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Generation function for one-loop tensor reduction

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Abstract

For loop integrals, reduction is the standard method. Having an efficient way to find reduction coefficients is an important topic in scattering amplitudes. In this paper, we present the generation functions of reduction coefficients for general one-loop integrals with an arbitrary tensor rank in their numerator.

Keywords: generation function, tensor reduction, one-loop integral

1. Motivation

As the bridge connecting the theoretical frame and the experiment data, scattering amplitude is always one of the central concepts in the quantum field theory. Its efficient computation (including higher order contributions in perturbation) becomes necessary, especially with the coming of the LHC experiment [1]. The on-shell program[1] in the scattering amplitudes, as the outcome of the challenge, has made the calculations of one-loop amplitude straightforward [10].

In general, the loop computation can be divided into two parts: the construction of integrands and then performing the integration. Although the construction of integrands using Feynman diagrams is well established, sometimes it is not the economic way to do so. Looking for a better way to construct integrands is one of the current research directions[2]. For the second part, i.e. doing the integration, a very useful method is the reduction. It has been shown that any integral can be written as the linear combination of some basis integrals (called the master integrals) with coefficients as the rational function of external momenta, polarization vectors, masses and spacetime dimension. Using the idea of reduction, loop integration can be separated into two parallel tasks: the computation of master integrals and the algorithm to efficiently extract the reduction coefficients. Progresses in any one task will enable us to do more and more complicated integrations (for a nice introduction of recent developments, see [11]).

For the reduction, it can be classified into two categories: the reduction at the integrand level and the reduction at the integral level. The reduction at the integrand level can be systematically solved using computational algebraic geometry [12–15]. For the reduction at the integral level, the first proposal is the famous Passarino-Veltman reduction (PV-reduction) method [16]. There are other proposals, such as the Integration-by-Part (IBP) method [17–23], the unitarity cut method [4, 5, 7, 24–30], and Intersection number [31–36]. Although there are a lot of developments for the reduction at the integral level, it is still desirable to improve them by the current complexity of computations.

In recent papers [37–41] we have introduced the auxiliary vector R to improve the traditional PV-reduction method. Using R we can construct the differential operators and then establish algebraic recurrence relation to determine reduction coefficients analytically. This method has also been generalized to a two-loop sunset diagram (see [40]) where the original PV-reduction method is hard to be used. When using the auxiliary vector R in the IBP method, the efficiency of reduction has also been improved as shown in [42, 43].

Although the advantage of using auxiliary vector R has been demonstrated from various aspects, the algebraic recursive structure makes it still hard to have a general understanding of reduction coefficients for higher and higher tensor ranks in the numerators of integrands. Could we get more understanding of the analytical structures of reduction coefficients by this method? As we will show in this paper, indeed we can get more if we probe the reduction problem

[1] There are many works on this topic. For an introduction, please see the following two books [2, 3]. Some early works with the on-shell concept are [4–9].

[2] For example, the unitarity cut method proposed in [4, 5, 7] uses on-shell tree-level amplitudes as input.

from a new angle. The key idea is the concept of **generation function**. In fact, the generation function is well-known in physics and mathematics. Sometimes the coefficients of a series is hard to imagine, but the series itself is easy to write down. For example, the Hermite Polynomial $\mathrm{H}_n(x)$ can be read out from the generation function

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \mathrm{H}_n(x)\frac{t^n}{n!}. \tag{1.1}$$

Thus we can ask that if we sum the reduction coefficients of different tensor ranks together, could we get a simpler answer? For the reduction problem, the numerator of tensor rank k is given by $(2\ell \cdot R)^k$ in our method and we need to see how to incorporate them together. There are many ways to do this. Two typical ones are

$$\psi_1(t) = \sum_{n=0}^{\infty} t^n(2\ell \cdot R)^n = \frac{1}{1-t(2\ell\cdot R)},$$
$$\psi_2(t) = \sum_{n=0}^{\infty} \frac{(2\ell\cdot R)^n t^n}{n!} = e^{t(2\ell\cdot R)}. \tag{1.2}$$

In this paper, we will focus on the generation function of the type $\psi_2(t)$ because it is invariant under the differential action, i.e. $\frac{de^x}{dx} = e^x$. We will see that the generation functions satisfy simple differential equations, which can be solved analytically.

The plan of the paper is following. In section two, we present the generation function of reduction coefficients of the tadpole integral. The tadpole example is a little bit trivial, thus in section three, we discuss carefully how to find generation functions for the bubble integral, which is the simplest non-trivial example. With the experience obtained for bubble, we present the solution for general one-loop integrals in section four. To demonstrate the frame established in section four, we discuss briefly the triangle example in section five. Finally, a brief summary and discussion are given in section six. Some technical details have been collected in the appendix, where in appendix A, the solution of two typical differential equations is presented, while the solution of the recursive relation for the bubble has been explained in appendix B.

2. Tadpole

With the above brief discussion, we start from the simplest case, i.e. the tadpole topology to discuss the generation function. Summing all tensor ranks properly we have[3]

$$I_{\text{tad}}(t, R) \equiv \int d\ell \frac{e^{t(2\ell\cdot R)}}{\ell^2 - M^2}$$
$$= c_{1\to 1}(t, R, M)\int d\ell\frac{1}{\ell^2 - M^2}, \tag{2.1}$$

where $c_{1\to 1}(t, R, M)$ is the generation function of reduction coefficients and for simplicity, we have defined $\int d\ell_i(\bullet) \equiv \int \frac{d^D\ell_i}{i\pi^{D/2}}(\bullet)$. To find the closed analytic expression for $c_{1\to 1}(t, R, M)$ we establish the corresponding differential equation. Acting with ∂_R we have

$$\frac{\partial}{\partial R}\cdot\frac{\partial}{\partial R}I_{\text{tad}}(t, R) = \int d\ell \frac{4t^2\ell^2 e^{t(2\ell\cdot R)}}{\ell^2 - M^2}$$
$$= 4t^2M^2 I_{\text{tad}}(t, R) \tag{2.2}$$

at one side, and $\left(\frac{\partial}{\partial R}\cdot\frac{\partial}{\partial R}c_{1\to 1}(t, R, M)\right)\int d\ell\frac{1}{\ell^2 - M^2}$ at the other side, thus we get

$$\frac{\partial}{\partial R}\cdot\frac{\partial}{\partial R}c_{1\to 1}(t, R, M) = 4t^2M^2 c_{1\to 1}(t, R, M). \tag{2.3}$$

By the Lorentz invariance, $c_{1\to 1}(t, R, M)$ is the function of $r = R\cdot R$ only, i.e. $c_{1\to 1}(t, R, M) = f(r)$. It is easy to see that the differential equation (2.3) becomes

$$\boxed{4rf'' + 2Df' - 4t^2M^2 f = 0}, \tag{2.4}$$

which is the form (A.14) studied in appendix A. This second order differential equation has two singular points $r = 0$ and $r = \infty$, where the singular point $r = 0$ is canonical. The solution has been given in (A.33). Putting $A = 4$, $B = 2D$, $C = -4t^2M^2$ in (A.29) and the boundary condition $c_0 = 1$, we get immediately

$$c_{1\to 1}(t, R, M) = \sum_{n=0}^{\infty}\frac{(t^2M^2r)^n}{n!\left(\frac{D}{2}\right)_n}$$
$$= {}_0F_1\left(\varnothing; \frac{D}{2}; t^2M^2r\right). \tag{2.5}$$

Before ending this section, let us mention that when we do the reduction for other topologies, we will meet the reduction of $\int d\ell\frac{e^{t(2\ell\cdot R)}}{(\ell - K)^2 - M^2}$. Using the momentum shifting, it is easy to see that

$$\int d\ell\frac{e^{t(2\ell\cdot R)}}{(\ell - K)^2 - M^2} = \int d\ell\frac{e^{t(2(\ell+K)\cdot R)}}{\ell^2 - M^2}$$
$$= e^{2t(K\cdot R)}c_{1\to 1}(t, R, M)\int d\ell\frac{1}{\ell^2 - M^2}. \tag{2.6}$$

The results show the advantage of using the generation function with exponential form.

3. Bubble

Having found the generation function of tadpole reduction, we move to the first nontrivial example, i.e. the generation function of bubble reduction, which is defined through

$$I_{\text{bub}}(t, R) \equiv \int d\ell\frac{e^{t(2\ell\cdot R)}}{(\ell^2 - M_0^2)((\ell - K)^2 - M_1^2)}$$
$$= c_{2\to 2}\int d\ell\frac{1}{(\ell^2 - M_0^2)((\ell - K)^2 - M_1^2)}$$
$$+ c_{2\to 1;\hat{1}}\int d\ell\frac{1}{(\ell^2 - M_0^2)} + c_{2\to 1;\hat{0}}$$
$$\times\int d\ell\frac{1}{((\ell - K)^2 - M_1^2)}. \tag{3.1}$$

[3] The mass dimension of parameter t is -2.

For simplicity, we have not written down the variables of reduction coefficients explicitly. If written explicitly, it will be $c(t,\ R,\ K; M_0,\ M_1)$ or $c(t,\ R^2,\ K\cdot R,\ K^2; M_0,\ M_1)$ if using the Lorentz contraction form.

The generation form (3.1) can produce some nontrivial relations among these generation functions of reduction coefficients easily. Noticing that[4]

$$
\begin{aligned}
I_{\rm bub}(t,\ R) &\equiv \int {\rm d}\ell \frac{{\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)((\ell-K)^2-M_1^2)}\\
&={\rm e}^{t(2K\cdot R)}\int {\rm d}\tilde{\ell}\frac{{\rm e}^{t(2\tilde{\ell}\cdot R)}}{((\tilde{\ell}+K)^2-M_0^2)(\tilde{\ell}^2-M_1^2)}\\
&={\rm e}^{t(2K\cdot R)}\{c_{2\to2}(t,\ R^2,\ -K\cdot R,\ K^2;\ M_1,\ M_0)\\
&\times\int {\rm d}\ell \frac{1}{(\ell^2-M_0^2)((\ell-K)^2-M_1^2)}\\
&+c_{2\to1;\hat{0}}(t,\ R^2,\ -K\cdot R,\ K^2;\ M_1,\ M_0)\int {\rm d}\ell\frac{1}{(\ell^2-M_0^2)}\\
&+c_{2\to1;\hat{1}}(t,\ R^2,\ -K\cdot R,\ K^2;\ M_1,\ M_0)\\
&\times\int {\rm d}\ell\frac{1}{((\ell-K)^2-M_1^2)}\Big\},
\end{aligned}\tag{3.2}
$$

we have

$$
\begin{aligned}
&c_{2\to2}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&={\rm e}^{t(2K\cdot R)}c_{2\to2}(t,\ R^2,\ -K\cdot R,\ K^2;\ M_1,\ M_0),
\end{aligned}\tag{3.3}
$$

$$
\begin{aligned}
&c_{2\to1;\hat{1}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&={\rm e}^{t(2K\cdot R)}c_{2\to1;\hat{0}}(t,\ R^2,-K\cdot R,\ K^2;\ M_1,\ M_0),
\end{aligned}\tag{3.4}
$$

$$
\begin{aligned}
&c_{2\to1;\hat{0}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&={\rm e}^{t(2K\cdot R)}c_{2\to1;\hat{1}}(t,\ R^2,\ -K\cdot R,\ K^2;\ M_1,\ M_0)\quad ,
\end{aligned}\tag{3.5}
$$

by comparing (3.2) with (3.1). The first relation (3.3) can be a consistent check for $c_{2\to2}$ while the second relation (3.4) and the third relation (3.5) tell us that we need to compute only one of $c_{2\to1;\hat{i}}$ functions. Another useful check is the mass dimension. Since the mass dimension of t is (-2), we have

$$
[c_{2\to2}]=0,\qquad [c_{2\to1}]=-2.\tag{3.6}
$$

3.1. Differential equations

Now we will write down differential equations for these generation functions. Acting $\partial_R\cdot\partial_R$ on both sides of (3.1) we have

$$
\begin{aligned}
\partial_R\cdot\partial_R I_{\rm bub}(t,\ R) &= \int {\rm d}\ell\frac{4t^2\ell^2{\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)((\ell-K)^2-M_1^2)}\\
&=4t^2M_0^2I_{\rm bub}(t,\ R)+4t^2{\rm e}^{t(2K\cdot R)}\int {\rm d}\ell\frac{{\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_1^2)},
\end{aligned}\tag{3.7}
$$

thus we derive

$$
\begin{aligned}
&\partial_R\cdot\partial_R c_{2\to2}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=4t^2M_0^2c_{2\to2}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1),
\end{aligned}\tag{3.8}
$$

$$
\begin{aligned}
&\partial_R\cdot\partial_R c_{2\to1;\hat{1}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=4t^2M_0^2c_{2\to1;\hat{1}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1),
\end{aligned}\tag{3.9}
$$

$$
\begin{aligned}
&\partial_R\cdot\partial_R c_{2\to1;\hat{0}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=4t^2M_0^2c_{2\to1;\hat{0}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&+4t^2{\rm e}^{t(2K\cdot R)}c_{1\to1}(t,\ R^2,\ M_1).
\end{aligned}\tag{3.10}
$$

