the former can be derived by comparing M_ν' in equation (13) and M_ν obtained via the reconstruction relation $M_\nu = U D_\nu U^{\rm T}$ (see equation (16)). By varying the lepton flavor mixing angles and neutrino mass squared differences in their 3σ ranges, we have calculated the ranges of $(M_\nu)_{\alpha\beta}$ and subsequently the ranges of the model parameters. The results are summarized as follows.

- In the NO case with $\sigma = 0$, we have $|(M_\nu)_{\mu\mu}| \sim |(M_\nu)_{\mu\tau}| \gg |(M_\nu)_{e\mu}| \gg |(M_\nu)_{ee}|$, $|a| \gg |b|$, $\phi_b \sim \arg[(M_\nu)_{e\mu}]$, and $(M_\nu)_{\mu\mu}$ being nearly purely real. And $|\eta_\mu|$ can take the special value of 1/2 that corresponds to the so-called TM1 mixing.
- In the NO case with $\sigma = \pi/2$, $|(M_\nu)_{ee}|$ becomes comparable to $|(M_\nu)_{e\mu}|$. Note that the results of η_μ are completely same as in the $\sigma = 0$ case.
- In the IO case with $\rho - \sigma = 0$, one arrives at $|(M_\nu)_{ee}| \sim 2|(M_\nu)_{\mu\mu}| \sim 2|(M_\nu)_{\mu\tau}| \gg |(M_\nu)_{e\mu}|$, $|a| \sim |b|$, $(M_\nu)_{e\mu}$ being nearly purely imaginary, and $(M_\nu)_{\mu\mu}$ and b being nearly purely real. In particular, η_μ is purely imaginary.
- In the IO case with $\rho - \sigma = \pi/2$, we have $|(M_\nu)_{e\mu}| \gg |(M_\nu)_{ee}| \sim |(M_\nu)_{\mu\mu}| \sim |(M_\nu)_{\mu\tau}|$, $|b| \gg |a|$, $(M_\nu)_{e\mu}$ being nearly purely real, and b being nearly purely imaginary. And the results of η_μ are completely the same as in the $\rho - \sigma = 0$ case.

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