

Riemann-Hilbert problem for the fifth-order modified Korteweg–de Vries equation with the prescribed initial and boundary values

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Abstract

In this paper, we investigate the fifth-order modified Korteweg–de Vries (mKdV) equation on the half-line via the Fokas unified transformation approach. We show that the solution $u(x, t)$ of the fifth-order mKdV equation can be represented by the solution of the matrix Riemann-Hilbert problem constructed on the plane of complex spectral parameter θ . The jump matrix $L(x, t, \theta)$ has an explicit representation dependent on x, t and it can be represented exactly by the two pairs of spectral functions $y(\theta), z(\theta)$ (obtained from the initial value $u_0(x)$) and $Y(\theta), Z(\theta)$ (obtained from the boundary conditions $v_0(t), \{v_k(t)\}_1^4$). Furthermore, the two pairs of spectral functions $y(\theta), z(\theta)$ and $Y(\theta), Z(\theta)$ are not independent of each other, but are related to the compatibility condition, the so-called global relation.

Keywords: Riemann-Hilbert problem, fifth-order modified Korteweg–de Vries equation, initial-boundary value problems, Fokas unified transformation approach

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1967, Gardner, Greene, Kruskal and Miura [1] first proposed the classical inverse scattering transform method when they analyzed the initial value problem of the Korteweg–de Vries (KdV) equation. In 1997, Fokas [2] proposed the unified transformation approach when studying the initial-boundary value problems (IBVPs) of nonlinear partial differential equations (PDEs), which can be used to analyze the IBVPs of linear and nonlinear PDEs on the half-line or the finite interval, such as the sine-Gordon equation [3], the nonlinear Schrödinger (NLS) equation [4], the KdV equation [5], the derivative NLS equation [6], etc (see [7–12] and references therein).

In 2012, Lenells popularized the Fokas method [13] and analyzed the IBVPs of the integrable evolution equations with a 3×3 matrix Lax pair on the half-line and the finite interval,

such as the Degasperis-Procesi equation [14], the Sasa-Satsuma equation [15], the Ostrovsky-Vakhnenko equation [16], the coupled NLS equations [17, 18], etc. After that, the idea was extended to study IBVPs of some integrable nonlinear evolution equations with a 4×4 matrix Lax pair on the half-line or the finite interval, such as the matrix Lakshmanan-Porsezian-Daniel system [19], the three-coupled Hirota system [20], the new two-component generalized Sasa-Satsuma equation [21], and the integrable spin-1 Gross–Pitaevskii equations [22] on the half-line, the integrable spin-1 Gross–Pitaevskii equations [23] and the general three-component NLS equation [24] on the finite interval.

It is well known that the modified KdV (mKdV) equation is one of the most important completely integrable nonlinear PDE [25], which has the form

$$u_t + \alpha(6u^2u_x + u_{xxx}) = 0, \quad (1.1)$$

where $u = u(x, t)$ is a real function with transverse variable x and evolution variable t . Equation (1.1) can describe internal ocean waves, waves in quantized films, transmission lines in

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the Schottky barrier, magnetohydrodynamic waves in plasmas, an Alfvén wave in a cold collision-free plasma, acoustic waves in certain anharmonic lattices, ultra-short pulses in nonlinear optics and other aspects [26–29]. Because of these important applications in physics, a series of results for the mKdV equation (1.1) have been reported, such as the Hirota bilinear technique [30], the inverse scattering transform [31], the Darboux transformation (DT) [32], etc.

In 1990, Marchant and Smyth [33] proposed the following extended mKdV equation

$$u_t + \varepsilon(6u^2u_x + u_{xxx}) + \kappa(30u^4u_x + 10u_x^3 + 40uu_xu_{xx} + 10u^2u_{xxx} + u_{xxxx}) = 0, \tag{1.2}$$

where $\varepsilon \ll 1$ and $\kappa \ll 1$ stand for the third-order and fifth-order dispersion coefficients matching with the relevant nonlinear terms, respectively. Equation (1.2) can describe the evolution of steeper waves with shorter wavelengths than the KdV equation [34]. The Painlevé test, the multi-soliton solutions [35], the infinitely many conservation laws, the periodic, the rational solutions [36], the long-time asymptotic behavior [37], the breather-soliton molecules and the breather-positons [38], the Painlevé-type asymptotics [39], the rational positons and rogue waves [40] for equation (1.2) have been studied, when $\varepsilon = 0$, the emKdV equation (1.2) reduces to the following fifth-order mKdV equation

$$u_t + \kappa(30u^4u_x + 10u_x^3 + 40uu_xu_{xx} + 10u^2u_{xxx} + u_{xxxx}) = 0. \tag{1.3}$$

In 2008, Kwon discussed the initial value problem of the fifth-order mKdV equation on the Sobolev spaces [41]. In 2017, Cheng et al studied the consistent Riccati expansion and nonlocal symmetry of the fifth-order mKdV equation [42]. In 2018, Kwak considered the low regularity of the Cauchy problem for the fifth-order mKdV equation on $[0, 2\pi]$ [43]. Recently, the soliton and breather solutions with a nonzero background [44], the long-time asymptotic behavior in the quarter plane [45] and in low regularity spaces [46] for equation (1.3) have been discussed. In this note, our aim is to construct the spectral analysis for equation (1.3) based on its Lax pair, and to derive the limit form solution of the IBVPs through the following Fokas method.

The design structure of this paper is as follows. In section 2, we will perform a spectral analysis of the Lax pair for the fifth-order mKdV equation (1.3). In section 3, we will discuss two pairs of spectral functions $y(\theta)$, $z(\theta)$ and $Y(\theta)$, $Z(\theta)$. In section 4, we will present the RH problem of the fifth-order mKdV equation (1.3). The last sections feature some conclusions and discussions.

2. The spectral analysis

The fifth-order mKdV equation (1.3) admits the Lax pair formulation [45]

$$\Psi_x = M(x, t, \theta)\Psi, \tag{2.1a}$$

$$\Psi_t = N(x, t, \theta)\Psi, \tag{2.1b}$$

where $\Psi = (\Psi_1, \Psi_2)^T$ is the vector eigenfunction and complex number θ is a spectral parameter and

$$M(x, t, \theta) = \begin{pmatrix} i\theta & u \\ -u & -i\theta \end{pmatrix} = i\theta\sigma_3 + U, \tag{2.2a}$$

$$N(x, t, \theta) = \kappa \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & -N_{11} \end{pmatrix} \\ = -16i\kappa\theta^5\sigma_3 - 16\kappa\theta^4U \\ - 8i\kappa\theta^3(U^2\sigma_3 - \sigma_3U_x) + \kappa\theta^2(4U_{xx} - 8U^3) \\ + i\kappa\theta\sigma_3(12U^2U_x - 2U_{xxx}) \\ + 2i\kappa\theta(2UU_{xx} - U_x^2 - 3U^4)\sigma_3 \\ + \kappa(-6U^5 + 10U^2U_{xx} + 10UU_x^2 - U_{xxxx}), \tag{2.2b}$$

with

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, U = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix}, \\ N_{11} = -16i\theta^5 + 8iu^2\theta^3 - (6iu^4 + 4iuu_{xx} - 2iu_x^2)\theta, \\ N_{12} = -16u\theta^4 + 8iu_x\theta^3 + (8u^3 + 4u_{xx})\theta^2 \\ - (12iu^2u_x + 2iu_{xxx})\theta \\ - 6u^5 - 10u^2u_{xx} - 10uu_x^2 - u_{xxxx}, \\ N_{21} = 16u\theta^4 + 8iu_x\theta^3 - (8u^3 + 4u_{xx})\theta^2 \\ - (12iu^2u_x + 2iu_{xxx})\theta \\ + 6u^5 + 10u^2u_{xx} + 10uu_x^2 + u_{xxxx}. \tag{2.3}$$

2.1. The exact 1-form

One can rewrite equations (2.1a)–(2.1b) as

$$\Psi_x - i\theta\sigma_3\Psi = U\Psi, \tag{2.4a}$$

$$\Psi_t + 16i\kappa\theta^5\sigma_3\Psi = V\Psi, \tag{2.4b}$$

with

$$V = -16\kappa\theta^4U + 8i\kappa\theta^3(\sigma_3U_x - U^2\sigma_3) \\ + 4\kappa\theta^2(U_{xx} - 2U^3) + 10\kappa(U^2U_{xx} + UU_x^2) \\ + 2i\kappa\theta[\sigma_3(6U^2U_x - U_{xxx}) \\ + (2UU_{xx} - U_x^2 - 3U^4)\sigma_3] - 6\kappa U^5 - \kappa U_{xxxx},$$

and introduce $\Phi(x, t, \theta)$ by

$$\Phi(x, t, \theta) = \Psi(x, t, \theta)e^{(-i\theta x + 16i\kappa\theta^5 t)\sigma_3}, \tag{2.5}$$

then, one can get the following equivalent Lax pair

$$\Phi_x - i\theta[\sigma_3, \Phi] = U\Phi, \tag{2.6a}$$

$$\Phi_t + 16i\kappa\theta^5[\sigma_3, \Phi] = V\Phi, \tag{2.6b}$$

where $[\sigma_3, \Phi] = \sigma_3\Phi - \Phi\sigma_3$, obviously, equation (2.6a)–(2.6b) can be written as the following full differential

$$d(e^{(-i\theta x + 16i\kappa\theta^5 t)\hat{\sigma}_3}\Phi(x, t, \theta)) \\ = e^{(-i\theta x + 16i\kappa\theta^5 t)\hat{\sigma}_3}(A(x, t, \theta)\Phi(x, t, \theta)), \tag{2.7}$$

where $A(x, t, \theta) = U(x, t)dx + V(x, t, \theta)dt$, and $\hat{\sigma}_3$ is a matrix operator (see [12]).