Acting with $K\cdot\partial_R$ we have

$$
\begin{aligned}
K\cdot\partial_R I_{\rm bub}(t,\ R) &= \int {\rm d}\ell\frac{t(2K\cdot\ell){\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)((\ell-K)^2-M_1^2)}\\
&=\int {\rm d}\ell\frac{t(D_0-D_1+f){\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)((\ell-K)^2-M_1^2)}\\
&=tfI_{\rm bub}(t,\ R)-t\int {\rm d}\ell\frac{{\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)}\\
&+t{\rm e}^{t(2K\cdot R)}\int {\rm d}\ell\frac{{\rm e}^{t(2\ell\cdot R)}}{(\ell^2-M_1^2)},
\end{aligned}\tag{3.11}
$$

where $f=K^2-M_1^2+M_0^2$, thus we derive

$$
\begin{aligned}
&K\cdot\partial_R c_{2\to2}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=tfc_{2\to2}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1),
\end{aligned}\tag{3.12}
$$

$$
\begin{aligned}
&K\cdot\partial_R c_{2\to1;\hat{1}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=tfc_{2\to1;\hat{1}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&-tc_{1\to1}(t,\ R^2,\ M_0),
\end{aligned}\tag{3.13}
$$

$$
\begin{aligned}
&K\cdot\partial_R c_{2\to1;\hat{0}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&=tfc_{2\to1;\hat{0}}(t,\ R^2,\ K\cdot R,\ K^2;\ M_0,\ M_1)\\
&+t{\rm e}^{t(2K\cdot R)}c_{1\to1}(t,\ R^2,\ M_1).
\end{aligned}\tag{3.14}
$$

The above two groups of differential equations can be uniformly written as

$$
\partial_R\cdot\partial_R c_T=4t^2M_0^2c_T+4t^2\xi_R h_T,\tag{3.15}
$$

$$
K\cdot\partial_R c_T=tfc_T+t\xi_K h_T,\tag{3.16}
$$

where h_T is the possible non-homogenous contribution coming from lower topology (tadpole). For different type T we have

$$
T=\{2\to2\}:\qquad h_T=0\qquad \text{or}\qquad \xi_R=\xi_K=0,\tag{3.17}
$$

$$
\begin{aligned}
&T=\{2\to1;\ \hat{1}\}:\qquad h_T=c_{1\to1}(t,\ R^2,\ M_0),\\
&\xi_R=0,\qquad \xi_K=-1,
\end{aligned}\tag{3.18}
$$

$$
\begin{aligned}
&T=\{2\to1;\ \hat{0}\}:\qquad h_T=e^{t(2K\cdot R)}c_{1\to1}(t,\ R^2,\ M_1),\\
&\xi_R=1,\qquad \xi_K=+1.
\end{aligned}\tag{3.19}
$$

Generation functions are functions of $f(r,\ p)$ with $r=R^2$ and $p=K\cdot R$. It is easy to work out that

$$
\begin{aligned}
\frac{\partial}{\partial R^\mu}&=2\eta_{\rho\mu}R^\rho\partial_r+K_\mu\partial_p,\\
K\cdot\frac{\partial}{\partial R^\mu}&=2p\partial_r+K^2\partial_p,
\end{aligned}\tag{3.20}
$$

$$
\eta^{\mu\nu}\frac{\partial}{\partial R^\nu}\frac{\partial}{\partial R^\mu}=(4r\partial_r^2+4p\partial_p\partial_r+K^2\partial_p^2+2D\partial_r),\tag{3.21}
$$

[4] Comparing to the shifting symmetry discussed in (3.2), one can also consider the symmetry with $\ell\to-\ell$. For this one, we have $R\to-R$ and $K\to-K$. If using the variables R^2, $K\cdot R$, it is invariant. In other words, the symmetry $\ell\to-\ell$ is trivial.

thus (3.15) and (3.16) can be written as

$$(4r\partial_r^2 + 4p\partial_p\partial_r + K^2\partial_p^2 + 2D\partial_r - 4t^2M_0^2)c_T = 4t^2\xi_R h_T, \tag{3.22}$$

$$(2p\partial_r + K^2\partial_p - tf)c_T = t\xi_K h_T. \tag{3.23}$$

Equations (3.22) and (3.23) are the differential equations we need to solve. We will present two ways to solve them. One is by the series expansion of the naive variables r, p. This is the method used in [37, 38]. However, as we will show, using the idea of the generation function, the powers of r, p are independent of each other, thus the recursion relations become simpler and can be solved explicitly. Another method is to solve the differential equation directly and get a more compact and analytical expression. An important lesson from the second method is that the right variables to do the series expansion are not r, p but their proper combination.

3.2. The series expansion

In this subsection, we will present the solution in the form of a series expansion of r, p. Writing[5]

$$c = \sum_{n,m=0}^{\infty} c_{n,m}r^np^m, \qquad h_c = \sum_{n,m=0}^{\infty} h_{n,m}r^np^m, \tag{3.24}$$

and putting them to (3.22) and (3.23) we get the following equations

$$\begin{aligned}0 = &2(n+1)(2n+2m+D)c_{n+1,m}\\ &+K^2(m+2)(m+1)c_{n,m+2}\\ &-4t^2M_0^2c_{n,m} - 4t^2\xi_R h_{n,m} \quad n, m \geqslant 0,\end{aligned} \tag{3.25}$$

$$\begin{aligned}0 = &2(n+1)c_{n+1,m} + K^2(m+2)c_{n,m+2} - tfc_{n,m+1}\\ &- t\xi_K h_{n,m+1} \qquad m, n \geqslant 0,\end{aligned} \tag{3.26}$$

$$0 = K^2c_{n,1} - tfc_{n,0} - t\xi_K h_{n,0} \qquad n \geqslant 0. \tag{3.27}$$

Using (3.25) and (3.26) we can solve

$$\begin{aligned}c_{n+1,m} = &\frac{4M_0^2t^2}{2(n+1)(D+2n+m-1)}c_{n,m}\\ &-\frac{tf(m+1)}{2(n+1)(D+2n+m-1)}c_{n,m+1}\\ &+\frac{4t^2\xi_R h_{n,m} - t\xi_K(m+1)h_{n,m+1}}{2(n+1)(D+2n+m-1)},\end{aligned} \tag{3.28}$$

$$\begin{aligned}c_{n,m+2} = &\frac{-4M_0^2t^2}{(m+2)(D+2n+m-1)K^2}c_{n,m}\\ &+\frac{tf(D+2n+2m)}{(m+2)(D+2n+m-1)K^2}c_{n,m+1}\\ &+\frac{-4t^2\xi_R h_{n,m} + t\xi_K(D+2n+2m)h_{n,m+1}}{2(n+1)(D+2n+m-1)}.\end{aligned} \tag{3.29}$$

[5] Since we consider the generation functions, the n, m are free indices, while in previous works [37, 38] with fixed tensor rank K, n, m are constrained by $2n + m = K$. One can see that many manipulations are simplified using the idea of generation functions.

Now setting $m = 0$ in (3.28) and combining (3.27) we can solve

$$\begin{aligned}c_{n+1,0} = &\frac{(-t^2f^2 + 4M_0^2t^2K^2)}{2(n+1)(D+2n-1)K^2}c_{n,0}\\ &-\frac{t^2(\xi_K f - 4\xi_R K^2)h_{n,0} + t\xi_K K^2 h_{n,1}}{2(n+1)(D+2n-1)K^2},\end{aligned} \tag{3.30}$$

$$c_{n,1} = \frac{tf}{K^2}c_{n,0} + \frac{t\xi_K h_{n,0}}{K^2}. \tag{3.31}$$

Using equation (3.30) we can recursively solve all $c_{n,0}$ starting from the boundary condition $c_{0,0} = 1$ for $c_{2\to 2}$ or $c_{0,0} = 0$ for $c_{2\to 1}$. Knowing all $c_{n,0}$ we can use (3.31) to get all $c_{n,1}$. To solve all $c_{n,m}$, we use (3.25) and (3.26) again, but now solve

$$\begin{aligned}c_{n,m+1} = &\frac{4M_0^2t}{f(m+1)}c_{n,m}\\ &-\frac{2(n+1)(D+2n+m-1)}{tf(m+1)}c_{n+1,m}\\ &+4t\xi_R h_{n,m} - \xi_K(m+1)h_{n,m+1},\end{aligned} \tag{3.32}$$

$$\begin{aligned}c_{n,m+2} = &\frac{-2(n+1)(D+2n+2m)}{(m+1)(m+2)K^2}c_{n+1,m}\\ &+\frac{4t^2M_0^2}{(m+1)(m+2)K^2}c_{n,m}\\ &+\frac{4t^2\xi_R h_{n,m}}{(m+1)(m+2)K^2}.\end{aligned} \tag{3.33}$$

Both equations can be used recursively to solve $c_{n,m}$. After using one of them to get all $c_{n,m}$, another one becomes a nontrivial consistent check. Among them (3.32) is better, since it solves $(m + 1)$ from m.

Using above algorithm, we present the first few terms of generation functions for comparison. For $c_{2\to 2}$ we have

$$\begin{aligned}c_{2,2} = &1 + \frac{ft}{K^2}p + \frac{(Df^2 - 4K^2M_0^2)t^2}{2(D-1)(K^2)^2}p^2\\ &+\frac{(4K^2M_0^2 - f^2)t^2}{2(D-1)K^2}r\\ &+\frac{f((2+D)f^2 - 12K^2M_0^2)t^3}{6(D-1)(K^2)^3}p^3\\ &-\frac{f(f^2 - 4K^2M_0^2)t^3}{2(D-1)(K^2)^2}rp + \ldots\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}c_{2\to 1;\hat{1}} = &0 - \frac{t}{K^2}p - \frac{Dft^2}{2(D-1)(K^2)^2}p^2 + \frac{ft^2}{2(D-1)K^2}r\\ &-\frac{(D(2+D)f^2 - 8(D-1)K^2M_0^2)t^3}{6(D-1)D(K^2)^3}p^3\\ &+\frac{(Df^2 - 4(D-1)K^2M_0^2)t^3}{2(D-1)D(K^2)^2}rp+\end{aligned} \tag{3.35}$$

and

$$
\begin{aligned}
c_{2\to 1;\hat{0}} &= 0 + \frac{t}{K^2}p + \frac{(-D\widetilde{f} + 4(D-1)K^2)t^2}{2(D-1)(K^2)^2}p^2 \\
&+ \frac{\widetilde{f}\,t^2}{2(D-1)K^2}r + \frac{(D\widetilde{f}\widetilde{f} + 4(D-1)K^2M_1^2)t^3}{2(D-1)D(K^2)^2}rp \\
&+ \frac{(6DK^2(K^2(D-2) + DM_0^2) - 2(D+2)(3D-2)K^2M_1^2 + D(D+2)\widetilde{f}^2)t^3}{6(D-1)(K^2)^2}p^3 + \ldots,
\end{aligned} \tag{3.36}
$$

where we have defined $\widetilde{f} = K^2 - M_0^2 + M_1^2 = -f + 2K^2$.

Here we have presented the general recursive algorithm. In appendix B, we will show that these recursion relations can be solved explicitly, i.e. we find explicit expressions for all coefficients $c_{n,m}$.

3.3. The analytic solution

In the previous section, we presented the solution using the series expansion. In this section, we will solve the two differential equations (3.22) and (3.23) directly.

Let us start from (3.23) first. To solve it, we define the following new variables

$$
x = K^2 r - p^2, \qquad y = p, \tag{3.37}
$$

then (3.23) becomes

$$
\begin{aligned}
&(2p\partial_r + K^2\partial_p - tf)c(r,p) = (K^2\partial_y - tf)c(x,y) \\
&= t\xi_K h_T,
\end{aligned} \tag{3.38}
$$

where we have used

$$
\begin{aligned}
&p = y, \qquad r = \frac{x + y^2}{K^2}, \qquad \partial_r = K^2\partial_x, \\
&\partial_p = -2y\partial_x + \partial_y.
\end{aligned} \tag{3.39}
$$

The differential equation (3.38) can be solved as (see the discussion in the appendix, for example, (A.5))

$$
\begin{aligned}
c(x,y) = \frac{1}{K^2}e^{\frac{tf}{K^2}y}\Big(&G(x) \\
&+ \int_0^y dw e^{-\frac{tf}{K^2}w}t\xi_K h_T(x,w)\Big),
\end{aligned} \tag{3.40}
$$

where the function G depends only on x, while $h_T \equiv h_T(x,y)$ is the function of both x, y.

Now we consider the equation (3.22). The first step is to simplify it by writing

$$
\begin{aligned}
&(4r\partial_r^2 + 4p\partial_p\partial_r + K^2\partial_p^2 + 2D\partial_r) \\
&= \frac{1}{K^2}(2p\partial_r + K^2\partial_p)(2p\partial_r + K^2\partial_p) \\
&+ \left(4r - \frac{4p^2}{K^2}\right)\partial_r^2 + 2(D-1)\partial_r.
\end{aligned} \tag{3.41}
$$

Thus using (3.23), (3.22) becomes

$$
\begin{aligned}
&\left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \frac{t^2(f^2 - 4K^2M_0^2)}{K^2}\right)c \\
&= \left(-t\xi_K\partial_y + \frac{4t^2\xi_R K^2 - t^2\xi_K f}{K^2}\right)h_T.
\end{aligned} \tag{3.42}
$$

Putting (3.40) to (3.42) and simplifying we get

$$
\begin{aligned}
&\left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \tfrac{t^2(f^2 - 4K^2M_0^2)}{K^2}\right)G(x) \\
&= K^2 e^{-\frac{tf}{K^2}y}\left(-t\xi_K\partial_y + \tfrac{4t^2\xi_R K^2 - t^2\xi_K f}{K^2}\right)h_T(x,y) \\
&- \left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \tfrac{t^2(f^2 - 4K^2M_0^2)}{K^2}\right) \\
&\times \int_0^y dw e^{-\frac{tf}{K^2}w}t\xi_K h_T(x,w).
\end{aligned} \tag{3.43}
$$

Equation (3.43) is the form of (A.14) which has been discussed in the appendix. One interesting point is that since the left-hand side is independent of y, the right-hand side should be zero under the action of ∂_y. One can check that it is indeed true.

Having laid out the frame, we can use it to solve various generation functions.

3.3.1. The generation function $c_{2\to 2}$. For this case, we have $h_T = 0$, thus using the result (A.29) we can immediately write down

$$
\begin{aligned}
c_{2\to 2}(t, r, p, K^2; M_0, M_1) = {}_0F_1\Big(&\varnothing; \tfrac{D-1}{2}; \\
&\left(\tfrac{t^2(4K^2M_0^2 - f^2)x}{4(K^2)^2}\right)e^{\frac{tf}{K^2}y}\big|_{x\to K^2r - p^2, y\to p}\Big).
\end{aligned} \tag{3.44}
$$

One can check it with the series expansion (B.3) given in appendix B. Compared to it, the result (3.44) is very simple and compact. This shows the power of using the generation function. Also, the differential equations (3.22) and (3.23) tell us the right variables for the series expansion should be x, y instead of the naive variables r, p.

3.3.2. The generation function $c_{2\to 1;\hat{1}}$. For this case, we have $\xi_R = 0$, $\xi_K = -1$ and

$$h_T(r) = c_{1\to 1}\left(t,\, r = \frac{x+y^2}{K^2},\, M_0\right) = \sum_{n=0}^{\infty} \frac{(t^2M_0^2)^n\left(\frac{x+y^2}{K^2}\right)^n}{n!\left(\frac{D}{2}\right)_n}, \tag{3.45}$$

which satisfies the differential equation (2.4). The (3.43) becomes

$$\begin{aligned}&\left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \frac{t^2(f^2-4K^2M_0^2)}{K^2}\right)G(x)\\ &= K^2 e^{-\frac{tf}{K^2}y} t\left(\partial_y + \frac{tf}{K^2}\right)h_T(x,y)\\ &+ t\left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \frac{t^2(f^2-4K^2M_0^2)}{K^2}\right)\\ &\times\int_0^y dw e^{-\frac{tf}{K^2}w} h_T(x,w).\end{aligned} \tag{3.46}$$

The first important check is that the right-hand side of (3.46) is y-independent. Acting $\frac{\partial}{\partial y}$ on the right-hand side we will get

$$\begin{aligned}te^{-\frac{tf}{K^2}y}\Bigg\{&-tf\left(\partial_y + \frac{tf}{K^2}\right)h_T(x,y)\\ &+K^2\left(\partial_y^2 + \frac{tf}{K^2}\partial_y\right)h_T(x,y)\\ &+\Bigg(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x\\ &+\frac{t^2(f^2-4K^2M_0^2)}{K^2}\Bigg)h_T(x,y)\Bigg\}.\end{aligned} \tag{3.47}$$

Using

$$\begin{aligned}\partial_x h_T(r) &= \frac{\partial\frac{x+y^2}{K^2}}{\partial x}\partial_r h_T = \frac{1}{K^2}\partial_r h_T,\\ \partial_y h_T(r) &= \frac{\partial\frac{x+y^2}{K^2}}{\partial y}\partial_r h_T = \frac{2y}{K^2}\partial_r h_T,\end{aligned} \tag{3.48}$$

one can check that (3.47) is reduced to the differential equation (2.4), thus we have proved the y-independent of (3.46).

Setting $y = 0$ in (3.46) we get

$$\left(4xK^2\partial_x^2 + 2(D-1)K^2\partial_x + \frac{t^2(f^2-4K^2M_0^2)}{K^2}\right)G(x) = t^2 f h_T(x, y=0), \tag{3.49}$$

where we have used $\partial_y h_T(x, y=0) = \frac{2y}{K^2}\partial_r h_T = 0$. The differential equation (3.49) is the form of (A.14) and we get the solution

$$\begin{aligned}G(x) &= \frac{t^2 f}{4K^2}G_0(x)\int_0^x dw w^{-\frac{(D-1)}{2}} e^{-2G_0(w)}\\ &\times\int_0^w d\xi h_T(\xi, y=0) G_0^{-1}(\xi) e^{2G_0(\xi)} \xi^{\frac{(D-1)}{2}-1},\end{aligned} \tag{3.50}$$

where

$$G_0(x) = {}_0F_1\left(\varnothing; \frac{(D-1)}{2}; \frac{t^2(4K^2M_0^2-f^2)}{4(K^2)^2}x\right). \tag{3.51}$$

Putting it all together we finally have

$$c_{2\to 1;\hat{1}}(x,y) = \frac{1}{K^2}e^{\frac{tf}{K^2}y}\left(G(x) - t\int_0^y dw e^{-\frac{tf}{K^2}w} h_T(x,w)\right). \tag{3.52}$$

Although we have a very compact expression (3.52) for the generation function, in practice it is more desirable to have the series expansion form. In appendix A, we have introduced three ways. Here we work out the expansion by direct integration. Using (3.45) we have

$$\int_0^y dw e^{-\frac{tf}{K^2}w} h_T(x,w) = \sum_{n=0}^{\infty}\frac{(t^2M_0^2)^n}{n!\left(\frac{D}{2}\right)_n} \times\int_0^y dw e^{-\frac{tf}{K^2}w}\left(\frac{x+w^2}{K^2}\right)^n. \tag{3.53}$$

To work out the integration, we see that

$$R(\alpha) \equiv \int_0^T du e^{\alpha u} = \frac{1}{\alpha}e^{\alpha u}\Big|_0^T = \frac{e^{\alpha T}-1}{\alpha}, \tag{3.54}$$

thus

$$\begin{aligned}\frac{d^n}{d\alpha^n}R(\alpha) &= \int_0^T du e^{\alpha u}u^n = \frac{(-)^n n!}{\alpha^{n+1}}\left(e^{\alpha T}\lfloor e^{-\alpha T}\rfloor_{\alpha^n} - 1\right)\\ &= \frac{(-)^n n!}{\alpha^{n+1}}\left(e^{\alpha T}\left(e^{-\alpha T} - \sum_{i=n+1}^{\infty}\frac{(-\alpha T)^i}{i!}\right) - 1\right)\\ &= \frac{-(-)^n n!}{\alpha^{n+1}}e^{\alpha T}\sum_{i=n+1}^{\infty}\frac{(-\alpha T)^i}{i!},\end{aligned} \tag{3.55}$$

where the symbol $\lfloor Y(x)\rfloor_{x^{n-1}}$ means to keep the Taylor expansion up to the order of $x^{(n-1)}$. Using (3.55) we have

$$\begin{aligned}&\int_0^T du e^{\alpha u}(\beta+\gamma u^2)^N\\ &= \sum_{i=0}^{N}\frac{N!}{i!(N-i)!}\beta^{N-i}\gamma^i\int_0^T du e^{\alpha u}u^{2i}\\ &= \sum_{i=0}^{N}\frac{N!}{i!(N-i)!}\beta^{N-i}\gamma^i\frac{-(2i)!}{\alpha^{2i+1}}e^{\alpha T}\\ &\times\sum_{j=2i+1}^{\infty}\frac{(-\alpha T)^j}{j!}.\end{aligned} \tag{3.56}$$

Using (3.56) we can evaluate (3.53) as

$$\int_0^y \mathrm{d}w \mathrm{e}^{-\frac{tf}{K^2}w} h_T(x, w) = y\mathrm{e}^{-\frac{tf}{K^2}y} \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{n} \times \frac{\left(\frac{t^2M_0^2}{K^2}\right)^n}{\left(\frac{D}{2}\right)_n} \frac{(2i)!x^{n-i}y^{2i}}{i!(n-i)!} \frac{\left(\frac{tfy}{K^2}\right)^j}{(j+2i+1)!}. \tag{3.57}$$

The evaluation of $G(x)$ can be found in (A.23) as

$$G(x) = \sum_{n=0}^{\infty}\sum_{i=0}^{n-1} \frac{\left(\frac{D-1}{2}\right)_i}{n!\left(\frac{D-1}{2}\right)_n \left(\frac{D}{2}\right)_i} \times \frac{f(M_0^2)^i(t^2x)^n(4K^2M_0^2 - f^2)^{n-i-1}}{4^{n-i}(K^2)^{2n-i-1}}. \tag{3.58}$$

Collecting all pieces together, we finally have

$$\begin{aligned} c_{2\to 1;\hat{1}}(t, r, p, K^2; M_0, M_1) &= \frac{1}{K^2}\mathrm{e}^{\frac{tf}{K^2}y}\sum_{n=0}^{\infty}\sum_{i=0}^{n-1} \\ &\times \frac{\left(\frac{D-1}{2}\right)_i}{n!\left(\frac{D-1}{2}\right)_n\left(\frac{D}{2}\right)_i} \frac{f(M_0^2)^i(t^2x)^n(4K^2M_0^2-f^2)^{n-i-1}}{4^{n-i}(K^2)^{2n-i-1}} \\ &- \frac{ty}{K^2}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{n} \frac{\left(\frac{t^2M_0^2}{K^2}\right)^n}{\left(\frac{D}{2}\right)_n} \\ &\times \frac{(2i)!x^{n-i}y^{2i}}{i!(n-i)!}\frac{\left(\frac{tfy}{K^2}\right)^j}{(j+2i+1)!}. \end{aligned} \tag{3.59}$$

This result can be checked with the one given in (B.14). One can see that the formula (3.59) is much more compact and manifest with various analytic structures.

4. The general frame

Having the detailed computations in the bubble, in this section we will set up the general frame to find generation functions for general one-loop integrals with $(n+1)$ propagators. The system has n external momenta K_i, $i=1,\ldots,n$ and $(n+1)$ masses M_j^2, $j=0, 1,\ldots,n$. Using the auxiliary vector R we have $(n+1)$ **auxiliary scalar products (ASP)** $r=R\cdot R$, $p_i=K_i\cdot R$, $i=1,\ldots,n$. From the experience in the bubble, we know that these ASPs are not good variables to solve differential equations produced by $\partial_R\cdot\partial_R$ and $K_i\cdot\partial_R$, $i=1,\ldots,n$. Thus we will discuss how to find these good variables in the first subsection. Then we discuss the differential equations in these new variables in the second subsection and finally their solutions in the third section.