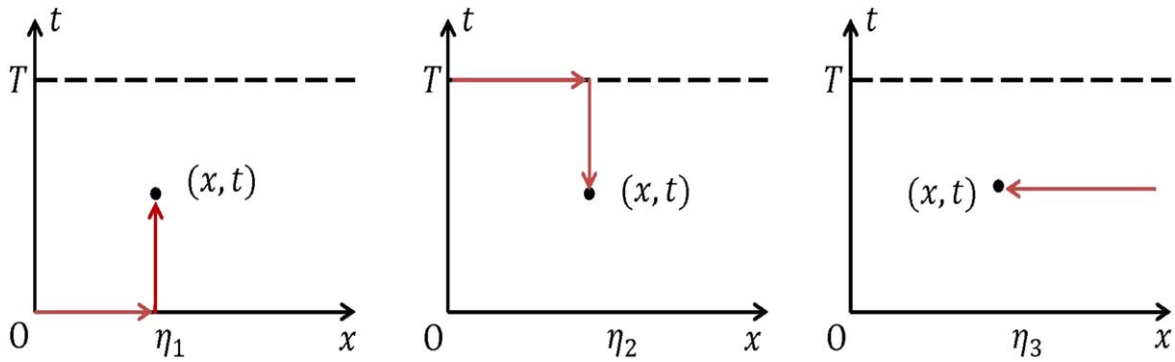


Figure 1. The three contours η_1, η_2, η_3 in the (x, t) -domain.

2.2. The three important eigenfunctions $\{\Phi_j(x, t, \theta)\}_1^3$

For $(x, t) \in D = \{(x, t) | 0 < x < +\infty, 0 < t < T\}$, let $u(x, t) \in \mathcal{S}$. We define three solutions $\{\Phi_j(x, t, \theta)\}_1^3$ of equation (2.6a)–(2.6b) by

$$\Phi_j(x, t, \theta) = I + \int_{(x_j, t_j)}^{(x, t)} e^{i\theta x - 16i\kappa\theta^5 t} \hat{\partial}_3 F(\zeta, \tau, \theta), \quad j = 1, 2, 3, \quad (2.8)$$

where $F(x, t, \theta) = e^{(-i\theta x + 16i\kappa\theta^5 t) \hat{\partial}_3} A(x, t, \theta) \Phi(x, t, \theta)$, $(x_1, t_1) = (0, 0)$, $(x_2, t_2) = (0, T)$, $(x_3, t_3) = (\infty, t)$. Because the 1-form $F(x, t, \theta)$ is exact, the integral of equation (2.8) is path independent, we choose the special curves depicted in figure 1, then we have

$$\begin{aligned} \Phi_1(x, t, \theta) &= I + \int_0^x e^{i\theta(x-\zeta) \hat{\partial}_3} (U\Phi_1)(\zeta, t, \theta) d\zeta \\ &+ e^{i\theta x \hat{\partial}_3} \int_0^t e^{-16i\kappa\theta^5(t-\tau) \hat{\partial}_3} (V\Phi_1)(0, \tau, \theta) d\tau, \end{aligned} \quad (2.9a)$$

$$\begin{aligned} \Phi_2(x, t, \theta) &= I + \int_0^x e^{i\theta(x-\zeta) \hat{\partial}_3} (U\Phi_2)(\zeta, t, \theta) d\zeta \\ &- e^{i\theta x \hat{\partial}_3} \int_t^T e^{-16i\kappa\theta^5(t-\tau) \hat{\partial}_3} (V\Phi_2)(0, \tau, \theta) d\tau, \end{aligned} \quad (2.9b)$$

$$\Phi_3(x, t, \theta) = I - \int_x^\infty e^{i\theta(x-\zeta) \hat{\partial}_3} (U\Phi_3)(\zeta, t, \theta) d\zeta. \quad (2.9c)$$

The choice of these integration paths shows that the following inequalities hold on the curves,

$$\eta_1: x - \zeta \geq 0, \quad t - \tau \geq 0, \quad (2.10a)$$

$$\eta_2: x - \zeta \geq 0, \quad t - \tau \leq 0, \quad (2.10b)$$

$$\eta_3: x - \zeta \leq 0. \quad (2.10c)$$

It is not difficult to find that the first column of equation (2.8) contains $\exp[-2i\theta(x-\zeta) + 32i\kappa\theta^5(t-\tau)]$, then, the bounded and analytical areas of eigenfunctions $\{\Phi_j(x, t, \theta)\}_1^3$ are as follows

$$[\Phi_1]_1(x, t, \theta): \{\text{Im } \theta \leq 0\} \cap \{\text{Im } \theta^5 \geq 0\}, \quad (2.11a)$$

$$[\Phi_2]_1(x, t, \theta): \{\text{Im } \theta \leq 0\} \cap \{\text{Im } \theta^5 \leq 0\}, \quad (2.11b)$$

$$[\Phi_3]_1(x, t, \theta): \{\text{Im } \theta \geq 0\}. \quad (2.11c)$$

Similarly, we have

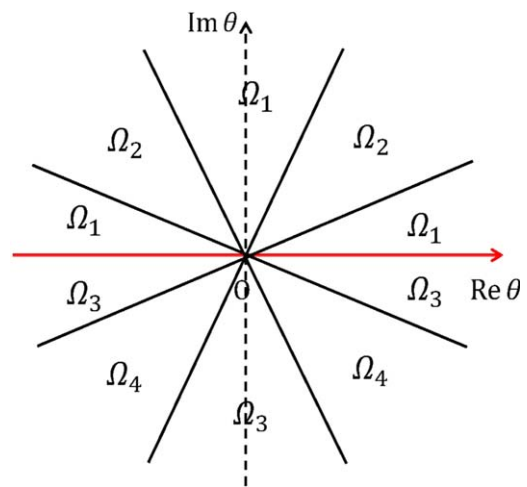


Figure 2. The areas $\Omega_i, i = 1, \dots, 4$ division on the complex θ -plane.

$$[\Phi_1]_2(x, t, \theta): \{\text{Im } \theta \geq 0\} \cap \{\text{Im } \theta^5 \leq 0\}, \quad (2.12a)$$

$$[\Phi_2]_2(x, t, \theta): \{\text{Im } \theta \geq 0\} \cap \{\text{Im } \theta^5 \geq 0\}, \quad (2.12b)$$

$$[\Phi_3]_2(x, t, \theta): \{\text{Im } \theta \leq 0\}, \quad (2.12c)$$

because the second column of equation (2.8) involves $\exp[2i\theta(x-\zeta) - 32i\kappa\theta^5(t-\tau)]$. Let $[\Phi_j]_k(x, t, \theta), k = 1, 2$ be k -column of $\Phi_j(x, t, \theta)$, we have

$$\begin{aligned} \Phi_1(x, t, \theta) &= ([\Phi_1]_1^{\Omega_4}(x, t, \theta), [\Phi_1]_2^{\Omega_2}(x, t, \theta)), \\ \Phi_2(x, t, \theta) &= ([\Phi_2]_1^{\Omega_3}(x, t, \theta), [\Phi_2]_2^{\Omega_1}(x, t, \theta)), \\ \Phi_3(x, t, \theta) &= ([\Phi_3]_1^{\mathbb{C}^+}(x, t, \theta), [\Phi_3]_2^{\mathbb{C}^-}(x, t, \theta)), \end{aligned} \quad (2.13)$$

where $\Phi_j^{\Omega_i}$ denotes that Ω_i is the bounded and analytical region of $\{\Phi_j\}_1^3$, $\mathbb{C}_+ = \Omega_1 \cup \Omega_2$ is the upper complex half-plane, $\mathbb{C}_- = \Omega_3 \cup \Omega_4$ is the lower complex half-plane, and $\Omega_i, i = 1, 2, 3, 4$ are depicted in figure 2.

In order to establish the RH problem of the fifth-order mKdV equation (1.3), we must define $f(\theta)$ and $g(\theta)$ by the following relations

$$\Phi_3(x, t, \theta) = \Phi_1(x, t, \theta) e^{i\theta x - 16i\kappa\theta^5 t} f(\theta), \quad (2.14a)$$

$$\Phi_2(x, t, \theta) = \Phi_1(x, t, \theta) e^{i\theta x - 16i\kappa\theta^5 t} g(\theta). \quad (2.14b)$$

By evaluation of equation (2.14a)–(2.14b) at $(x, t) = (0, 0)$

and $(x, t) = (0, T)$, respectively, we obtain

$$\begin{aligned} g(\theta) &= \Phi_2(0, 0, \theta) = (e^{16i\kappa\theta^5 T \hat{\sigma}_3} \Phi_1(0, T, \theta))^{-1}, \\ f(\theta) &= \Phi_3(0, 0, \theta), \end{aligned} \tag{2.15}$$

thus, from equation (2.14a)–(2.14b) and equation (2.15), we have

$$\Phi_2(x, t, \theta) = \Phi_3(x, t, \theta) e^{i\theta x - 16i\kappa\theta^5 t \hat{\sigma}_3} (f(\theta))^{-1} g(\theta), \tag{2.16}$$