4.1. Finding good variables

We will denote the good variables by x and y_i, $i=1,\ldots,n$. To simplify the differential equations, we need to impose the following conditions

$$(K_i\cdot\partial_R)x = 0, \qquad (K_i\cdot\partial_R)y_j \sim \delta_{ij}, \qquad \forall\, i, j = 1,\ldots,n. \tag{4.1}$$

To see if there is indeed a solution for (4.1), let us define the Gram matrix G and the row vector P^T as

$$G_{ij} = K_i\cdot K_j, \qquad (P^T)_i = K_i\cdot R. \tag{4.2}$$

Putting $y_j=\sum_t\beta_{jt}p_t$ to (4.1), it is easy to see that the condition becomes

$$K_i\cdot\partial_R y_j = \sum_t \beta_{jt}(K_t\cdot K_i) \sim \delta_{ij}, \tag{4.3}$$

thus the matrix β can be solved as

$$\beta = |G|G^{-1}, \tag{4.4}$$

where $|G|$ is the Gram determinant. For x, let us assume

$$x = |G|r + P^TAP, \qquad A^T = A. \tag{4.5}$$

Since

$$K_i\cdot\partial_R x = |G|2p_i + 2(P^T)_{R\to K_i}AP, \tag{4.6}$$

where $P_{R\to K_i}$ means to replace vector R by the vector K_i, when collecting all of i together, the right-hand side of (4.6) is just $(2|G|I+2GA)P$, thus we have the solution

$$A = -|G|G^{-1}. \tag{4.7}$$

Putting everything together, we finally have

$$x = |G|(r - P^TG^{-1}P), \qquad Y = |G|G^{-1}P, \tag{4.8}$$

where the mass dimensions of various quantities are

$$[|G|] = 2n, \quad [(G^{-1})_{ij}] = -2, \quad [A_{ij}] = 2(n-1), \quad [x] = 2(n+1), \quad [y_i] = 2n. \tag{4.9}$$

From (4.8) we can solve

$$P = \frac{1}{|G|}GY, \qquad r = \frac{|G|x + Y^TGY}{|G|^2}. \tag{4.10}$$

4.2. The differential equations

Having found the good variables, we express differential operators $\partial_R\cdot\partial_R$ and $K_i\cdot\partial_R$, $i=1,\ldots,n$ using them. The first step is to use (4.8) to write

$$\frac{\partial}{\partial R^\mu} = (2|G|R_\mu - 2|G|\mathcal{K}_\mu^TG^{-1}P)\partial_x + |G|\mathcal{K}_\mu^TG^{-1}\partial_Y, \tag{4.11}$$

where we have defined $\mathcal{K}^T=(K_1,\ldots,K_n)$ and $\partial_Y^T=(\partial_{y_1},\ldots,\partial_{y_n})$. Thus we find

$$\mathcal{K}\cdot\frac{\partial}{\partial R^\mu} = |G|\partial_Y, \qquad \partial_R\cdot\partial_R = 2|G|(D-n)\partial_x + 4|G|x\partial_x^2 + |G|^2\partial_Y^TG^{-1}\partial_Y. \tag{4.12}$$

The differential equations for c_T, where T denotes different types of generation functions, have the following pattern

$$K_i \cdot \partial_R c_T = \alpha_i c_T + H_{T;i}, \quad i = 1, 2, \ldots, n, \tag{4.13}$$

$$\partial_R \cdot \partial_R c_T = \alpha_R c_T + H_{T;R}, \tag{4.14}$$

where α_R, α_i are constant (which are independent of T) and $H_{T;R}$, $H_{T;i}$ are known functions coming from lower topologies. Using the result (4.12) and (4.13), we find

$$|G|^2 \partial_Y^T G^{-1} \partial_Y c_T = \alpha_{\mathcal{K}}^T G^{-1} \alpha_{\mathcal{K}} c_T + H_{T;\mathcal{K}}^T G^{-1} \alpha_{\mathcal{K}} + |G| \partial_Y^T G^{-1} H_{T;\mathcal{K}}, \tag{4.15}$$

where $\alpha_{\mathcal{K}}^T = (\alpha_1, \ldots, \alpha_n)$ and $H_{T;\mathcal{K}}^T = (H_{T;1}, \ldots, H_{T;n})$. Thus the differential equations (4.13) can be written as

$$\left(\partial_{y_i} - \frac{\alpha_i}{|G|}\right) c_T = \frac{1}{|G|} H_{T;i} \quad i = 1, 2, \ldots, n, \tag{4.16}$$

while (4.14) becomes

$$(4|G|x\partial_x^2 + 2|G|(D-n)\partial_x + \widetilde{\alpha}_R) c_T = \mathcal{H}_{T;R} \tag{4.17}$$

with

$$\widetilde{\alpha}_R = \alpha_{\mathcal{K}}^T G^{-1} \alpha_{\mathcal{K}} - \alpha_R,$$
$$\mathcal{H}_{T;R} = -H_{T;\mathcal{K}}^T G^{-1} \alpha_{\mathcal{K}} - |G| \partial_Y^T G^{-1} H_{T;\mathcal{K}} + H_{T;R}. \tag{4.18}$$

Having given the differential equations (4.16) and (4.17), there is an important point to be mentioned. For (4.16) and (4.17) to have a solution, functions H are not arbitrary but must satisfy the **integrability conditions**, which are

$$\left(\partial_{y_j} - \frac{\alpha_j}{|G|}\right) H_{T;i} = \left(\partial_{y_i} - \frac{\alpha_i}{|G|}\right) H_{T;j}, \quad \forall\, i, j = 1, 2, \ldots, n, \tag{4.19}$$

and

$$(4|G|x\partial_x^2 + 2|G|(D-n)\partial_x + \widetilde{\alpha}_R) \frac{1}{|G|} H_{T;i} = \left(\partial_{y_i} - \frac{\alpha_i}{|G|}\right) \mathcal{H}_{T;R}, \quad \forall\, i = 1, 2, \ldots, n. \tag{4.20}$$

Differential equations (4.16) and (4.17) are the type of (A.1) and (A.14) respectively in appendix A, for which the solution has been presented. In the next subsection, we will solve them analytically.

4.3. Analytic solution

In this part, we will present the necessary steps for solving the above differential equations (4.16) and (4.17). Let us solve them one by one. For differential equation (4.16) with $i = 1$, using the result (A.5) in appendix A, we have

$$c_T(x, y) = e^{\frac{\alpha_1}{|G|} y_1} \left(F_T(x, y_2, \ldots, y_n) + \frac{1}{|G|} \times \int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|} w_1} H_{T;1}(x, w_1, y_2, \ldots, y_n) \right), \tag{4.21}$$

where $F_T(x, y_2, \ldots, y_n)$ does not depend on the variable y_1. Now we act with $\left(\partial_{y_2} - \frac{\alpha_2}{|G|}\right)$ on both sides of (4.21) to get the differential equation

$$\left(\partial_{y_2} - \frac{\alpha_2}{|G|}\right) F_T(x, y_2, \ldots, y_n) = -\frac{1}{|G|} \times \int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|} w_1} \left(\partial_{y_2} - \frac{\alpha_2}{|G|}\right) H_{T;1}(x, w_1, y_2, \ldots, y_n) + e^{-\frac{\alpha_1}{|G|} y_1} \frac{1}{|G|} H_{T;2}(x, y_1, y_2, \ldots, y_n). \tag{4.22}$$

Using (A.5) we find

$$F_T(x, y_2, \ldots, y_n) = e^{\frac{\alpha_2}{|G|} y_2} F_T(x, y_3, \ldots, y_n) + e^{\frac{\alpha_2}{|G|} y_2} e^{-\frac{\alpha_1}{|G|} y_1} \frac{1}{|G|} \times \int_0^{y_2} dw_2 e^{-\frac{\alpha_2}{|G|} w_2} H_{T;2}(x, y_1, w_2, \ldots, y_n) - e^{\frac{\alpha_2}{|G|} y_2} \frac{1}{|G|} \int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|} w_1} (e^{-\frac{\alpha_2}{|G|} y_2} H_{T;1}(x, w_1, y_2, \ldots, y_n) - H_{T;1}(x, w_1, y_2 = 0, \ldots, y_n)). \tag{4.23}$$

Putting (4.23) back to (4.21) and doing some algebraic manipulations, we get

$$c_T(x, y) = e^{\frac{\alpha_1}{|G|} y_1} e^{\frac{\alpha_2}{|G|} y_2} F_T(x, y_3, \ldots, y_n) + e^{\frac{\alpha_2}{|G|} y_2} \frac{1}{|G|} \int_0^{y_2} dw_2 e^{-\frac{\alpha_2}{|G|} w_2} H_{T;2}(x, y_1, w_2, \ldots, y_n) + e^{\frac{\alpha_1}{|G|} y_1} e^{\frac{\alpha_2}{|G|} y_2} \frac{1}{|G|} \int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|} w_1} H_{T;1} \times (x, w_1, y_2 = 0, \ldots, y_n). \tag{4.24}$$

Repeating above procedure with the action $\left(\partial_{y_2} - \frac{\alpha_2}{|G|}\right)$ we can solve $F_T(x, y_3, \ldots, y_n)$ and then find

$$c_T(x, y) = e^{\frac{\alpha_1}{|G|} y_1} e^{\frac{\alpha_2}{|G|} y_2} e^{\frac{\alpha_3}{|G|} y_3} F_T(x, y_4, \ldots, y_n) + e^{\frac{\alpha_3}{|G|} y_3} \frac{1}{|G|} \int_0^{y_3} dw_3 e^{-\frac{\alpha_3}{|G|} w_3} H_{T;3}(x, y_1, y_2, w_3, \ldots, y_n) + e^{\frac{\alpha_2}{|G|} y_2} e^{\frac{\alpha_3}{|G|} y_3} \frac{1}{|G|} \int_0^{y_2} dw_2 e^{-\frac{\alpha_2}{|G|} w_2} H_{T;2} \times (x, y_1, w_2, y_3 = 0, \ldots, y_n) + e^{\frac{\alpha_1}{|G|} y_1} e^{\frac{\alpha_2}{|G|} y_2} e^{\frac{\alpha_3}{|G|} y_3} \frac{1}{|G|} \int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|} w_1} H_{T;1} \times (x, w_1, y_2 = 0, y_3 = 0, \ldots, y_n). \tag{4.25}$$

By checking (4.24) and (4.25) we can see that after solving n first order differential equations (4.16) we get

$$c_T(x, y_1, \ldots, y_n) = e^{\frac{\sum_{i=1}^n \alpha_i y_i}{|G|}} F(x) + \mathcal{H}_{T;K} \tag{4.26}$$

with

$$\mathcal{H}_{T;K} = \frac{1}{|G|} \sum\nolimits^{\sim}{}_{i=n}^{1} e^{\frac{\sum^{\sim}{}_{j=n}^{i} \alpha_j y_j}{|G|}} \times \int_0^{y_i} dw_i e^{\frac{-\alpha_i w_i}{|G|}} H_{T;i}(x, y_1, \ldots, y_{i-1}, w_i, 0, \ldots, 0), \tag{4.27}$$

where for simplicity we have defined the sum $\sum\nolimits^{\sim}{}_{i=a}^{b}$ to mean to take the sum over $(a, a-1, a-2, \ldots, b)$ with $a \geqslant b$.

Before going to solve the only unknown function $F(x)$, let us check that the form (4.26) does satisfy the differential equations (4.16). When acting with $\left(\partial_{y_k}-\frac{\alpha_k}{|G|}\right)$ on the both sides, it is easy to see that the first term at the right-hand side of (4.26) and the terms in $\mathcal{H}_{T;K}$ with $i<k$ give zero contributions since they contain only the factor $e^{\frac{-\alpha_k y_k}{|G|}}$ depending on y_k. For the term $i=k$ in $\mathcal{H}_{T;K}$, the action gives

$$\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+1}\alpha_j y_j}{|G|}}H_{T;k}(x,\,y_1,\ldots,y_{k-1},\,y_k,\,0,\ldots,0). \tag{4.28}$$

For the term $i=k+1$ in $\mathcal{H}_{T;K}$, the action gives

$$\begin{aligned}&\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+1}\alpha_j y_j}{|G|}}\int_0^{y_{k+1}}\mathrm{d}w_{k+1}e^{\frac{-\alpha_{k+1}w_{k+1}}{|G|}}\\&\times\left(\partial_{y_k}-\frac{\alpha_k}{|G|}\right)H_{T;k+1}(x,\,y_1,\ldots,y_k,\,w_{k+1},\,0,\ldots,0)\\&=\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+1}\alpha_j y_j}{|G|}}\int_0^{y_{k+1}}\mathrm{d}w_{k+1}e^{\frac{-\alpha_{k+1}w_{k+1}}{|G|}}\\&\times\left(\partial_{w_{k+1}}-\frac{\alpha_{k+1}}{|G|}\right)H_{T;k}(x,\,y_1,\ldots,y_k,\,w_{k+1},\,0,\ldots,0),\end{aligned} \tag{4.29}$$

where in the second line we have used the integrability condition (4.19). After partial integration we get

$$\begin{aligned}&-\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+1}\alpha_j y_j}{|G|}}H_{T;k}(x,\,y_1,\ldots,y_{k-1},\,y_k,\,0,\ldots,0)\\&+\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+2}\alpha_j y_j}{|G|}}H_{T;k}(x,\,y_1,\ldots,y_k,\,y_{k+1},\,0,\ldots,0).\end{aligned} \tag{4.30}$$

The first term in (4.30) cancels the term in (4.28) and we are left with the second term in (4.30). Now the pattern is clear. The $i=k+2$ term in $\mathcal{H}_{T;K}$ will produce two terms after using the integrability condition and partial integration, the first term will cancel the second term in (4.30), while the second term will be the form

$$\frac{1}{|G|}e^{\frac{\Sigma\sim_{j=n}^{k+3}\alpha_j y_j}{|G|}}H_{T;k}(x,\,y_1,\ldots,y_k,\,y_{k+1},\,y_{k+2},\,0,\ldots,0). \tag{4.31}$$

Continuing to the term $i=n$ in $\mathcal{H}_{T;K}$ we will be left with $\frac{1}{|G|}H_{T;k}(x,\,y_1,\ldots,y_n)$, thus we have proved that (4.26) does satisfy the differential equations (4.16).

Now we consider the differential equation (4.17). Using the form (4.26), we derive

$$\begin{aligned}&(4|G|x\partial_x^2+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)F(x)\\&=e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}\\&\times(\mathcal{H}_{T;R}-(4|G|x\partial_x^2+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)\mathcal{H}_{T;K}).\end{aligned} \tag{4.32}$$

One important point of (4.32) is that the right-hand side must be y_i-independent. To check this point, we act ∂_{y_k} on the right-hand side to give

$$\begin{aligned}&-\frac{\alpha_k}{|G|}e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}(\mathcal{H}_{T;R}-(4|G|x\partial_x^2\\&+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)\mathcal{H}_{T;K})\\&+e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}(\partial_{y_k}\mathcal{H}_{T;R}-(4|G|x\partial_x^2\\&+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)\partial_{y_k}\mathcal{H}_{T;K}).\end{aligned} \tag{4.33}$$

Since we have proved

$$\left(\partial_{y_k}-\frac{\alpha_k}{|G|}\right)\mathcal{H}_{T;K}=\frac{1}{|G|}H_{T;k}, \tag{4.34}$$

(4.33) is simplified to

$$\begin{aligned}&-\frac{\alpha_k}{|G|}e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}\mathcal{H}_{T;R}+e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}\\&\times(\partial_{y_k}\mathcal{H}_{T;R}-(4|G|x\partial_x^2+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)\\&\times\frac{1}{|G|}H_{T;k}\Big).\end{aligned} \tag{4.35}$$

Using the integrability condition (4.20) we get

$$\begin{aligned}&-\frac{\alpha_k}{|G|}e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}\mathcal{H}_{T;R}+e^{\frac{-\Sigma_{i=1}^n\alpha_i y_i}{|G|}}\\&\times\left(\partial_{y_k}\mathcal{H}_{T;R}-\left(\partial_{y_k}-\frac{\alpha_k}{|G|}\right)\mathcal{H}_{T;R}\right)=0.\end{aligned} \tag{4.36}$$

Having checked the y-independent, we can take y_i to be any values at the right-hand side of (4.32). From the expression (4.27) one can see that if we take $y_1=y_2=\ldots=y_n=0$, we have $\mathcal{H}_{T;K}=0$, thus (4.32) is simplified to

$$\begin{aligned}&(4|G|x\partial_x^2+2|G|(D-n)\partial_x+\widetilde{\alpha}_R)F(x)\\&=\mathcal{H}_{T;R}(x,\,0,\,0,\ldots,0).\end{aligned} \tag{4.37}$$

The above differential equation is the form of (A.14) in appendix A and we can write down the solution immediately (see (A.33))

$$\begin{aligned}F(x)=&F_0(x)\Big(f_0+\int_0^x\mathrm{d}ww^{-\frac{(D-n)}{2}}e^{-2F_0(w)}\\&\times\int_0^w\mathrm{d}\xi\frac{1}{A}\mathcal{H}_{T;R}(x,\,0,\,0,\ldots,0)F_0^{-1}(\xi)e^{2F_0(\xi)}\xi^{\frac{(D-n)}{2}-1}\Big)\end{aligned}, \tag{4.38}$$

where

$$F_0(x)=\ {}_0F_1(\varnothing;\,\frac{(D-n)}{2};\,\frac{-\widetilde{\alpha}_R x}{4|G|}). \tag{4.39}$$

Putting (4.39) back to (4.26) we get the final analytic expression of generation functions

$$\begin{aligned}c_T(x,\,y_1,\ldots,y_n)=&\mathcal{H}_{T;K}+e^{\frac{\Sigma_{i=1}^n\alpha_i y_i}{|G|}}F_0(x)\\&\times\Big(f_0+\int_0^x\mathrm{d}ww^{-\frac{(D-n)}{2}}e^{-2F_0(w)}\int_0^w\mathrm{d}\xi\\&\times\frac{1}{A}\mathcal{H}_{T;R}(x,\,0,\,0,\ldots,0)F_0(\xi)^{-1}e^{2F_0(\xi)}\xi^{\frac{(D-n)}{2}-1}\Big).\end{aligned} \tag{4.40}$$

When we consider the generation functions of reduction coefficients of one-loop integrals with $(n+1)$ propagators, there is a special case, where all $H_{T;i}$, $H_{T;R}$ are zero. For this case, we can write down immediately the generation function

$$c_{n+1\to n+1}(R, K_1,\ldots,K_n) = {}_0F_1 \times\left(\varnothing;\ \frac{(D-n)}{2};\ \frac{-\widetilde{\alpha}_R x}{4\mid G\mid}\right)\mathrm{e}^{\frac{\sum_{i=1}^n \alpha_i y_i}{|G|}}. \tag{4.41}$$

5. Triangle

In this part we present another example, i.e. the triangle, to demonstrate the general frame laid down in the previous section. The seven generation functions are defined by

$$\begin{aligned} I_{\rm tri}(t, R) &\equiv\int \mathrm{d}\ell\frac{\mathrm{e}^{t(2\ell\cdot R)}}{(\ell^2-M_0^2)((\ell-K_1)^2-M_1^2)((\ell-K_2)^2-M_2^2)}\\ &\equiv\int \mathrm{d}\ell\frac{\mathrm{e}^{t(2\ell\cdot R)}}{D_0D_1D_2}\\ &=c_{3\to 3}\int \mathrm{d}\ell\frac{1}{D_0D_1D_2} + c_{3\to 2;\hat{i}}\\ &\times\int \mathrm{d}\ell\frac{1}{\prod_{j\neq i,0}^2 D_j} + c_{3\to 1;i}\int \mathrm{d}\ell\frac{1}{D_i}. \end{aligned} \tag{5.1}$$

Using the permutation symmetry and the shifting of loop momentum we can find nontrivial relations among these seven generation functions. The first group of relations is

$$\begin{aligned} &c_{3\to 3}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 3}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 2;\hat{0}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 2;\hat{0}}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 2;\hat{1}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 2;\hat{2}}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 2;\hat{2}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 2;\hat{1}}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 1;0}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 1;0}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 1;1}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 1;2}(t, R; M_0; K_2, M_2; K_1, M_1),\\ &c_{3\to 1;2}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=c_{3\to 1;1}(t, R; M_0; K_2, M_2; K_1, M_1). \end{aligned} \tag{5.2}$$

The second group of relations is

$$\begin{aligned} &c_{3\to 3}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 3}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 2;\hat{0}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 2;\hat{1}}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 2;\hat{1}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 2;\hat{0}}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 2;\hat{2}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 2;\hat{1}}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 1;0}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 1;1}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 1;1}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 1;0}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2),\\ &c_{3\to 1;2}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_1\cdot R}c_{3\to 1;1}(t, R; M_1; -K_1, M_0; K_2-K_1, M_2). \end{aligned} \tag{5.3}$$

The third group of relations is

$$\begin{aligned} &c_{3\to 3}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 3}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 2;\hat{0}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 2;\hat{2}}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 2;\hat{1}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 2;\hat{1}}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 2;\hat{2}}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 2;\hat{0}}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 1;0}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 1;2}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 1;1}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 1;1}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0),\\ &c_{3\to 1;2}(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=\mathrm{e}^{2K_2\cdot R}c_{3\to 1;0}(t, R; M_2; K_1-K_2, M_1; -K_2, M_0). \end{aligned} \tag{5.4}$$

Using these three groups of relations, we just need to compute three generation functions, for example, $c_{3\to 3}$, $c_{3\to 2;\hat{1}}$ and $c_{3\to 1;0}$. The mass dimensions of them are

$$[t]=-2,\qquad [c_{3\to 3}]=0,\qquad [c_{3\to 2;\hat{i}}]=-2,\qquad [c_{3\to 1;i}]=-4. \tag{5.5}$$

5.1. The differential equations

Since we have given enough details in the section of the bubble, here we will be more brief. Using $\partial_R\cdot\partial_R$, $K_1\cdot\partial_R$, $K_2\cdot\partial_R$ operators, we can find

$$\begin{aligned} &\partial_R\cdot\partial_R c_T(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=4t^2M_0^2c_T(t, R; M_0; K_1, M_1; K_2, M_2) + 4t^2\xi_T h_T, \end{aligned} \tag{5.6}$$

$$\begin{aligned} &K_1\cdot\partial_R c_T(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=tf_1c_T(t, R; M_0; K_1, M_1; K_2, M_2) - t\widetilde{\xi}_T\widetilde{h}_T + t\xi_T h_T, \end{aligned} \tag{5.7}$$

$$\begin{aligned} &K_2\cdot\partial_R c_T(t, R; M_0; K_1, M_1; K_2, M_2)\\ &\quad=tf_2c_T(t, R; M_0; K_1, M_1; K_2, M_2) - t\widehat{\xi}_T\widehat{h}_T + t\xi_T h_T, \end{aligned} \tag{5.8}$$

where T is the index for different generation functions and $f_1=K_1^2-M_1^2+M_0^2$, $f_2=K_2^2-M_2^2+M_0^2$. The various constants ξ, $\widetilde{\xi}$, $\widehat{\xi}$ are given in the table

T	ξ_T	h_T	$\widetilde{\xi}_T$	$\widetilde{h}_T$	$\widehat{\xi}_T$	$\widehat{h}_T$
$\{3\to 3\}$	0	0	0	0	0	0
$\{3\to 2;\widehat{0}\}$	1	$h_{3\to 2;\widehat{0}}$	0	0	0	0
$\{3\to 2;\widehat{1}\}$	0	0	1	$\widetilde{h}_{3\to 2;\widehat{1}}$	0	0
$\{3\to 2;\widehat{2}\}$	0	0	0	0	1	$\widehat{h}_{3\to 2;\widehat{2}}$
$\{3\to 1;0\}$	0	0	1	$\widetilde{h}_{3\to 1;0}$	1	$\widehat{h}_{3\to 1;0}$
$\{3\to 1;1\}$	1	$h_{3\to 1;1}$	0	0	1	$\widehat{h}_{3\to 1;1}$
$\{3\to 1;2\}$	1	$h_{3\to 1;2}$	1	$\widetilde{h}_{3\to 1;2}$	0	0

(5.9)

while h_T are given by

$$
\begin{aligned}
h_{3\to 2;\hat{0}} &= e^{2K_1\cdot R}c_{2\to 2}(t, R^2, (K_2-K_1)\cdot R,\\
&\quad (K_2-K_1)^2; M_1, M_2),\\
h_{3\to 1;1} &= e^{2K_1\cdot R}c_{2\to 2;\hat{1}}(t, R^2, (K_2-K_1)\cdot R,\\
&\quad (K_2-K_1)^2; M_1, M_2),\\
h_{3\to 1;2} &= e^{2K_1\cdot R}c_{2\to 2;\hat{0}}(t, R^2, (K_2-K_1)\cdot R,\\
&\quad (K_2-K_1)^2; M_1, M_2),
\end{aligned} \tag{5.10}
$$

and $\widetilde{h}_T$ are given by

$$
\begin{aligned}
\widetilde{h}_{3\to 2;\hat{1}} &= c_{2\to 2}(t, R^2, K_2\cdot R, K_2^2; M_0, M_2),\\
\widetilde{h}_{3\to 1;0} &= c_{2\to 1;\hat{1}}(t, R^2, K_2\cdot R, K_2^2; M_0, M_2),\\
\widetilde{h}_{3\to 1;2} &= c_{2\to 1;\hat{0}}(t, R^2, K_2\cdot R, K_2^2; M_0, M_2),
\end{aligned} \tag{5.11}
$$

and $\widehat{h}_T$ are given by

$$
\begin{aligned}
\widehat{h}_{3\to 2;\hat{2}} &= c_{2\to 2}(t, R^2, K_1\cdot R, K_1^2; M_0, M_1),\\
\widehat{h}_{3\to 1;0} &= c_{2\to 1;\hat{1}}(t, R^2, K_1\cdot R, K_1^2; M_0, M_1),\\
\widehat{h}_{3\to 1;1} &= c_{2\to 1;\hat{0}}(t, R^2, K_1\cdot R, K_1^2; M_0, M_1).
\end{aligned} \tag{5.12}
$$

The differential equations (5.6), (5.7) and (5.8) are indeed the form (4.13) and (4.14) in the previous section.