Furthermore, we also get $\Phi_1(x, t, \theta)$, $\Phi_2(x, t, \theta)$ at $x = 0$

$$\begin{aligned} \Phi_1(0, t, \theta) &= ([\Phi_1]_1^{\Omega_1 \cup \Omega_4}(0, t, \theta), [\Phi_1]_2^{\Omega_2 \cup \Omega_3}(0, t, \theta)) \\ &= I + \int_0^t e^{-16i\kappa\theta^5(t-\tau)\hat{\sigma}_3} (V\Phi_1)(0, \tau, \theta) d\tau, \end{aligned} \tag{2.17a}$$

$$\begin{aligned} \Phi_2(0, t, \theta) &= ([\Phi_2]_1^{\Omega_2 \cup \Omega_3}(0, t, \theta), [\Phi_2]_2^{\Omega_1 \cup \Omega_4}(0, t, \theta)) \\ &= I - \int_t^T e^{-16i\kappa\theta^5(T-\tau)\hat{\sigma}_3} (V\Phi_2)(0, \tau, \theta) d\tau, \end{aligned} \tag{2.17b}$$

and get $\Phi_1(x, t, \theta)$, $\Phi_3(x, t, \theta)$ at $t = 0$

$$\begin{aligned} \Phi_1(x, 0, \theta) &= ([\Phi_1]_1^{\mathbb{C}_-}(x, 0, \theta), [\Phi_1]_2^{\mathbb{C}_+}(x, 0, \theta)), \\ &= I + \int_0^x e^{i\theta(x-\zeta)\hat{\sigma}_3} (U\Phi_1)(\zeta, 0, \theta) d\zeta, \end{aligned} \tag{2.18a}$$

$$\begin{aligned} \Phi_3(x, 0, \theta) &= ([\Phi_3]_1^{\mathbb{C}_+}(x, 0, \theta), [\Phi_3]_2^{\mathbb{C}_-}(x, 0, \theta)) \\ &= I - \int_x^\infty e^{i\theta(x-\zeta)\hat{\sigma}_3} (U\Phi_3)(\zeta, 0, \theta) d\zeta, \end{aligned} \tag{2.18b}$$

Assume that $u_0(x) = u(x, t = 0)$, $v_0(t) = u(x = 0, t)$, $v_1(t) = u_x(x = 0, t)$, $v_2(t) = u_{xx}(x = 0, t)$, $v_3(t) = u_{xxx}(x = 0, t)$, $v_4(t) = u_{xxxx}(x = 0, t)$ are the initial and boundary values, then, we have

$$\begin{aligned} U(x, 0, \theta) &= \begin{pmatrix} 0 & u_0 \\ -u_0 & 0 \end{pmatrix}, \\ V(0, t, \theta) &= \begin{pmatrix} V_{11}(0, t, \theta) & V_{12}(0, t, \theta) \\ V_{21}(0, t, \theta) & -V_{11}(0, t, \theta) \end{pmatrix}. \end{aligned} \tag{2.19}$$

with

$$\begin{aligned} V_{11}(0, t, \theta) &= i\kappa[8v_0^2\theta^3 - (6v_0^4 + 4v_0v_2 - 2v_1^2)\theta], \\ V_{12}(0, t, \theta) &= \kappa[-16v_0\theta^4 \\ &+ 8iv_1\theta^3 + (8v_0^3 + 4v_2)\theta^2 - (12iv_0^2v_1 + 2iv_3)\theta \\ &- 6v_0^5 - 10v_0^2v_2 - 10v_0v_1^2 - v_4], \\ V_{21}(0, t, \theta) &= \kappa[16v_0\theta^4 + 8iv_1\theta^3 - (8v_0^3 + 4v_2)\theta^2 \\ &- (12iv_0^2v_1 + 2iv_3)\theta \\ &+ 6v_0^5 + 10v_0^2v_2 + 10v_0v_1^2 + v_4], \end{aligned}$$

2.3. The other properties of the eigenfunctions

Proposition 2.1. (Symmetries) Let $\Phi(x, t, \theta) = \Phi_j(x, t, \theta)$, $j = 1, 2, 3$, then $\Phi(x, t, \theta)$ admits the following

symmetry relations

$$\begin{aligned} \overline{\Phi(x, t, \bar{\theta})} &= \Phi(x, t, -\theta) = \sigma_2 \Phi(x, t, \theta) \sigma_2, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \end{aligned} \tag{2.20}$$

as well as

$$\begin{aligned} \Phi_{11}(x, t, \theta) &= \overline{\Phi_{22}(x, t, \bar{\theta})}, \\ \Phi_{21}(x, t, \theta) &= -\overline{\Phi_{12}(x, t, \bar{\theta})} \end{aligned} \tag{2.21a}$$

$$\begin{aligned} \Phi_{11}(x, t, -\theta) &= -\Phi_{11}(x, t, \theta), \\ \Phi_{22}(x, t, -\theta) &= -\Phi_{22}(x, t, \theta), \end{aligned} \tag{2.21b}$$

$$\begin{aligned} \Phi_{12}(x, t, -\theta) &= \Phi_{12}(x, t, \theta), \\ \Phi_{21}(x, t, -\theta) &= \Phi_{21}(x, t, \theta), \end{aligned} \tag{2.21c}$$

Proposition 2.2. The eigenfunctions $\Phi_j(x, t, \theta) = ([\Phi_j]_1(x, t, \theta), [\Phi_j]_2(x, t, \theta))$, $j = 1, 2, 3$ admit the following properties

- $\det \Phi_j(x, t, \theta) = 1$, $j = 1, 2, 3$,
- $[\Phi_1]_1$ is analytical for $\theta \in \Omega_4$, as well as continuous to $\overline{\Omega}_4$, $[\Phi_1]_2$ is analytical for $\theta \in \Omega_2$, as well as continuous to $\overline{\Omega}_2$,
- $[\Phi_2]_1$ is analytical for $\theta \in \Omega_3$, as well as continuous to $\overline{\Omega}_3$, $[\Phi_2]_2$ is analytical for $\theta \in \Omega_1$, as well as continuous to $\overline{\Omega}_1$,
- $[\Phi_3]_1$ is analytical for $\theta \in \mathbb{C}_+$, as well as continuous to $\mathbb{C}_+ \cup \text{Re } \theta$, $[\Phi_3]_2$ is analytical for $\theta \in \mathbb{C}_-$, as well as continuous to $\mathbb{C}_- \cup \text{Re } \theta$,
- $[\Phi_j]_1(x, t, \theta) \rightarrow (1, 0)^T$, $[\Phi_j]_2(x, t, \theta) \rightarrow (0, 1)^T$, as $\theta \rightarrow \infty$.

Indeed, according to equation (2.15), we can get the expressions for $f(\theta)$ and $g(\theta)$ as follows:

$$f(\theta) = I - \int_0^\infty e^{-i\theta(\zeta-x)\hat{\sigma}_3} (U\Phi_3)(\zeta, 0, \theta) d\zeta, \tag{2.22a}$$

$$g^{-1}(\theta) = I + \int_0^T e^{16i\kappa\theta^5(\tau-t)\hat{\sigma}_3} (V\Phi_1)(0, \tau, \theta) d\tau, \tag{2.22b}$$

then, by the symmetry of Proposition 2.1, we can define

$$\begin{aligned} f(\theta) &= \begin{pmatrix} \overline{y(\bar{\theta})} & z(\theta) \\ -z(\bar{\theta}) & y(\theta) \end{pmatrix}, \\ g(\theta) &= \begin{pmatrix} \overline{Y(\bar{\theta})} & Z(\theta) \\ -Z(\bar{\theta}) & Y(\theta) \end{pmatrix}. \end{aligned} \tag{2.23}$$

Proposition 2.3. It follows from equations (2.15), (2.22a)–(2.22b) that the $y(\theta)$, $z(\theta)$ and $Y(\theta)$, $Z(\theta)$ have properties as follows

- $\begin{pmatrix} z(\theta) \\ y(\theta) \end{pmatrix} = [\Phi_3]_2^{\mathbb{C}_-}(0, 0, \theta) = \begin{pmatrix} (\Phi_3)_{12}^{\mathbb{C}_-}(0, 0, \theta) \\ (\Phi_3)_{22}^{\mathbb{C}_-}(0, 0, \theta) \end{pmatrix}$,
- $\begin{pmatrix} -e^{32i\kappa\theta^5 T} Z(\theta) \\ \overline{Y(\bar{\theta})} \end{pmatrix} = [\Phi_1]_2^{\Omega_2 \cup \Omega_3}(0, T, \theta) = \begin{pmatrix} (\Phi_1)_{12}^{\Omega_2 \cup \Omega_3}(0, T, \theta) \\ (\Phi_1)_{22}^{\Omega_2 \cup \Omega_3}(0, T, \theta) \end{pmatrix}$,
- $y(-\theta) = \overline{y(\bar{\theta})}$, $z(-\theta) = \overline{z(\bar{\theta})}$.

- $Y(-\theta) = \overline{Y(\overline{\theta})}$, $Z(-\theta) = \overline{Z(\overline{\theta})}$.
- $\det f(\theta) = y(\theta)y(\overline{\theta}) + z(\theta)z(\overline{\theta}) = 1$, for $\theta \in \mathbb{R}$.
- $\det g(\theta) = Y(\theta)\overline{Y(\overline{\theta})} + Z(\theta)\overline{Z(\overline{\theta})} = 1$, for $\theta \in \mathbb{C}$ ($\text{Im } \theta^5 = 0$, if $T = \infty$).
- $y(\theta) = 1 + O(\theta^{-1})$, $z(\theta) = O(\theta^{-1})$, as $\theta \rightarrow \infty$, $\text{Im } \theta > 0$.
- $Y(\theta) = 1 + O(\theta^{-1})$, $Z(\theta) = O(\theta^{-1})$, as $\theta \rightarrow \infty$, $\text{Im } \theta^5 > 0$.