For triangle, the natural variables for generation functions are

$$
r = R\cdot R, \qquad p_1 = K_1\cdot R, \qquad p_2 = K_2\cdot R. \tag{5.13}
$$

However, the good variables for differential equations (5.6), (5.7) and (5.8) are x, y_1, y_2 as

$$
\begin{aligned}
x &= [K_1^2K_2^2-(K_1\cdot K_2)^2]r - K_2^2p_1^2 - K_1^2p_2^2\\
&\quad + (2K_1\cdot K_2)p_1p_2\\
&= |G|r - p_1y_1 - p_2y_2,\\
y_2 &= -(K_1\cdot K_2)p_1 + K_1^2p_2,\\
y_1 &= -(K_1\cdot K_2)p_2 + K_2^2p_1,
\end{aligned} \tag{5.14}
$$

which are defined in (4.8) with Gram matrix

$$
G(K_1, K_2) \equiv \begin{pmatrix} K_1^2 & K_1\cdot K_2\\ K_1\cdot K_2 & K_2^2\end{pmatrix},\qquad
G^{-1} = \frac{1}{|G|}\begin{pmatrix} K_2^2 & -K_1\cdot K_2\\ -K_1\cdot K_2 & K_1^2\end{pmatrix}. \tag{5.15}
$$

Using the new variables, the differential equations are

$$
\left(\partial_{y_1} - \frac{\alpha_1}{|G|}\right)c_T = \frac{1}{|G|}H_{T;1},\qquad
\left(\partial_{y_2} - \frac{\alpha_2}{|G|}\right)c_T = \frac{1}{|G|}H_{T;2}, \tag{5.16}
$$

$$
(4|G|x\partial_x^2 + 2(D-2)|G|\partial_x + \widetilde{\alpha}_R)c_T = \mathcal{H}_{T;R}, \tag{5.17}
$$

where

$$
\begin{aligned}
\alpha_1 &= tf_1, \qquad \alpha_2 = tf_2,\\
\widetilde{\alpha}_R &= \frac{t^2f_1^2K_2^2 + t^2f_2^2K_1^2 - 2t^2f_1f_2(K_1\cdot K_2)}{|G|}\\
&\quad - 4t^2M_0^2,
\end{aligned} \tag{5.18}
$$

and

$$
\begin{aligned}
\mathcal{H}_{T;R} &= -\bigg(K_2^2\partial_{y_1} - (K_1\cdot K_2)\partial_{y_2}\\
&\quad + \frac{\alpha_1K_2^2 - \alpha_2(K_1\cdot K_2)}{|G|}\bigg)H_{T;1}\\
&\quad - \bigg(K_1^2\partial_{y_2} - (K_1\cdot K_2)\partial_{y_1}\\
&\quad + \frac{\alpha_2K_1^2 - \alpha_1(K_1\cdot K_2)}{|G|}\bigg)H_{T;2} + H_{T;R}.
\end{aligned} \tag{5.19}
$$

The solution is given by

$$
\begin{aligned}
c_T(x, y_1, y_2) &= e^{\frac{\alpha_1}{|G|}y_1}e^{\frac{\alpha_2}{|G|}y_2}F_T^{tri}(x) + e^{\frac{\alpha_2}{|G|}y_2}\frac{1}{|G|}\\
&\quad\times\int_0^{y_2} dw_2 e^{-\frac{\alpha_2}{|G|}w_2}H_{T;2}(x, y_1, w_2)\\
&\quad + e^{\frac{\alpha_1}{|G|}y_1}e^{\frac{\alpha_2}{|G|}y_2}\frac{1}{|G|}\int_0^{y_1} dw_1 e^{-\frac{\alpha_1}{|G|}w_1}H_{T;1}(x, w_1, 0),
\end{aligned} \tag{5.20}
$$

where

$$
F_0^{\rm tri}(x) = {}_0F_1\left(\varnothing; \frac{(D-2)}{2}; \frac{-\widetilde{\alpha}_Rx}{4|G|}\right) \tag{5.21}
$$

and

$$
\begin{aligned}
F^{\rm tri}(x) &= F_0^{\rm tri}(x)\bigg(f_0 + \int_0^x dw w^{\frac{-(D-2)}{2}}e^{-2F_0^{\rm tri}(w)}\\
&\quad\times\int_0^w d\xi\frac{1}{A}\mathcal{H}_{T;R}(x, 0,\ldots,0)F_0^{\rm tri}(\xi)^{-1}e^{2F_0^{\rm tri}(\xi)}\xi^{\frac{(D-2)}{4}}\bigg).
\end{aligned} \tag{5.22}
$$

The generation function $c_{3\to 3}$: for this one, from table (5.9), we see that $H_{T;1} = H_{T;2} = H_{T;R} = 0$, thus we write down immediately

$$
c_{3\to 3} = e^{\frac{\alpha_1}{|G|}y_1}e^{\frac{\alpha_2}{|G|}y_2}\ {}_0F_1\left(\varnothing; \frac{(D-2)}{2}; \frac{-\widetilde{\alpha}_Rx}{4|G|}\right). \tag{5.23}
$$

The generation function $c_{3\to 2}$: there are three generation functions $c_{3\to 2;\hat{i}}$. We want to choose one of them with the simplest H_1, H_2, H_R. Checking with table (5.9), we see that if we consider $c_{3\to 2;\hat{1}}$, we will have $H_R = 0$, $H_2 = 0$ and

$$
\begin{aligned}
H_{3\to 2;\hat{1};1}(x, y_1, y_2) &= -c_{2\to 2}(t, R^2, K_2\cdot R, K_2^2; M_0, M_2)\\
&= -{}_0F_1\bigg(\varnothing; \frac{D-1}{2};\\
&\quad \left(\frac{t^2(4K_2^2M_0^2 - f_2^2)(K_2^2r - p_2^2)}{4(K_2^2)^2}\right)\bigg)e^{\frac{tf_2}{K_2^2}p_2},
\end{aligned} \tag{5.24}
$$

where we have used the result (3.44) and expressed r, p_1, p_2 using (4.10)

$$
p_1 = \frac{K_1^2y_1 + (K_1\cdot K_2)y_2}{|G|},\qquad p_2 = \frac{K_2^2y_2 + (K_1\cdot K_2)y_1}{|G|},\qquad
r = \frac{|G|x + K_1^2y_1^2 + K_2^2y_2^2 + 2(K_1\cdot K_2)y_1y_2}{|G|^2}. \tag{5.25}
$$

Thus by (5.19) we can find

$$\begin{aligned}\mathcal{H}_{3\to2;\hat{1};R} &= \left(\frac{\alpha_1 K_2^2 - \alpha_2 (K_1\cdot K_2)}{|G|}\right) \mathrm{e}^{\frac{tf_2}{K_2^2}p_2}\ {}_0F_1\\ &\times\left(\varnothing;\ \frac{D-1}{2};\ \left(\frac{t^2(4K_2^2M_0^2-f_2^2)(K_2^2r-p_2^2)}{4(K_2^2)^2}\right)\right)\\ &+\frac{t^2(4K_2^2M_0^2-f_2^2)}{4(K_2^2)^2}\frac{2K_2^2y_1}{|G|}\mathrm{e}^{\frac{tf_2}{K_2^2}p_2}\ {}_0F_1^{(1)}\\ &\times\left(\varnothing;\ \frac{D-1}{2};\ \left(\frac{t^2(4K_2^2M_0^2-f_2^2)(K_2^2r-p_2^2)}{4(K_2^2)^2}\right)\right), \end{aligned} \tag{5.26}$$

where we have defined

$$\begin{aligned} &{}_AF_B^{(n)}(a_1,\ldots,a_A;\ b_1,\ldots,b_B;\ x)\\ &=\frac{\mathrm{d}^n}{\mathrm{d}x^n}\ {}_AF_B(a_1,\ldots,a_A;\ b_1,\ldots,b_B;\ x). \end{aligned} \tag{5.27}$$

Putting them back to (5.20) we can find the analytic expression. Using it we can get the explicit series expansion as discussed in appendix A.

For generation functions $c_{3\to1;i}$, we can do similar calculations. The key is to find $H_{T;i}$ and $H_{T;R}$, which is the generation functions of one order lower topology. Thus we see the recursive structures of generation functions from lower topologies to higher topologies. The logic is clear although working out details takes some effects.

6. Conclusion

In this paper, we have introduced the concept of generation function for reduction coefficients of loop integrals. For one-loop integrals, using the recent proposal of auxiliary vector R, we can construct two types of differential operators $\frac{\partial}{\partial R}\cdot\frac{\partial}{\partial R}$ and $K_i\cdot\frac{\partial}{\partial R}$. Using these operators, we can establish corresponding differential equations for generation functions. By proper changing of variables, these differential equations can be written into the decoupled form, thus one can solve them one by one analytically. Obviously, one could try to apply the same idea to discuss the reduction problem for two and higher loop integrals. But with the appearance of irreducible scalar products, the problem becomes harder. One can try to use the IBP relation in the Baikov representation [44] as shown [42, 43].

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Appendix A. Solving differential equations

As shown in the paper, differential equations for generation functions can be reduced to two typical types. In this appendix, we present details of solving these typical differential equations.

The first order differential equation: the first typical differential equation is following first order differential equation

$$\boxed{\left(A\frac{\mathrm{d}}{\mathrm{d}x}+B\right)F(x)=H(x)}, \tag{A.1}$$

where A, B are independent of x and $H(x)$ is the known function of x. To solve it, first, we solve the homogenous part

$$\begin{aligned} &\left(A\tfrac{\mathrm{d}}{\mathrm{d}x}+B\right)F_0(x)=0,\\ &\Longrightarrow\quad F_0(x)=\mathrm{e}^{-\frac{B}{A}x}. \end{aligned} \tag{A.2}$$

Then we write $F(x)=F_0(x)F_1(x)$ in (A.1) to get the differential equation for $F_1(x)$ as

$$A\frac{\mathrm{d}}{\mathrm{d}x}F_1(x)=F_0^{-1}H(x), \tag{A.3}$$

thus we have

$$F_1(x)=F_1(x=0)+\int_0^x \mathrm{d}t\frac{1}{A}\mathrm{e}^{\frac{B}{A}t}H(t). \tag{A.4}$$

Knowing the special solution of $F(x)$ in (A.4), the general solution of (A.1) is given by

$$\boxed{F(x)=F_0(x)F_1(x)+\tilde{\alpha}F_0(x)=\mathrm{e}^{-\frac{B}{A}x}\left(\alpha+\int_0^x\mathrm{d}w\frac{1}{A}\mathrm{e}^{\frac{B}{A}w}H(w)\right)}, \tag{A.5}$$

where the constant α is determined by the boundary condition.

If we want to have the series expansion of x, there are several ways to do this. The first one is to carry out the integration

$$\begin{aligned}\int_0^x \mathrm{d}t\tfrac{1}{A}\mathrm{e}^{\frac{B}{A}t}H(t) &= \tfrac{x}{A}\sum_{n=0}^{\infty}\sum_{a=0}^{\infty}h_n x^n\\ &\times\frac{1}{(n+a+1)}\frac{\left(\frac{Bx}{A}\right)^a}{a!}, \end{aligned} \tag{A.6}$$

where we have used the expansion of $H(x)=\sum_{n=0}^{\infty}h_n x^n$. The second way is to put $F(x)=\sum_{n=0}^{\infty}f_n x^n$ directly to (A.1) to arrive at the recursion relation

$$\begin{aligned} f_{n+1} &= \frac{-B}{(n+1)A}f_n+\frac{h_n}{(n+1)A}\\ &\equiv\gamma(n)f_n+\rho(n),\qquad n\geqslant 0. \end{aligned} \tag{A.7}$$

The recursive relation (A.7) can be solved as

$$f_{n+1}=f_0\prod_{i=0}^{n}\gamma(i)+\sum_{i=0}^{n}\rho(i)\prod_{j=i+1}^{n}\gamma(j). \tag{A.8}$$

It is easy to compute

$$\Xi[n+1]=\prod_{i=0}^{n}\gamma(i)=\frac{(-B)^{n+1}}{(n+1)!A^{n+1}} \tag{A.9}$$

and

$$\begin{aligned} \rho(i)\textstyle\prod_{j=i+1}^{n-1}\gamma(j) &= \frac{h_i}{(i+1)A}\textstyle\prod_{j=i+1}^{n-1}\gamma(j)\\ &=\frac{h_i}{-B}\textstyle\prod_{j=i}^{n-1}\gamma(j)\\ &=\frac{h_i}{-B}\frac{\Xi[n]}{\Xi[i]}=\frac{h_i}{-B}\frac{i!(-B)^{n-i}}{n!A^{n-i}}, \end{aligned} \tag{A.10}$$

thus we have

$$F(x)=\sum_{n=0}\left(f_0\frac{(-B)^n}{n!A^n}+\sum_{i=0}^{n-1}\frac{h_i}{A}\frac{i!(-B)^{n-1-i}}{n!A^{n-1-i}}\right)x^n. \tag{A.11}$$

The third way is to use analytic expression (A.5). We need to compute $\frac{d^nF(x)}{dx^n}$ and then set $x=0$. One can see that, for example,

$$\frac{dF(x)}{dx}=-\frac{B}{A}e^{-\frac{B}{A}x}\left(\alpha+\int_0^x dw\frac{1}{A}e^{\frac{B}{A}w}H(w)\right)+\frac{1}{A}H(x), \tag{A.12}$$

thus

$$\frac{dF(x)}{dx}|_{x=0}=-\frac{B}{A}\alpha+\frac{1}{A}H(x=0). \tag{A.13}$$

The important point is that when setting $x=0$ at the end of differentiation, the integration $\int_0^x dw\,...=0$, thus we have got rid of integration and all we need to do are the differentiations over $e^{\frac{B}{A}x}$ and $H(x)$.

The second order differential equation: the second typical differential equation met in this paper is the following second order differential equation

$$\boxed{\left(Ax\frac{d^2}{dx^2}+B\frac{d}{dx}+C\right)F(x)=H(x)}, \tag{A.14}$$

where A, B, and C are independent of x and $H(x)$ is a known function of x. Let us solve it using the series expansion. Writing

$$F(x)=\sum_{n=0}^{\infty}f_nx^n,\qquad H(x)=\sum_{n=0}^{\infty}h_nx^n, \tag{A.15}$$

and putting back to (A.14) we have

$$\sum_{n=0}^{\infty}h_nx^n=\sum_{n=0}^{\infty}(An(n+1)f_{n+1}x^n+B(n+1)f_{n+1}x^n+Cf_nx^n), \tag{A.16}$$

thus we have the relation

$$h_n=(n+1)(B+An)f_{n+1}+Cf_n,\qquad n\geqslant 0. \tag{A.17}$$

Using it, we can solve[6]

$$f_{n+1}=\frac{h_n}{(n+1)(B+An)}-C\frac{1}{(n+1)(B+An)}f_n\equiv\gamma(n)f_n+\rho(n),\qquad n\geqslant 0. \tag{A.18}$$

The recursive relation (A.18) can be solved as

$$f_{n+1}=f_0\prod_{i=0}^{n}\gamma(i)+\sum_{i=0}^{n}\rho(i)\prod_{j=i+1}^{n}\gamma(j). \tag{A.19}$$

[6] An important point is that although (A.14) is a second order differential equation, because the x is in front of $\frac{d^2}{dx^2}$, around $x=0$, it is essentially a first order differential equation. This explains why using only f_0 and known $H(x)$ we can determine $F(x)$ using (A.18).

It is easy to compute

$$\Xi[n+1]=\prod_{i=0}^{n}\gamma(i)=\frac{(-C)^{n+1}}{(n+1)!A^{n+1}\prod_{i=0}^{n}(B/A+i)}=\frac{(-C)^{n+1}}{(n+1)!A^{n+1}\left(\frac{B}{A}\right)_{n+1}}, \tag{A.20}$$

where we have used the **Pochhammer symbol** to simplify the expression[7]

$$(x)_n=\frac{\Gamma(x+n)}{\Gamma(x)}=\prod_{i=1}^{n}(x+(i-1)). \tag{A.21}$$

Using

$$\rho(i)\prod_{j=i+1}^{n-1}\gamma(j)=\frac{h_i}{(i+1)(B+Ai)}\prod_{j=i+1}^{n-1}\gamma(j)=\frac{-h_i}{C}\prod_{j=i}^{n-1}\gamma(j)=\frac{-h_i}{C}\frac{\Xi[n]}{\Xi[i]} \tag{A.22}$$

we have

$$F(x)=\sum_{n=0}^{\infty}f_nx^n=\sum_{n=0}^{\infty}x^n\Xi[n]\left\{f_0+\sum_{i=0}^{n-1}\frac{-h_i}{C\Xi[i]}\right\}. \tag{A.23}$$

Let us define the following function

$$H_{A,B,C}(x)=\sum_{i=0}^{\infty}\frac{-h_ix^i}{C\Xi[i]}=\sum_{i=0}^{\infty}\frac{i!A^i}{(-C)^{i+1}}\left(\frac{B}{A}\right)_ih_ix^i, \tag{A.24}$$

which can be considered as the 'dual function' of $H(x)$ corresponding to the differential equation (A.14), then we can write

$$\sum_{i=0}^{n-1}\frac{-h_i}{C\Xi[i]}=\Big\lfloor H_{A,B,C}(x=1)\Big\rfloor_{x^{n-1}}, \tag{A.25}$$

where the symbol $\lfloor Y(x)\rfloor_{x^{n-1}}$ means to keep the Taylor series up to the order of $x^{(n-1)}$. Thus $F(x)$ can be written compactly as

$$F(x)=\sum_{n=0}^{\infty}f_nx^n=\sum_{n=0}^{\infty}x^n\Xi[n]\times\{f_0+\lfloor H_{A,B,C}(x=1)\rfloor_{x^{n-1}}\}. \tag{A.26}$$

For the special case $H(x)=0$, it is easy to see that

$$F(x)=f_0F_0,\qquad F_0(x)\equiv\sum_{n=0}^{\infty}\frac{(-C)^nx^n}{n!A^n\left(\frac{B}{A}\right)_n}. \tag{A.27}$$

The expression (A.27) is nothing, but the special case of **generalized hypergeometric function** (see (C.2) of [11]), which is defined as

$$_AF_B(a_1,\ldots,a_A;\,b_1,\ldots,b_B;\,x)=\sum_{n=0}^{\infty}\frac{(a_1)_n\ldots(a_A)_n}{(b_1)_n\ldots(b_B)_n}\frac{x^n}{n!}, \tag{A.28}$$

[7] From the definition one can see that $(x)_{n=0}=1,\ \forall x$.

thus we have

$$F(x) = f_0 F_0(x), \qquad F_0 = {}_0F_1\left(\varnothing; \frac{B}{A}; \frac{-Cx}{A}\right). \tag{A.29}$$

The solution in (A.23) is given in the series expansion. We can also write it in the analytic expression. Writing $F(x) = F_0(x)F_1(x)$ we can find the differential equation of $F_1(x)$ as

$$H(x) = F_0(x)\left(Ax\frac{d^2}{dx^2} + \left(B + 2Ax\frac{dF_0(x)}{dx}\right)\frac{d}{dx}\right)F_1(x), \tag{A.30}$$

which is the first order differential equation of $\frac{dF_1(x)}{dx} = U(x)$. Using a similar method as for the differential equation (A.1) we can solve

$$U(x) = x^{-\frac{B}{A}} e^{-2F_0(x)}\left(\alpha_1 + \int_0^x dw \frac{1}{A} H(w) F_0^{-1}(w) e^{2F_0(w)} w^{\frac{B}{A}-1}\right). \tag{A.31}$$

Integrating $U(x)$ to get $F_1(x)$ we obtain

$$F(x) = F_0(x)\left(\alpha_2 + \int_0^x dw w^{-\frac{B}{A}} e^{-2F_0(w)} \times\left(\alpha_1 + \int_0^w d\xi \frac{1}{A} H(\xi) F_0^{-1}(\xi) e^{2F_0(\xi)} \xi^{\frac{B}{A}-1}\right)\right), \tag{A.32}$$

where the α_1, α_2 can be determined using the initial condition of $F(x=0)$ and $\frac{dF(x=0)}{dx}$. Using the expansion of $F(x)$ we see that $\alpha_2 = f_0$ and $\alpha_1 = 0$ thus we have

$$\boxed{F(x) = F_0(x)\left(f_0 + \int_0^x dw w^{-\frac{B}{A}} e^{-2F_0(w)} \int_0^w d\xi \frac{1}{A} H(\xi) F_0^{-1}(\xi) e^{2F_0(\xi)} \xi^{\frac{B}{A}-1}\right)}. \tag{A.33}$$

The analytic form (A.33) is very compact, but it is hard to carry out the integration in general. However, we can use it to get series expansion just like before. One can see that

$$\frac{dF(x)}{dx} = \frac{dF_0(x)}{dx}\left(f_0 + \int_0^x dw w^{-\frac{B}{A}} e^{-2F_0(w)} \int_0^w dt \frac{1}{A} H(t) F_0(t)^{-1} e^{2F_0(t)} t^{\frac{B}{A}-1}\right) + F_0(x) x^{-\frac{B}{A}} e^{-2F_0(x)} \times \int_0^x dt \frac{1}{A} H(t) F_0(t)^{-1} e^{2F_0(t)} t^{\frac{B}{A}-1}. \tag{A.34}$$

When taking the value at the $x = 0$, we need to be careful with the second line and evaluate it as

$$\begin{aligned}\int_0^x dt \frac{1}{A} H(t) F_0(t)^{-1} e^{2F_0(t)} t^{\frac{B}{A}-1} &\to \frac{1}{A} H(x=0) F_0^{-1}(x=0) e^{2F_0(x=0)} \int_0^x dt t^{\frac{B}{A}-1} \\ &= \frac{1}{A} H(x=0) F_0^{-1}(x=0) e^{2F_0(x=0)} \frac{A}{B} x^{\frac{B}{A}}\Big|_0^x \\ &= H(x=0) F_0^{-1}(x=0) e^{2F_0(x=0)} \frac{1}{B} x^{\frac{B}{A}}.\end{aligned} \tag{A.35}$$

With the result (A.35), (A.34) gives

$$\frac{dF(x)}{dx} = f_0 \frac{dF_0(x)}{dx}\Big|_{x=0} + \frac{1}{B} H(x=0). \tag{A.36}$$

From (A.14) we have

$$\frac{dF}{dx}\Big|_{x=0} = \frac{1}{B} H(x=0) - \frac{C}{B} F(x=0), \tag{A.37}$$

thus we see that

$$f_0 = -\frac{C}{B} F(x=0)\left(\frac{dF_0(x)}{dx}\Big|_{x=0}\right)^{-1}. \tag{A.38}$$

For general $\frac{d^n F(x)}{dx^n}$ we do the same thing to get the series expansion of x. At each step, we use (A.35) and there is no integration to be done for $x = 0$.

Appendix B. The explicit solutions of $c_{n,m}$ for bubble reduction

In this part, we will show how to get explicit solutions for the recursion relations (3.30), (3.31), (3.32) and (3.33).

B.1. The generation function $c_{2\to2}$

For this case, we have $h_T = 0$, i.e. all $h_{n,m} = 0$ in (3.30), (3.31), (3.32) and (3.33). Using (3.30) it is easy to find

$$\begin{aligned}c_{N,0} &= \prod_{n=1}^{N} \frac{(-f^2 + 4K^2M_0^2)t^2}{2K^2 n(D + 2n - 3)} \\ &= \prod_{n=0}^{N-1} \frac{K^2(\beta - \alpha^2)}{2(n+1)(D + 2n - 1)},\end{aligned} \tag{B.1}$$

where we have used the initial condition $c_{0,0} = 1$ and defined

$$\alpha = \frac{tf}{K^2}, \qquad \beta = \frac{4t^2M_0^2}{K^2}. \tag{B.2}$$

Using (3.32) to compute the first few $c_{n,m}$, one can see the pattern

$$c_{N,m} = \frac{1}{m!} d_{N,m} c_{N,0}, \tag{B.3}$$

where $d_{N,m}$ depends on both N, m and the first few $d_{N,m}$ are

$$d_{N,0} = 1, \quad d_{N,1} = \alpha, \quad d_{N,2} = \frac{(D+2N)\alpha^2 - \beta}{(D+2N-1)}. \tag{B.4}$$

Putting the form (B.3) to (3.32) we get the recursion relation

$$\mathrm{d}_{N,m+1} = \frac{\beta}{\alpha}\mathrm{d}_{N,m} - \frac{(\beta-\alpha^2)(D+2N+m-1)}{\alpha(D+2N-1)}\mathrm{d}_{N+1,m}. \tag{B.5}$$

Using the formally defined operator $\widehat{P}$ such that

$$\widehat{P}f(N,m) = f(N+1,m), \tag{B.6}$$

the solution of (B.5) can be formally given by

$$\begin{aligned}\mathrm{d}_{N,M} &= \prod_{m=1}^{M}\left(\frac{\beta}{\alpha} - \frac{(\beta-\alpha^2)(D+2N+m-2)}{\alpha(D+2N-1)}\widehat{P}\right),\\ M &\geqslant 1,\\ &:=\left(\frac{\beta}{\alpha} - \frac{(\beta-\alpha^2)(D+2N+M-2)}{\alpha(D+2N-1)}\widehat{P}\right)\\ &\times\left(\frac{\beta}{\alpha} - \frac{(\beta-\alpha^2)(D+2N+(M-1)-2)}{\alpha(D+2N-1)}\widehat{P}\right)\\ &\times\ldots\ldots\left(\frac{\beta}{\alpha} - \frac{(\beta-\alpha^2)(D+2N+2-2)}{\alpha(D+2N-1)}\widehat{P}\right)\\ &\times\left(\frac{\beta}{\alpha} - \frac{(\beta-\alpha^2)(D+2N+1-2)}{\alpha(D+2N-1)}\widehat{P}\right)\mathrm{d}_{N,0},\end{aligned} \tag{B.7}$$

where since the appearance of the operator $\widehat{P}$, the ordering of the multiple factors is given explicitly. With a little computation, one can see that

$$\mathrm{d}_{N,M} = \sum_{i=0}^{M}\frac{(-)^i\beta^{M-i}(\beta-\alpha^2)^i}{\alpha^M}\mathrm{d}_{N,M;i}, \tag{B.8}$$

where

$$\begin{aligned}\mathrm{d}_{N,M;i} &= \sum_{1;[t_1,\ldots,t_i]}^{M}\prod_{s=1}^{i}\\ &\times\frac{D+2(N+s-1)+(M+1-t_s)-2}{D+2(N+s-1)-1},\\ i &\geqslant 1;\quad \mathrm{d}_{N,M;0} = 1\end{aligned} \tag{B.9}$$

and the summation sign is defined as

$$\sum_{1\leqslant t_1<t_2\ldots<t_i\leqslant M} := \sum_{1;[t_1,\ldots,t_i]}^{M}. \tag{B.10}$$

Combining (B.3) with (B.1) and (B.8), we have the explicit solution for the generation function of $c_{2\to 2}(t, r, p, K^2; M_0, M_1)$.