2.4. The basic RH problem

For the convenience of calculation, we introduce the following expressions

$$\begin{aligned} \mu(\theta) &= -\theta x + 16\kappa\theta^5 t, \\ \alpha(\theta) &= y(\theta)\overline{Y(\overline{\theta})} + z(\theta)\overline{Z(\overline{\theta})}, \quad \theta \in \overline{\Omega}_2, \\ \beta(\theta) &= y(\theta)Z(\theta) - z(\theta)Y(\theta), \quad \theta \in \overline{\Omega}_4, \\ S(\theta) &= \frac{\overline{Z(\overline{\theta})}}{y(\theta)\alpha(\theta)}, \quad \theta \in \overline{\Omega}_3, \\ s(\theta) &= \frac{\overline{\beta(\overline{\theta})}}{\alpha(\theta)} = \frac{\overline{z(\overline{\theta})}}{y(\theta)} - S(\theta), \quad \theta \in \mathbb{R}, \end{aligned} \tag{2.24}$$

then, we get

- $S(-\theta) = \overline{S(\overline{\theta})}$, $s(-\theta) = \overline{s(\overline{\theta})}$.
- $g^{-1}(\theta)f(\theta) = \begin{pmatrix} \overline{\alpha(\overline{\theta})} & \beta(\theta) \\ -\overline{\beta(\overline{\theta})} & \alpha(\theta) \end{pmatrix}$,
- $\alpha(-\theta) = \overline{\alpha(\overline{\theta})}$, $\beta(-\theta) = \overline{\beta(\overline{\theta})}$.
- $\det[g^{-1}(\theta)f(\theta)] = \alpha(\theta)\overline{\alpha(\overline{\theta})} + \beta(\theta)\overline{\beta(\overline{\theta})} = 1$,
- $\alpha(\theta) = 1 + O\left(\frac{1}{\theta}\right)$, $\beta(\theta) = O\left(\frac{1}{\theta}\right)$ as $\theta \rightarrow \infty$.

Next, we define the function $H(x, t, \theta)$ by

$$H_+^{(1)}(x, t, \theta) = \left([\Phi_3]_1^{\mathbb{C}^+}(x, t, \theta), \frac{[\Phi_2]_2^{\Omega_1}(x, t, \theta)}{\alpha(\overline{\theta})} \right), \tag{2.25a}$$

$\theta \in \Omega_1$,

$$H_-^{(1)}(x, t, \theta) = \left([\Phi_3]_1^{\mathbb{C}^+}(x, t, \theta), \frac{[\Phi_1]_2^{\Omega_2}(x, t, \theta)}{y(\overline{\theta})} \right), \tag{2.25b}$$

$\theta \in \Omega_2$,

$$H_+^{(2)}(x, t, \theta) = \left(\frac{[\Phi_1]_1^{\Omega_4}(x, t, \theta)}{y(\theta)}, [\Phi_3]_2^{\mathbb{C}^-}(x, t, \theta) \right), \tag{2.25c}$$

$\theta \in \Omega_4$,

$$H_-^{(2)}(x, t, \theta) = \left(\frac{[\Phi_2]_1^{\Omega_3}(x, t, \theta)}{\alpha(\theta)}, [\Phi_3]_2^{\mathbb{C}^-}(x, t, \theta) \right), \tag{2.25d}$$

$\theta \in \Omega_3$.

Therefore, we have

$$\det H(x, t, \theta) = 1, H(x, t, \theta) \rightarrow I, \theta \rightarrow \infty. \tag{2.26}$$

Theorem 2.4. Let $u(x, t) \in \mathbb{S}$, on curve $\overline{\Omega}_i$, $i = 1, \dots, 4$, the function $H(x, t, \theta)$ defined by equation (2.25a)–(2.25d)

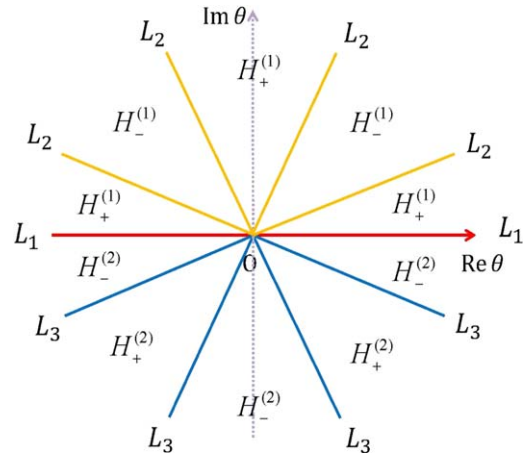


Figure 3. The contour for the RH problem on the complex θ -plane.

satisfies the jump condition as follows

$$\begin{aligned} H_+(x, t, \theta) &= H_-(x, t, \theta)L(x, t, \theta), \\ \theta \in \Pi &= \{\theta \in \mathbb{C} | \text{Im } \theta^5 = 0\}, \end{aligned} \tag{2.27}$$

where

$$L(x, t, \theta) = \begin{cases} L_1(x, t, \theta), & \theta \in \mathbb{R}, \\ L_2(x, t, \theta), & \theta \in \partial\Omega_2, \\ L_3(x, t, \theta), & \theta \in \partial\Omega_4, \end{cases} \tag{2.28}$$

with

$$\begin{aligned} L_1(x, t, \theta) &= \begin{pmatrix} 1 & -\overline{s(\overline{\theta})}e^{-2i\mu(\theta)} \\ s(\theta)e^{2i\mu(\theta)} & 1 - s(\theta)\overline{s(\overline{\theta})} \end{pmatrix}, \\ L_2(x, t, \theta) &= \begin{pmatrix} 1 & \overline{S(\overline{\theta})}e^{-2i\mu(\theta)} \\ 0 & 1 \end{pmatrix}, \\ L_3(x, t, \theta) &= \begin{pmatrix} 1 & 0 \\ S(\theta)e^{2i\mu(\theta)} & 1 \end{pmatrix}. \end{aligned}$$

The contour for this RH problem is shown in figure 3.

Proof. It follows from the equations (2.14a)–(2.14b) and equation (2.23) that

$$\begin{aligned} \overline{y(\overline{\theta})}[\Phi_1]_1^{\Omega_4}(x, t, \theta) - \overline{z(\overline{\theta})}e^{2i\mu(\theta)} \\ \times [\Phi_1]_2^{\Omega_2}(x, t, \theta) &= [\Phi_3]_1^{\mathbb{C}^+}(x, t, \theta), \end{aligned} \tag{2.29a}$$

$$\begin{aligned} z(\theta)e^{-2i\mu(\theta)}[\Phi_1]_1^{\Omega_4}(x, t, \theta) + y(\theta) \\ \times [\Phi_1]_2^{\Omega_2}(x, t, \theta) &= [\Phi_3]_2^{\mathbb{C}^-}(x, t, \theta), \end{aligned} \tag{2.29b}$$

and

$$\begin{aligned} \overline{Y(\overline{\theta})}[\Phi_1]_1^{\Omega_4}(x, t, \theta) - \overline{Z(\overline{\theta})}e^{2i\mu(\theta)} \\ \times [\Phi_1]_2^{\Omega_2}(x, t, \theta) &= [\Phi_2]_1^{\Omega_3}(x, t, \theta), \end{aligned} \tag{2.30a}$$

$$\begin{aligned} Z(\theta)e^{-2i\mu(\theta)}[\Phi_1]_1^{\Omega_4}(x, t, \theta) + Y(\theta) \\ \times [\Phi_1]_2^{\Omega_2}(x, t, \theta) &= [\Phi_2]_2^{\Omega_1}(x, t, \theta), \end{aligned} \tag{2.30b}$$

then, the equations (2.29a)–(2.30b) and equation (2.24) show

that

$$\alpha(\theta)[\Phi_3]_1^{C^+}(x, t, \theta) - \overline{\beta(\bar{\theta})}e^{2i\mu(\theta)}[\Phi_3]_2^{C^-}(x, t, \theta) = [\Phi_2]_1^{\Omega_3}(x, t, \theta), \tag{2.31a}$$

$$-\beta(\theta)e^{-2i\mu(\theta)}[\Phi_3]_1^{C^+}(x, t, \theta) + \overline{\alpha(\bar{\theta})}[\Phi_3]_2^{C^-}(x, t, \theta) = [\Phi_2]_2^{\Omega_1}(x, t, \theta). \tag{2.31b}$$

By equations (2.25a)–(2.25d) and equation (2.27), we have

$$\left([\Phi_3]_1^{C^+}(x, t, \theta), \frac{[\Phi_2]_2^{\Omega_1}(x, t, \theta)}{\alpha(\bar{\theta})} \right) = \left(\frac{[\Phi_2]_1^{\Omega_3}(x, t, \theta)}{\alpha(\theta)}, [\Phi_3]_2^{C^-}(x, t, \theta) \right) L_1(x, t, \theta), \tag{2.32a}$$

$$\left([\Phi_3]_1^{C^+}(x, t, \theta), \frac{[\Phi_2]_2^{\Omega_1}(x, t, \theta)}{\alpha(\bar{\theta})} \right) = \left([\Phi_3]_1^{C^+}(x, t, \theta), \frac{[\Phi_1]_2^{\Omega_2}(x, t, \theta)}{y(\bar{\theta})} \right) L_2(x, t, \theta), \tag{2.32b}$$

$$\left(\frac{[\Phi_1]_1^{\Omega_4}(x, t, \theta)}{y(\theta)}, [\Phi_3]_2^{C^-}(x, t, \theta) \right) = \left(\frac{[\Phi_2]_1^{\Omega_3}(x, t, \theta)}{\alpha(\theta)}, [\Phi_3]_2^{C^-}(x, t, \theta) \right) L_3(x, t, \theta). \tag{2.32c}$$

Therefore, the equations (2.32a)–(2.32c) give rise to the jump matrices $\{L_j(x, t, \theta)\}_1^3$ defined by equation (2.28).

Furthermore, we can get the jump relation between $H_+^{(2)}(x, t, \theta)$ and $H_-^{(1)}(x, t, \theta)$ as follows

$$H_+^{(2)}(x, t, \theta) = H_-^{(1)}(x, t, \theta)L_4(x, t, \theta) = H_-^{(1)}(L_3^{-1}L_1L_2^{-1})(x, t, \theta),$$

where

$$L_4(x, t, \theta) = \begin{pmatrix} 1 & -\frac{z(\theta)}{y(\bar{\theta})}e^{-2i\mu(\theta)} \\ -\frac{\overline{z(\bar{\theta})}}{y(\theta)}e^{2i\mu(\theta)} & \frac{1}{y(\theta)y(\bar{\theta})} \end{pmatrix}.$$

Assumption 2.5. Suppose that

- $y(\theta)$ enjoys λ simple zeros $\{\gamma_k\}_{k=1}^\lambda$, $\lambda = 2\lambda_1$, as well as $\{\bar{\gamma}_k\}_1^\lambda \in \Omega_4$, and $\{\bar{\gamma}_k\}_1^\lambda \in \Omega_2$.
- $\alpha(\theta)$ enjoys ξ simple zeros $\{\delta_k\}_{k=1}^\xi$, $\xi = 2\xi_1 + \xi_2$, as well as $\{\bar{\delta}_k\}_1^\xi \in \Omega_3$, and $\{\bar{\delta}_k\}_1^\xi \in \Omega_1$.
- None of the zeros of $y(\theta)$ coincides with a zero of $\alpha(\theta)$.