B.1.1. The generation function $c_{2\to 1;\widehat{1}}$

For this case, we have $h_T = c_{1\to 1}(t, r, M_0)$. Using (2.5) to (3.30) one can write down

$$c_{n+1,0} = \gamma(n)c_{n,0} + \rho(n) \tag{B.11}$$

with

$$\begin{aligned}\gamma(n) &= \frac{K^2(\beta-\alpha^2)}{2(n+1)(D+2n-1)},\\ \rho(n) &= \frac{t^{2n+1}(M_0^2)^n\alpha}{2(D+2n-1)(n+1)!\left(\frac{D}{2}\right)_n},\end{aligned} \tag{B.12}$$

thus we can solve

$$c_{n+1,0} = c_{0,0}\prod_{i=0}^{n}\gamma(i) + \sum_{i=0}^{n}\rho(i)\prod_{j=i+1}^{n}\gamma(j), \tag{B.13}$$

where for the current case, the initial condition is $c_{0,0}=0$. To find $c_{n,m}$ we write

$$c_{N,m} = \frac{1}{m!}d_{N,m}c_{N,0} - \frac{1}{m!}\frac{t\beta^N(K^2)^{N-1}}{N!4^N\left(\frac{D}{2}\right)_N}b_{N,m} \tag{B.14}$$

with the first few $\mathrm{d}_{N,m}$ and $b_{N,m}$

$$\begin{aligned}&\mathrm{d}_{N,0} = 1, \qquad b_{N,0} = 0,\\ &\mathrm{d}_{N,1} = \alpha, \qquad b_{N,1} = 1,\\ &\mathrm{d}_{N,2} = \frac{-\beta+\alpha^2(D+2N)}{(D+2N-1)},\\ &b_{N,2} = \frac{\alpha(D+2N)}{(D+2N-1)}.\end{aligned} \tag{B.15}$$

Using the form (B.14) to (3.32) we get the recursion relations

$$\mathrm{d}_{N,m+1} = \frac{\beta}{\alpha}\mathrm{d}_{N,m} - \frac{(D+2N+m-1)(\beta-\alpha^2)}{\alpha(D+2N-1)}\mathrm{d}_{N+1,m}, \tag{B.16}$$

$$b_{N,m+1} = \frac{\beta}{\alpha}b_{N,m} - \frac{\beta(D+2N+m-1)}{\alpha(D+2N)}b_{N+1,m} + \frac{(D+2N+m-1)}{(D+2N-1)}\mathrm{d}_{N+1,m}. \tag{B.17}$$

The solution of $\mathrm{d}_{N,m}$ can be similarly solved using the operator language as in the previous subsubsection and we find

$$\mathrm{d}_{N,M} = \sum_{i=0}^{M}\frac{(-)^i\beta^{M-i}(\beta-\alpha^2)^i}{\alpha^M}\mathrm{d}_{N,M;i} \tag{B.18}$$

with

$$\begin{aligned}\mathrm{d}_{N,M;i} &= \sum_{1;[t_1,\ldots,t_i]}^{M}\prod_{s=1}^{i}\\ &\times\frac{D+2(N+s-1)+(M+1-t_s)-2}{D+2(N+s-1)-1},\\ i &\geqslant 1;\quad \mathrm{d}_{N,M;0} = 1.\end{aligned} \tag{B.19}$$

The solution for $b_{N,m}$ is a little bit complicated because the third term is on the right-hand side of (B.17). To solve it, we write

$$b_{N,m+1} = \widetilde{\gamma}_N(m)b_{N,m} + \widetilde{\rho}_N(m), \tag{B.20}$$

where

$$\begin{aligned}\widetilde{\gamma}_N(m) &= \left(\frac{\beta}{\alpha} - \frac{\beta(D+2N+m-1)}{\alpha(D+2N)}\widehat{P}\right),\\ \widetilde{\rho}_N(m) &= \frac{(D+2N+m-1)}{(D+2N-1)}\mathrm{d}_{N+1,m}.\end{aligned} \tag{B.21}$$

Iterating (B.20) with proper ordering we have

$$\begin{aligned} b_{N,m+1} = {} & \widetilde{\gamma}_N(m)\widetilde{\gamma}_N(m-1)\ldots\widetilde{\gamma}_N(0)b_{N,0} \\ & + \widetilde{\gamma}_N(m)\widetilde{\gamma}_N(m-1)\ldots\widetilde{\gamma}_N(1)\widetilde{\rho}_N(0) \\ & + \widetilde{\gamma}_N(m)\widetilde{\gamma}_N(m-1)\ldots\widetilde{\gamma}_N(2)\widetilde{\rho}_N(1)+\ldots \\ & + \widetilde{\gamma}_N(m)\widetilde{\rho}_N(m-1) + \widetilde{\rho}_N(m). \end{aligned} \tag{B.22}$$

Using

$$(\widehat{P})^a\widetilde{\rho}_N(m) = \frac{(D+2(N+a)+m-1)}{(D+2(N+a)-1)}\mathrm{d}_{N+1+a,m} \tag{B.23}$$

and

$$\begin{aligned} & \widetilde{\gamma}_N(m)\widetilde{\gamma}_N(m-1)\ldots\widetilde{\gamma}_N(m-k)\widetilde{\rho}_N(m-k-1) \\ & = \sum_{i=0}^{k+1}(-)^i\left(\frac{\beta}{\alpha}\right)^{k+1} \\ & \times\left(\sum_{1;[t_1,\ldots,t_i]}^{k+1}\prod_{s=1}^{i}\right. \\ & \left.\times\frac{D+2(N+s-1)+(m+1-t_s)-1}{D+2(N+s-1)}\right) \\ & \times\frac{(D+2(N+i)+(m-k-1)-1)}{(D+2(N+i)-1)} \\ & \mathrm{d}_{N+1+i,m-k-1}, \end{aligned} \tag{B.24}$$

$$\begin{aligned} F[N+2] = {} & c^{(N+2)} \\ & -\left(\frac{(D+2N)fpc^{(N+1)} - (N+1)(4M_0^2p^2 + (f^2 - 4M_0^2K^2)r)c^{(N)}}{(D+N-1)K^2}\right), \end{aligned} \tag{B.28}$$

we finally reach

$$\begin{aligned} b_{N,m+1} = {} & \frac{(D+2N+m-1)}{(D+2N-1)}\mathrm{d}_{N+1,m} \\ & + \sum_{k=0}^{m-1}\left(\frac{\beta}{\alpha}\right)^{k+1}\sum_{i=0}^{k+1}(-)^i \\ & \times\frac{(D+2(N+i)+(m-k-1)-1)}{(D+2(N+i)-1)}\mathrm{d}_{N+1+i,m-k-1} \\ & \times\left(\sum_{1;[t_1,\ldots,t_i]}^{k+1}\prod_{s=1}^{i}\frac{D+2(N+s-1)+(m+1-t_s)-1}{D+2(N+s-1)}\right). \end{aligned} \tag{B.25}$$

where the condition $b_{N,0} = 0$ has been used. The formula (B.14) plus (B.13), (B.18) and (B.25) gives the explicit solution for the generation function $c_{2\to 1;\hat{1}}$.

Knowing $c_{2\to 1;\hat{1}}$, we can use (3.5) to get the generation function $c_{2\to 1;\hat{0}}$ or directly compute it using (3.30), (3.31), (3.32) and (3.33).

B.1.2. The proof of one useful relation

When we use the improved PV-reduction method with auxiliary vector R to discuss the reduction of sunset topology, an important reduction relation between different tensor ranks has been observed in [40]. Later this relation has been studied in [41, 42, 45]. For the bubble, it is given explicitly by

$$\begin{aligned} I_2^{(r)} = {} & \frac{(D+2r-4)fp}{(D+r-3)K^2}I_2^{(r-1)} \\ & - \frac{(r-1)(4M_0^2p^2 + (f^2 - 4M_0^2K^2)r)}{(D+r-3)K^2}I_2^{(r-2)} \\ & + \frac{p}{K^2}I_{2;\hat{0}}^{(r-1)} + \frac{(r-1)((K^2 - M_0^2 + M_1^2)r - 2p^2)}{(D+r-3)K^2}I_{2;\hat{0}}^{(r-2)} \\ & + \frac{-p}{K^2}I_{2;\hat{1}}^{(r-1)} + \frac{(r-1)fr}{(D+r-3)K^2}I_{2;\hat{1}}^{(r-2)}, \end{aligned} \tag{B.26}$$

where

$$\begin{aligned} I_2^{(r)} &= \int \mathrm{d}\ell\frac{(2\ell\cdot R)^r}{(\ell^2 - M_0^2)((\ell-K)^2 - M_1^2)}, \\ I_{2;\hat{1}}^{(r)} &= \int \mathrm{d}\ell\frac{(2\ell\cdot R)^r}{(\ell^2 - M_0^2)}, \\ I_{2;\hat{0}}^{(r)} &= \int \mathrm{d}\ell\frac{(2\ell\cdot R)^r}{(\ell-K)^2 - M_1^2}. \end{aligned} \tag{B.27}$$

We can use the series form to check the relation (B.26). Let us define

where

$$c^{(N)} = \frac{N!}{t^N}\sum_{n=0}^{[N/2]}c_{n,N-2n}r^np^{N-2n} \tag{B.29}$$

and c can be $c_{2\to 2}$ or $c_{2\to 1}$. Depending on if N is even or odd, the computation details have some differences, thus we consider N even only, while N odd will be similar. Expanding (B.28) we have

$$\begin{aligned} F[2N+2] = {} & \frac{(2N+2)!}{t^{2N+2}}r^0p^{2N+2} \\ & \times\left(c_{0,2N+2} - \left(\frac{(D+4N)tfc_{0,2N+1}}{(2N+2)(D+2N-1)K^2}\right.\right. \\ & \left.\left. - \frac{4M_0^2t^2c_{0,2N}}{2(N+1)(D+2N-1)K^2}\right)\right) \\ & + \frac{(2N+2)!}{t^{2N+2}}r^{N+1}p^0(c_{N+1,0} \\ & -\left(-\frac{t^2(f^2 - 4M_0^2K^2)}{(2N+2)(D+2N-1)K^2}c_{N,0}\right)\Bigg) \\ & + \frac{(2N+2)!}{t^{2N+2}}\textstyle\sum_{n=1}^{N} r^np^{2N+2-2n}(c_{n,2N+2-2n} \\ & -\left(\frac{(D+4N)tf}{(2N+2)(D+2N-1)K^2}c_{n,2N+1-2n}\right. \\ & - \frac{4M_0^2t^2}{(2N+2)(D+2N-1)K^2}c_{n,2N-2n} \\ & \left.\left. - \frac{t^2(f^2 - 4M_0^2K^2)}{(2N+2)(D+2N-1)K^2}c_{n-1,2N+2-2n}\right)\right). \end{aligned} \tag{B.30}$$

For the term with $r^0 p^{2N+2}$, using the relation (3.29) with $n=0$, $m=2N$, we find the part inside the bracket is simplified to

$$\frac{-4\xi_R t^2 h_{0,2N} + \xi_K t(D+4N)h_{0,2N+1}}{2(N+1)(D+2N-1)K^2}. \tag{B.31}$$

For the term with $r^{N+1}p^0$, using the relation (3.30) with $n=N$, we find the part inside the bracket is simplified to

$$-\frac{t^2(\xi_K f - \xi_R 4K^2)h_{N,0} + \xi_K t K^2 h_{N,1}}{2(N+1)(D+2N-1)K^2}. \tag{B.32}$$

For the term with $r^n p^{2N+2-2n}$, the computation is a little bit complicated. First, we use (3.28) with $n\to n-1$ and $m\to 2N-2n+2,\ 2N-2n+1,\ 2N-2n$ to write all $c_{i,j}$ with $i=n-1$. Then we use (3.29) with $n\to n-1$ and $m\to 2N-2n+1,\ 2N-2n$. After doing the above two steps and making algebraic simplification, we get

$$\begin{aligned}&\frac{t^3(\xi_K M_0^2 - \xi_R f)}{(N+1)(D+2N-1)nK^2}h_{n-1,2N-2n+1}\\ &+\frac{t^2(-\xi_K nf + \xi_R 4K^2(N+1))}{2(N+1)(D+2N-1)nK^2}h_{n-1,2N-2n+2}\\ &+\frac{-\xi_K(2N-2n+3)t}{2n(D+2N-1)}h_{n-1,2N-2n+3}.\end{aligned} \tag{B.33}$$

For different c we use the different known h_T as given in (3.17), (3.18) and (3.19), thus one can see the relation (B.26) is satisfied.

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