Proposition 2.6. (The residue formula) Let $\dot{y}(\theta) = \frac{dy}{d\theta}$, it holds that the following residue formulae:

$$\text{Res}\{[H(x, t, \theta)]_1, \gamma_k\} = \frac{1}{z(\gamma_k)\dot{y}(\gamma_k)}e^{2i\mu(\gamma_k)}[H(x, t, \gamma_k)]_2, \tag{2.33a}$$

$$k = 1, \dots, 2\lambda_1.$$

$$\text{Res}\{[H(x, t, \theta)]_2, \bar{\gamma}_k\} = -\frac{1}{z(\bar{\gamma}_k)\dot{y}(\bar{\gamma}_k)}e^{-2i\mu(\bar{\gamma}_k)} \times [H(x, t, \bar{\gamma}_k)]_1, k = 1, \dots, 2\lambda_1. \tag{2.33b}$$

$$\text{Res}\{[H(x, t, \theta)]_1, \delta_k\} = -\frac{\overline{Z(\bar{\delta}_k)}}{y(\delta_k)\dot{\alpha}(\delta_k)}e^{2i\mu(\delta_k)}[H(x, t, \delta_k)]_1, \tag{2.33c}$$

$$k = 1, \dots, \xi.$$

$$\text{Res}\{[H(x, t, \theta)]_2, \bar{\delta}_k\} = \frac{Z(\bar{\delta}_k)}{y(\bar{\delta}_k)\dot{\alpha}(\bar{\delta}_k)}e^{-2i\mu(\bar{\delta}_k)}[H(x, t, \bar{\delta}_k)]_2, \tag{2.33d}$$

$$k = 1, \dots, \xi.$$

Proof. We shall only prove equation (2.33a). the proofs of equation (2.33b)–(2.33d) are analogous. It follows from $H(x, t, \theta) = (\frac{[\Phi_1]_1^{\Omega_4}}{y(\theta)}, [\Phi_3]_2^{C^-})$ that the zeros $\{\gamma_k\}_1^{2\lambda_1}$ of $y(\theta)$ are the poles of $\frac{[\Phi_1]_1^{\Omega_4}}{y(\theta)}$, therefore, we have

$$\text{Res}\left\{\frac{[\Phi_1]_1^{\Omega_4}(x, t, \theta)}{y(\theta)}, \gamma_k\right\} = \lim_{\theta \rightarrow \gamma_k} (\theta - \gamma_k) \frac{[\Phi_1]_1^{\Omega_4}(x, t, \theta)}{y(\theta)} = \frac{[\Phi_1]_1^{\Omega_4}(x, t, \gamma_k)}{\dot{y}(\gamma_k)}, \tag{2.34}$$

taking $\theta = \gamma_k$ into the equation (2.29b) yields

$$[\Phi_1]_1^{\Omega_4}(x, t, \gamma_k) = \frac{1}{z(\gamma_k)}e^{2i\mu(\gamma_k)}[\Phi_3]_2^{C^-}(x, t, \gamma_k), \tag{2.35}$$

thus, equation (2.34) and equation (2.35) give rise to

$$\text{Res}\left\{\frac{[\Phi_1]_1^{\Omega_4}(x, t, \theta)}{y(\theta)}, \gamma_k\right\} = \frac{1}{z(\gamma_k)\dot{y}(\gamma_k)}e^{2i\mu(\gamma_k)}[\Phi_3]_2^{C^-}(x, t, \gamma_k), \tag{2.36}$$

which can lead to equation (2.33a).

2.5. The global relation

In this subsection, we show that $y(\theta)$, $z(\theta)$, $Y(\theta)$, $Z(\theta)$ are not independent but admit a significant relationship. In fact, the integral of the 1-form $A(\zeta, \tau, \theta)$ defined by the equation (2.7) is vanished for the boundary of the region $(\zeta, \tau): 0 < \zeta < \infty, 0 < \tau < t$. Let $\Phi(\zeta, \tau, \theta) = \Phi_3(\zeta, \tau, \theta)$ in the 1-form $A(\zeta, \tau, \theta)$,

we have

$$\begin{aligned} & \int_{\infty}^0 e^{-i\theta\zeta\hat{\sigma}_3}(U\Phi_3)(\zeta, 0, \theta)d\zeta \\ & + \int_0^t e^{16i\kappa\theta^5\tau\hat{\sigma}_3}(V\Phi_3)(0, \tau, \theta)d\tau \\ & + e^{16i\kappa\theta^5t\hat{\sigma}_3} \int_0^{\infty} e^{-i\theta\zeta\hat{\sigma}_3}(U\Phi_3)(\zeta, t, \theta)d\zeta \\ & = x \lim_{x \rightarrow \infty} e^{-i\theta x\hat{\sigma}_3} \int_0^t e^{16i\kappa\theta^5\tau\hat{\sigma}_3}(V\Phi_3)(x, \tau, \theta)d\tau. \end{aligned} \quad (2.37)$$

Since $f(\theta) = \Phi_3(0, 0, \theta)$, equation (2.18b) mean that

$$\int_{\infty}^0 e^{-i\theta\zeta\hat{\sigma}_3}(U\Phi_3)(\zeta, 0, \theta)d\zeta = f(\theta) - I.$$

Evaluation of equation (2.14a) at $x = 0$ gives the expressions

$$\Phi_3(0, \tau, \theta) = \Phi_1(0, \tau, \theta)e^{-16i\kappa\theta^5\tau\hat{\sigma}_3}f(\theta), \quad (2.38)$$

as well as

$$e^{16i\kappa\theta^5\tau\hat{\sigma}_3}(V\Phi_3)(0, \tau, \theta) = [e^{16i\kappa\theta^5\tau\hat{\sigma}_3}(V\Phi_1)(0, \tau, \theta)]f(\theta). \quad (2.39)$$

Furthermore, equation (2.39) and equation (2.17a) imply that

$$\begin{aligned} & \int_0^t e^{16i\kappa\theta^5\tau\hat{\sigma}_3}(V\Phi_3)(0, \tau, \theta)d\tau \\ & = [e^{16i\kappa\theta^5t\hat{\sigma}_3}\Phi_1(0, t, \theta) - I]f(\theta). \end{aligned}$$

Let $u(x, t) \in \mathbb{S}$ for $x \rightarrow \infty$, we find that equation (2.37) becomes

$$\begin{aligned} & g^{-1}(t, \theta)f(\theta) + e^{16i\kappa\theta^5t\hat{\sigma}_3} \\ & \times \int_0^{\infty} e^{-i\theta\zeta\hat{\sigma}_3}(U\Phi_3)(\zeta, t, \theta)d\zeta = I, \end{aligned} \quad (2.40)$$

where the first column of equation (2.40) is valid for $\theta \in \mathbb{C}_-$, the second column of equation (2.40) is valid for $\theta \in \mathbb{C}_-$ and

$$g^{-1}(t, \theta) = e^{16i\kappa\theta^5t\hat{\sigma}_3}\Phi_1(0, t, \theta).$$

Due to $g(\theta) = g(T, \theta)$ and denoting $t = T$, equation (2.40) becomes

$$\begin{aligned} & g^{-1}(\theta)f(\theta) + e^{16i\kappa\theta^5T\hat{\sigma}_3} \\ & \times \int_0^{\infty} e^{-i\theta\zeta\hat{\sigma}_3}(U\Phi_3)(\zeta, T, \theta)d\zeta = I. \end{aligned} \quad (2.41)$$

Hence, the (12)-component of equation (2.41) is

$$y(\theta)Z(\theta) - Y(\theta)z(\theta) = e^{32i\kappa\theta^5T}J(\theta), \quad (2.42)$$

where

$$J(\theta) = \int_0^{\infty} e^{-2i\theta\zeta}(U\Phi_3)_{12}(\zeta, T, \theta)d\zeta, \quad (2.43)$$

Indeed, equation (2.42) and equation (2.43) are the so-called global relation.

3. The functions $y(\theta)$, $z(\theta)$ and $Y(\theta)$, $Z(\theta)$

Definition 3.1. ($y(\theta)$ and $z(\theta)$) Given $u_0(x) = u(x, 0) \in \mathbb{S}$, one defines the mapping

$$\varphi_1: \{u_0(x)\} \rightarrow \{y(\theta), z(\theta)\},$$

by

$$(z(\theta), y(\theta))^T = [\Phi_3]_2^{\mathbb{C}-}(x, 0, \theta),$$

where $\Phi_3(x, 0, \theta)$ is the unique solution of the following Volterra integral equation

$$\Phi_3(x, 0, \theta) = I - \int_x^{\infty} e^{-i\theta(\zeta-x)\hat{\sigma}_3}(U\Phi_3)(\zeta, 0, \theta)d\zeta,$$

where $U(x, 0, \theta)$ is given in terms of $u_0(t)$ expressed by equation (2.19)

Proposition 3.2. The $y(\theta)$ and $z(\theta)$ possess the following properties

- (i) $y(\theta)$, $z(\theta)$ are analytical for $\text{Im } \theta > 0$ as well as bounded and continuous for $\text{Im } \theta \geq 0$.
- (ii) $y(\theta) = 1 + O(\frac{1}{\theta})$, $z(\theta) = O(\frac{1}{\theta})$, $\theta \rightarrow \infty$, $\text{Im } \theta \geq 0$.
- (iii) $y(\theta)\overline{y(\bar{\theta})} + z(\theta)\overline{z(\bar{\theta})} = 1$, $\theta \in \mathbb{R}$.
- (iv) $y(-\theta) = \overline{y(\bar{\theta})}$, $z(-\theta) = -\overline{z(\bar{\theta})}$, $\text{Im } \theta \geq 0$.
- (v) The mapping $\varphi_1^{-1} = \phi_1: \{y(\theta), z(\theta)\} \rightarrow \{u_0(x)\}$, inverse to φ_1 , is defined by

$$u_0(x) = -2i \lim_{\theta \rightarrow \infty} (\theta H^{(x)}(x, \theta))_{12},$$

where $H^{(x)}(x, \theta)$ is the unique solution to the RH problem as follows

- $H^{(x)}(x, \theta) = \begin{cases} H_-^{(x)}(x, \theta), & \text{Im } \theta \leq 0, \\ H_+^{(x)}(x, \theta), & \text{Im } \theta \geq 0, \end{cases}$ is a sectionally analytical function.
- $H_+^{(x)}(x, \theta) = H_-^{(x)}(x, \theta)L^{(x)}(x, \theta)$, $\theta \in \mathbb{R}$, where

$$L^{(x)}(x, \theta) = \begin{pmatrix} 1 & -\frac{z(\theta)}{y(\theta)}e^{-2i\theta x} \\ -\frac{\overline{z(\bar{\theta})}}{y(\theta)}e^{2i\theta x} & \frac{1}{y(\theta)\overline{y(\bar{\theta})}} \end{pmatrix}. \quad (3.1)$$

- $H^{(x)}(x, \theta) = I + O(\frac{1}{\theta})$, $\theta \rightarrow \infty$.
- $y(\theta)$ has λ simple zeros $\{\gamma_k\}_1^\lambda$, $\lambda = 2\lambda_1$, such that $\text{Im } \gamma_k < 0$, $k = 1, 2, \dots, \lambda$, where $\{\gamma_k\}_1^{2\lambda_1} \in \Omega_4$.
- The first column of $H_+^{(x)}(x, \theta)$ possesses simple poles at $\theta = \{\bar{\gamma}_k\}_1^{2\lambda_1}$. The second column of $H_-^{(x)}(x, \theta)$ possesses simple poles at $\theta = \{\gamma_k\}_1^{2\lambda_1}$. The associated residues expression are

$$\begin{aligned} \text{Res}\{[H^{(x)}(x, \theta)]_1, \gamma_k\} &= \frac{e^{2i\gamma_k x}}{y'(\gamma_k)z(\gamma_k)}[H^{(x)}(x, \gamma_k)]_2, \\ k &= 1, 2, \dots, 2\lambda_1, \end{aligned} \quad (3.2a)$$

$$\text{Res}\{[H^{(x)}(x, \theta)]_2, \bar{\gamma}_k\} = -\frac{e^{-2i\bar{\gamma}_k x}}{\dot{y}(\gamma_k)\bar{z}(\gamma_k)}[H^{(x)}(x, \bar{\gamma}_k)]_1,$$

$$k = 1, 2, \dots, 2\lambda_1.$$

$$(3.2b)$$

Proof. (i)–(iv) follow from the analysis in section 2.3, and the deduction of (v) can be obtained like [6], where the derivation of $u_0(x)$ is given in the appendix.

Definition 3.3. ($Y(\theta)$ and $Z(\theta)$). Let $v_0(t), \{v_k(t)\}_1^4 \in \mathbb{S}$, the mapping

$$\varphi_2: \{v_0(t), v_1(t), v_2(t), v_3(t), v_4(t)\} \rightarrow \{Y(\theta), Z(\theta)\},$$

in terms of

$$(Z(\theta), Y(\theta))^T = [\Phi_1]_2^{\Omega_2}(0, t, \theta),$$

where $\Phi_1(0, t, \theta)$ is the unique solution of the following Volterra integral equation

$$\Phi_1(0, t, \theta) = I - \int_t^T e^{16i\kappa\theta^5(\tau-t)\hat{\sigma}_3}(V\Phi_1)(0, \tau, \theta)d\tau,$$

and $V(0, t, \theta)$ is given in terms of $\{v_0(t), v_1(t), v_2(t), v_3(t), v_4(t)\}$ expressed by equation (2.19).

Proposition 3.4. The $Y(\theta)$ and $Z(\theta)$ possess the following properties

- (i) $Y(\theta), Z(\theta)$ are bounded for $\text{Im } \kappa\theta^5 \geq 0$, if $T = \infty$, as well as the $Y(\theta), Z(\theta)$ are defined only for $\text{Im } \kappa\theta^5 \geq 0$.
- (ii) $Y(\theta) = 1 + O\left(\frac{1}{\theta}\right), Z(\theta) = O\left(\frac{1}{\theta}\right), \theta \rightarrow \infty, \text{Im } \kappa\theta^5 \geq 0$.
- (iii) $Y(\theta)\overline{Y(\bar{\theta})} + Z(\theta)\overline{Z(\bar{\theta})} = 1, \theta \in \mathbb{C}(\kappa\theta^5 \in \mathbb{R}, \text{if } T = \infty)$.
- (iv) $Y(-\theta) = \overline{Y(\bar{\theta})}, Z(-\theta) = -\overline{Z(\bar{\theta})}, \text{Im } \kappa\theta^5 \geq 0$.
- (v) The mapping $\varphi_2^{-1} = \varphi_2: \{Y(\theta), Z(\theta)\} \rightarrow \{v_0(t), v_1(t), v_2(t), v_3(t), v_4(t)\}$, inverse to φ_2 , is defined by

$$\begin{aligned} v_0(t) &= v_0(t) = -2i\phi_{12}^{(1)}(t), \\ v_1(t) &= 4\phi_{12}^{(2)}(t) - 2iv_0(t)\phi_{22}^{(1)}(t) + v_0^2(t), \\ v_2(t) &= 8i\phi_{12}^{(3)}(t) + 4v_0(t)\phi_{22}^{(2)}(t) - 2iv_1(t)\phi_{22}^{(1)}(t) - v_0^3(t), \\ v_3(t) &= -16\phi_{12}^{(4)}(t) + 8iv_0(t)\phi_{22}^{(3)}(t) + 4v_1(t)\phi_{22}^{(2)}(t) \\ &\quad - 2i(v_2(t) + v_0^3(t))\phi_{22}^{(1)}(t) - 5v_0^2(t)v_1(t), \\ v_4(t) &= -32i\phi_{12}^{(5)}(t) - 16v_0(t)\phi_{22}^{(4)}(t) + 8iv_0(t)\phi_{22}^{(3)}(t) \\ &\quad + 4(v_2(t) + v_0^3(t))\phi_{22}^{(2)}(t) - 2i(5v_0^2(t)v_1(t) \\ &\quad + v_3(t))\phi_{22}^{(1)}(t) \\ &\quad - (10v_0^2(t)v_1(t) + 10v_0(t)v_1^2(t) + 6v_0^5(t)), \end{aligned} \tag{3.3}$$

where the functions $\phi^{(j)}(t), j = 1, 2, 3, 4, 5$ are determined by

$$H^{(t)}(t, \theta) = I + \sum_{j=1}^5 \frac{\phi^{(j)}(t)}{\theta^j} + O\left(\frac{1}{\theta^6}\right), \quad \theta \rightarrow \infty,$$

where $H^{(t)}(t, \theta)$ is the unique solution to the RH problem as follows

- $H^{(t)}(t, \theta) = \begin{cases} H_{-}^{(t)}(t, \theta), & \text{Im } \kappa\theta^5 \leq 0, \\ H_{+}^{(t)}(t, \theta), & \text{Im } \kappa\theta^5 \geq 0, \end{cases}$ is a sectionally analytical function.
- $H_{-}^{(t)}(t, \theta) = H_{+}^{(t)}(t, \theta)L^{(t)}(t, \theta), \theta^5 \in \mathbb{R}$, where

$$H^{(t)}(t, \theta) = \begin{pmatrix} 1 & -\frac{Z(\theta)}{Y(\bar{\theta})}e^{-32i\kappa\theta^5 t} \\ \frac{Z(\bar{\theta})}{Y(\theta)}e^{32i\kappa\theta^5 t} & \frac{1}{Y(\theta)\overline{Y(\bar{\theta})}} \end{pmatrix}. \tag{3.4}$$

- $H^{(t)}(t, \theta) = I + O\left(\frac{1}{\theta}\right), \theta \rightarrow \infty$.
- $Y(\theta)$ has $2m$ simple zeros $\{\omega_k\}_1^{2m}$ such that $\text{Im } \omega_k^5 > 0, k = 1, 2, \dots, 2m$.
- The first column of $H_{+}^{(t)}(t, \theta)$ possesses simple poles at $\theta = \{\bar{\omega}_k\}_1^{2m}$, the second column of $H_{-}^{(t)}(t, \theta)$ possesses simple poles at $\theta = \{\omega_k\}_1^{2m}$. The associated residues expression are

$$\text{Res}\{[H^{(t)}(t, \theta)]_1, \omega_k\} = \frac{e^{32i\kappa\omega_k^5 t}}{Y(\omega_k)Z(\omega_k)}[H^{(t)}(t, \omega_k)]_2,$$

$$k = 1, 2, \dots, 2m,$$

$$(3.5a)$$

$$\text{Res}\{[H^{(t)}(t, \theta)]_2, \bar{\omega}_k\} = -\frac{e^{-32i\kappa\bar{\omega}_k^5 t}}{Y(\bar{\omega}_k)Z(\bar{\omega}_k)}[H^{(t)}(t, \bar{\omega}_k)]_1,$$

$$k = 1, 2, \dots, 2m.$$

$$(3.5b)$$

Proof. (i)–(iv) follow from the analysis in section 2.3, and the deduction of (v) can be obtained like [6], where the derivation of $v_0(t), v_1(t), v_2(t), v_3(t)$ and $v_4(t)$ are given in the appendix.

4. The RH problem

Theorem 4.1. Given $u_0(x) \in \mathbb{S}(\mathbb{R}^+)$, we define two pairs of spectral functions $y(\theta), z(\theta)$ and $Y(\theta), Z(\theta)$ by $u_0(x), v_0(t)$ and $\{v_k(t)\}_1^4$ according to definitions 3.1 and definitions 3.3, as well as the matrix-value functions $f(\theta)$ and $g(\theta)$ are defined by equation (2.23) in terms of $y(\theta), z(\theta)$ and $Y(\theta), Z(\theta)$, respectively. Suppose that the possible simple zeros $\{\gamma_k\}_{k=1}^\lambda$ of $y(\theta)$ and $\{\delta_k\}_{k=1}^\xi$ of $\alpha(\theta)$ are as in assumption 2.5 so that we can establish the matrix-value function $H(x, t, \theta)$ as the solution of the 2×2 matrix RH problem as follows:

- $H(x, t, \theta)$ is a sectionally analytical function for $\theta \in \mathbb{C} \setminus \{\theta^5 \in \mathbb{R}\}$.
- $H(x, t, \theta)$ has the jump relation as in theorem 2.4, i.e

$$H_+(x, t, \theta) = H_-(x, t, \theta)L(x, t, \theta), \quad \theta^5 \in \mathbb{R},$$

where $H_{\pm}(x, t, \theta)$ are defined by equations (2.25a)–(2.25d), and $L(x, t, \theta)$ is a jump matrix expressed by equation (2.28).

- $H(x, t, \theta) = I + O(\frac{1}{\theta})$, $\theta \rightarrow \infty$.
- $H(x, t, \theta)$ associated residues satisfy the relations in proposition 2.6.

Therefore, the matrix-value function $H(x, t, \theta)$ exists and is unique. Hence, the potential function $u(x, t)$ is a solution of the fifth-order mKdV equation (1.3) in terms of $H(x, t, \theta)$ by

$$u(x, t) = -2i \lim_{\theta \rightarrow \infty} (\theta H(x, t, \theta))_{12}. \quad (4.1)$$

Furthermore, $u(x, 0) = u_0(x)$, $u(0, t) = v_0(t)$, $u_x(0, t) = v_1(t)$, $u_{xx}(0, t) = v_2(t)$, $u_{xxx}(0, t) = v_3(t)$, $u_{xxxx}(0, t) = v_4(t)$.

Proof. Firstly, it is not difficult to find that equation (4.1) can be obtained by the large θ asymptotic property of the characteristic functions in the appendix. Secondly, when $y(\theta)$ and $\alpha(\theta)$ have no zeros, $H(x, t, \theta)$ can be reduced to a non-regular RH problem, so we can get that $H(x, t, \theta)$ is unique from the following Vanishing Lemma. Finally, when $y(\theta)$ and $\alpha(\theta)$ have zeros, $H(x, t, \theta)$ is a regular RH problem, which can be mapped to an algebraic equation without zeros, and the uniqueness can be obtained by solving this algebraic equation.

Therefore, we have the Vanishing lemma as follows

Lemma 4.2. (Vanishing Lemma) Assume that $H(x, t, \theta) \rightarrow 0$ as $\theta \rightarrow \infty$, then the 2×2 matrix RH problem in theorem 4.1 has only the zero solution.

Proof Assume that $H(x, t, \theta)$ is the solution to the RH problem given by Theorem 4.1, and $H_{\pm}(x, t, \theta) \rightarrow 0$ when $\theta \rightarrow \infty$. Define

$$\begin{aligned} W_+(\theta) &= H_+(\theta)H_+^\dagger(-\bar{\theta}), \text{Im } \kappa\theta^5 \geq 0, \\ W_-(\theta) &= H_-(\theta)H_-^\dagger(-\bar{\theta}), \text{Im } \kappa\theta^5 \leq 0, \end{aligned} \quad (4.2)$$

where H_{\pm}^\dagger denote the complex conjugate transposition of the 2×2 matrix H_{\pm} , the x and t are dependent. Then, $W_+(\theta)$ is analytic in $\{\theta \in \mathbb{C} \setminus \text{Im } \kappa\theta^5 > 0\}$, and $W_-(\theta)$ is analytic in $\{\theta \in \mathbb{C} \setminus \text{Im } \kappa\theta^5 < 0\}$. Combining the symmetry property $y(-\theta) = \overline{y(\bar{\theta})}$, $z(-\theta) = \overline{z(\bar{\theta})}$, $Y(-\theta) = \overline{Y(\bar{\theta})}$, $Z(-\theta) = \overline{Z(\bar{\theta})}$ in because the RH problem is equivalent to Proposition 2.3

and equation (2.28), we have

$$L_4^\dagger(-\bar{\theta}) = L_1(\theta), L_3^\dagger(-\bar{\theta}) = L_3(\theta), L_2^\dagger(-\bar{\theta}) = L_2(\theta). \quad (4.3)$$

Therefore

$$\begin{aligned} W_+(\theta) &= H_-(\theta)L(\theta)H_+^\dagger(-\bar{\theta}), \text{Im } \kappa\theta^5 \in \mathbb{R}, \\ W_-(\theta) &= H_-(\theta)L^\dagger(-\bar{\theta})H_-^\dagger(-\bar{\theta}), \text{Im } \kappa\theta^5 \in \mathbb{R}. \end{aligned} \quad (4.4)$$

For $\text{Im } \kappa\theta^5 \in \mathbb{R}$, equation (4.3) and equation (4.4) mean that $W_+(\theta) = W_-(\theta)$. Thus, $W_+(\theta)$ and $W_-(\theta)$ are defined by an entire function vanishing at infinity, and $W_{\pm}(\theta) = 0$. Since $L_1(i\varepsilon)$ ($\varepsilon \in \mathbb{R}$) is a Hermitian matrix with unit determinant and (2, 2) entry 1 for any $\varepsilon \in \mathbb{R}$. Therefore, $L_1(i\varepsilon)$ ($\varepsilon \in \mathbb{R}$) is a positive definite matrix, and because $W_+(\varepsilon)$ vanishes identically for $\varepsilon \in i\mathbb{R}$, this shows that

$$H_-(i\varepsilon)L_1(i\varepsilon)H_-^\dagger(i\varepsilon) = 0, \quad \varepsilon \in \mathbb{R}. \quad (4.5)$$

Furthermore, equation (4.5) means $H_-(i\varepsilon) = 0$ for $\varepsilon \in \mathbb{R}$. Thus, $H_+(\theta)$ and $H_-(\theta)$ are vanish identically.

5. Conclusions and discussions

In [45], the RH problem of the fifth-order mKdV equation (1.3) in the quarter plane was presented and the asymptotic behavior of the solution using the Deift-Zhou method (see [47, 48]) was analyzed. In this paper, we have discussed the IBVPs of the fifth-order mKdV equation (1.3) on the half-line. Compared with the results in [45], the difference is that we get the two pairs of spectral functions $y(\theta)$, $z(\theta)$ and $Y(\theta)$, $Z(\theta)$ admitting the global relation equation (2.42) is not the same, and we can also investigate the IBVPs of the fifth-order mKdV equation (1.3) on a finite interval. Furthermore, because the RH problem is equivalent to Gel'fand–Levitan–Marchenko (GLM) theory, we can get the soliton solution of the fifth-order mKdV equation (1.3) by solving the GLM equation following [49], these questions are the subject of our future research.

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Conflict of interest

The authors declare that they have no conflict of interest.

Appendix. Recovering $v_0(t)$ and $\{v_j(t)\}_1^4$

In this appendix, we will give a proof of equation (3.3), i.e., derive $v_0(t)$ and $v_j(t), j = 1, 2, 3, 4$ from $H^{(t)}$. Assume that $\Phi(x, t, \theta)$ is a solution of equation (2.6a)–(2.6b) and $\Phi(x, t, \theta)$ has the following asymptotic expansion

$$\Phi(x, t, \theta) = \Phi_0 + \frac{\Phi_1}{\theta} + \frac{\Phi_2}{\theta^2} + \frac{\Phi_3}{\theta^3} + \frac{\Phi_4}{\theta^4} + \frac{\Phi_5}{\theta^5} + O\left(\frac{1}{\theta^6}\right), \theta \rightarrow \infty, \tag{A.1}$$

substituting equation (A.1) into equation (2.6a) and comparing the coefficients for θ^k , we get

$$\begin{aligned} O(\theta^1): i[\sigma_3, \Phi_0] &= 0, \\ O(\theta^0): \Phi_{0x} - i[\sigma_3, \Phi_1] &= U\Phi_0. \end{aligned} \tag{A.2}$$

From $O(\theta^1)$ and $O(\theta^0)$, one find that Φ_0 enjoys a diagonal matrix form denoting as

$$\Phi_0 = \begin{pmatrix} \Phi_0^{11} & 0 \\ 0 & \Phi_0^{22} \end{pmatrix}, \Phi_1^{(od)} = \frac{i}{2}\sigma_3 U\Phi_0 = \begin{pmatrix} \frac{i}{2}u\Phi_0^{22} \\ \frac{i}{2}u\Phi_0^{11} \end{pmatrix}, \Phi_{0x} = 0, \tag{A.3}$$

where $\Phi_1^{(od)}$ denotes the off-diagonal part of Φ_1 .

At the same time, substituting equation (A.1) into equation (2.6b) and comparing the coefficient for θ^k yields

$$\begin{aligned} O(\theta^5): 16i[\sigma_3, \Phi_0] &= 0, \\ O(\theta^4): 16i[\sigma_3, \Phi_1] &= -16U\Phi_0, \\ O(\theta^3): 16i[\sigma_3, \Phi_2] &= -16U\Phi_1 + 8i(\sigma_3 U_x - U^2\sigma_3)\Phi_0, \\ O(\theta^2): 16i[\sigma_3, \Phi_3] &= -16U\Phi_2 + 8i(\sigma_3 U_x - U^2\sigma_3)\Phi_1 \\ &\quad + 4(U_{xx} - 2U^3)\Phi_0, \\ O(\theta^1): 16i[\sigma_3, \Phi_4] &= -16U\Phi_3 + 8i(\sigma_3 U_x - U^2\sigma_3)\Phi_2 \\ &\quad + 4(U_{xx} - 2U^3)\Phi_1 \\ &\quad + 2i[\sigma_3(6U^2U_x - U_{xxx}) \\ &\quad + (2UU_{xx} - U_x^2 - 3U^4)\sigma_3]\Phi_0, \end{aligned}$$

$$\begin{aligned} O(\theta^0): \Phi_{0t} + 16i\kappa[\sigma_3, \Phi_5] &= \kappa\{-16U\Phi_4 \\ &\quad + 8i(\sigma_3 U_x - U^2\sigma_3)\Phi_3 \\ &\quad + 4(U_{xx} - 2U^3)\Phi_2 \\ &\quad + 2i[\sigma_3(6U^2U_x - U_{xxx}) \\ &\quad + (2UU_{xx} - U_x^2 - 3U^4)\sigma_3]\Phi_1 \\ &\quad + [10(U^2U_{xx} \\ &\quad + UU_x^2) - 6U^5 - U_{xxx}]\Phi_0\}. \end{aligned} \tag{A.4}$$

Through tedious calculation, we have

$$\begin{aligned} \Phi_2^{12} &= \frac{i}{2}u\Phi_1^{22} + \frac{1}{4}u_x\Phi_0^{22}, & \Phi_2^{21} &= \frac{i}{2}u\Phi_1^{11} - \frac{1}{4}u_x\Phi_0^{11}, \\ \Phi_3^{12} &= \frac{i}{2}u\Phi_2^{22} + \frac{1}{4}u_x\Phi_1^{22} - \frac{i}{8}(u_{xx} + u^3)\Phi_0^{22}, \\ \Phi_3^{21} &= \frac{i}{2}u\Phi_2^{11} - \frac{1}{4}u_x\Phi_1^{11} - \frac{i}{8}(u_{xx} + u^3)\Phi_0^{11}, \\ \Phi_4^{12} &= \frac{i}{2}u\Phi_3^{22} + \frac{1}{4}u_x\Phi_2^{22} \\ &\quad - \frac{i}{8}(u_{xx} + u^3)\Phi_1^{22} - \frac{1}{16}(5u^2u_x + u_{xxx})\Phi_0^{22}, \\ \Phi_4^{21} &= \frac{i}{2}u\Phi_3^{11} - \frac{1}{4}u_x\Phi_2^{11} - \frac{i}{8}(u_{xx} + u^3)\Phi_1^{11} \\ &\quad + \frac{1}{16}(5u^2u_x + u_{xxx})\Phi_0^{11}, \\ \Phi_5^{12} &= \frac{i}{2}u\Phi_4^{22} + \frac{1}{4}u_x\Phi_3^{22} - \frac{i}{8}(u_{xx} + u^3)\Phi_2^{22} \\ &\quad - \frac{1}{16}(5u^2u_x + u_{xxx})\Phi_1^{22} \\ &\quad + \frac{i}{32}(10u^2u_{xx} + 10uu_x^2 + 6u^5 + u_{xxxx})\Phi_0^{22}, \\ \Phi_5^{21} &= \frac{i}{2}u\Phi_4^{11} - \frac{1}{4}u_x\Phi_3^{11} - \frac{i}{8}(u_{xx} + u^3)\Phi_2^{11} \\ &\quad + \frac{1}{16}(5u^2u_x + u_{xxx})\Phi_1^{11} \\ &\quad + \frac{i}{32}(10u^2u_{xx} + 10uu_x^2 + 6u^5 + u_{xxxx})\Phi_0^{11}, \\ \Phi_{0t} &= 0. \end{aligned} \tag{A.5}$$

Let $\Phi_0 = I$, according to equation (A.5), we obtain

$$\sigma_3 U_x = 4\sigma_3\Phi_2^{(od)} - 2iU\Phi_1^{(d)} + U^2\sigma_3, \tag{A.6a}$$

$$\begin{aligned} U_{xx} &= 8i\sigma_3\Phi_3^{(od)} + 4U\Phi_2^{(d)} - 2i\sigma_3 U_x\Phi_1^{(d)} \\ &\quad + 2iU^2\sigma_3\Phi_1^{(od)} + 2U^3, \end{aligned} \tag{A.6b}$$

$$\begin{aligned} \sigma_3 U_{xxx} &= 16\sigma_3\Phi_4^{(od)} + 8iU\Phi_3^{(d)} + 4\sigma_3 U_x\Phi_2^{(d)} - 4U^2\sigma_3\Phi_2^{(od)} \\ &\quad - 2i(U_{xx} - 2U^3)\Phi_1^{(d)} \\ &\quad + 6\sigma_3 U^2 U_x + (2UU_{xx} - U_x^2 - 3U^4)\sigma_3, \end{aligned} \tag{A.6c}$$

$$\begin{aligned} U_{xxxx} &= -32i\sigma_3\Phi_5^{(od)} - 16U\Phi_4^{(d)} + 8i\sigma_3 U_x\Phi_3^{(d)} \\ &\quad - 8iU^2\sigma_3\Phi_3^{(od)} \\ &\quad + 4(U_{xx} - 2U^3)\Phi_2^{(d)} + 2i\sigma_3(6U^2U_x - U_{xxx})\Phi_3^{(d)} \\ &\quad + 2i(2UU_{xx} - U_x^2 - 3U^4)\sigma_3\Phi_1^{(od)} \\ &\quad + 10(U^2U_{xx} + UU_x^2) - 6U^5, \end{aligned} \tag{A.6d}$$

where $\Phi(x, t, \theta)$ is the solution of equation (2.7). Let

$$\begin{aligned} \Phi(x, t, \theta) &= I + \frac{\phi^{(1)}}{\theta} + \frac{\phi^{(2)}}{\theta^2} + \frac{\phi^{(3)}}{\theta^3} \\ &\quad + \frac{\phi^{(4)}}{\theta^4} + \frac{\phi^{(5)}}{\theta^5} + O\left(\frac{1}{\theta^6}\right), \theta \rightarrow \infty, \end{aligned} \tag{A.7}$$

(A.4) the (12)-entry of equation (A.6a)–(A.6d) gives rise to

$$u_x = 4\phi_{12}^{(2)} - 2iu\phi_{22}^{(1)} + u^2, \quad (\text{A.8a})$$

$$u_{xx} = 8i\phi_{12}^{(3)} + 4u\phi_{22}^{(2)} - 2iu_x\phi_{22}^{(1)} - u^3, \quad (\text{A.8b})$$

$$u_{xxx} = -16\phi_{12}^{(4)} + 8iu\phi_{22}^{(3)} + 4u_x\phi_{22}^{(2)} - 2i(u_{xx} + u^3)\phi_{22}^{(1)} - 5u^2u_x, \quad (\text{A.8c})$$

$$u_{xxxx} = -32i\phi_{12}^{(5)} - 16u\phi_{22}^{(4)} + 8iu\phi_{22}^{(3)} + 4(u_{xx} + u^3)\phi_{22}^{(2)} - 2i(5u^2u_x + u_{xxx})\phi_{22}^{(1)} - (10u^2u_{xx} + 10uu_x^2 + 6u^5). \quad (\text{A.8d})$$

Furthermore, equation (A.3) implies that

$$u(x, t) = -2i\phi(x, t) = -2i \lim_{\theta \rightarrow \infty} (\theta\Phi(x, t, \theta))_{12}, \quad (\text{A.9})$$

then, we have

$$u(x, t) = -2i\phi_{12}^{(1)}(x, t). \quad (\text{A.10})$$

Evaluating equations (A.8a)–(A.8d), (A.10) at $x = 0$ gives the expressions

$$v_0(t) = -2i\phi_{12}^{(1)}(t), \quad (\text{A.11a})$$

$$v_1(t) = 4\phi_{12}^{(2)}(t) - 2iv_0(t)\phi_{22}^{(1)}(t) + v_0^2(t), \quad (\text{A.11b})$$

$$v_2(t) = 8i\phi_{12}^{(3)}(t) + 4v_0(t)\phi_{22}^{(2)}(t) - 2iv_1(t)\phi_{22}^{(1)}(t) - v_0^3(t), \quad (\text{A.11c})$$

$$v_3(t) = -16\phi_{12}^{(4)}(t) + 8iv_0(t)\phi_{22}^{(3)}(t) + 4v_1(t)\phi_{22}^{(2)}(t) - 2i(v_2(t) + v_0^3(t))\phi_{22}^{(1)}(t) - 5v_0^2(t)v_1(t), \quad (\text{A.11d})$$

$$v_4(t) = -32i\phi_{12}^{(5)}(t) - 16v_0(t)\phi_{22}^{(4)}(t) + 8iv_0(t)\phi_{22}^{(3)}(t) + 4(v_2(t) + v_0^3(t))\phi_{22}^{(2)}(t) - 2i(5v_0^2(t)v_1(t) + v_3(t))\phi_{22}^{(1)}(t) - (10v_0^2(t)v_1(t) + 10v_0(t)v_1^2(t) + 6v_0^5(t)), \quad (\text{A.11e})$$

where $\phi^{(j)}(t)$, $j = 1, 2, 3, 4, 5$ are determined by

$$H^{(t)}(t, \theta) = \mathbf{I} + \sum_{j=1}^5 \frac{\phi^{(j)}(t)}{\theta^j} + \mathcal{O}\left(\frac{1}{\theta^6}\right), \quad \theta \rightarrow \infty. \quad (\text{A.12})$$

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