

Solitons and quasi-Grammians of the generalized lattice Heisenberg magnet model

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Abstract

In this paper, we study the discrete Darboux and standard binary Darboux transformation for the generalized lattice Heisenberg magnet model. We calculate the quasi-Grammian solutions by the iteration of standard binary Darboux transformation. Furthermore, we derive the explicit matrix solutions for the binary Darboux matrix and then reduce them to the elementary Darboux matrix and plot the dynamics of solutions.

Keywords: quasi-Grammian, discrete integrable systems, binary Darboux transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

During the past few decades, there has been a lot of interest in the study of continuous and lattice Heisenberg magnet models. The continuous Heisenberg magnet model is completely integrable and exhibits the exact soliton solutions. Similarly, the lattice Heisenberg magnet model also preserves the integrability. The soliton solutions of this model have been studied using the inverse scattering transform, Bäcklund transformation, Darboux transformation and other solution-generating methods (see, e.g. [1–9]). The lattice Heisenberg magnet model has been studied in many works (see [10–13]). The existence of a Lax pair, Bäcklund transformation and other symmetries of the lattice Heisenberg magnet model explains many aspects of integrability [10–17]. Darboux transformation of the generalized lattice Heisenberg magnet model is studied in [25] and soliton solutions are presented.

Discrete integrable systems have received much attention from modern researchers. Many techniques, such as the Darboux transformation, the Hirota method, the Bäcklund transformation, etc, have been employed to calculate the exact solutions of many nonlinear partial differential equations ([30–36]). Binary Darboux transformation is a well-known technique used to compute the Grammian-type multisolitons of integrable systems [19, 33]. The general mechanism of this method is to

keep both the spectral problem and the corresponding adjoint spectral problem associated with the nonlinear equations, which are invariant with respect to the action of the binary Darboux transformation. Furthermore, the solutions can be expressed in terms of Grammian, quasi-Grammian and quasideterminants in the literature [18, 20, 33, 37, 38].

In this paper, we study the discrete Darboux and binary Darboux transformation of the generalized lattice Heisenberg magnet (GLHM) model. For this purpose, we operate the discrete Darboux matrix on a Lax pair of GLHM models for both the direct and adjoint space to calculate the multi-soliton solutions. For the representation of solutions, we use the quasideterminant approach. Furthermore, by the iteration of binary Darboux transformation, we derive the general expressions of multi quasi-Grammian solutions. Finally, we obtain the explicit solutions for the GLHM model based upon Lie group $SU(2)$ transformation and present the solutions, which also include the soliton solution.

2. Lax pair

The Lax pair of the GLHM model is given as,

$$\begin{aligned}\Psi_{n+1} &= A_n \Psi_n, \\ \frac{d}{dt} \Psi_n &= B_n \Psi_n,\end{aligned}\tag{2.1}$$

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having matrices A_n and B_n given by,

$$\begin{aligned} A_n &= I + \lambda U_n, \\ B_n &= \frac{\lambda}{1 - \lambda^2} J_n + \frac{\lambda^2}{1 - \lambda^2} J_n U_n, \end{aligned} \tag{2.2}$$

where the matrix function $U_n \equiv U_n(t)$ take values from Lie group \mathcal{G} in the Lie algebra \mathfrak{g} and $\Psi_n \equiv \Psi_n(\lambda)$ is an $N \times N$ eigen matrix, which depends upon variable n written in subscripts defined over a lattice. The matrix function U_n is subjected to constraints given by $U_n^2 = I$ and $J_n U_n = U_{n-1} J_n$, which also implies $J_n A_n = A_{n-1} J_n$. The compatibility condition $dA_n/dt + A_n B_n - B_{n+1} A_n = 0$, is operated on (2.1), which gives the equation of motion of the GLHM model given by,

$$\frac{d}{dt} U_n = J_{n+1} - J_n. \tag{2.3}$$

Equation of motion (2.3) implies that,

$$J_{n+1}(U_{n+1} + U_n) = (U_n + U_{n-1})J_n, \tag{2.4}$$

which is satisfied if we take,

$$J_n = 2iaU_{n-1}(U_n + U_{n-1})^{-1} + 2b(U_n + U_{n-1})^{-1}. \tag{2.5}$$

By substituting equation (2.5) in (2.3), we derive the following form of the equation of motion:

$$\frac{d}{dt} U_n = \Delta_n [2iaU_{n-1}(U_n + U_{n-1})^{-1} - 1 + 2b(U_n + U_{n-1})^{-1}], \tag{2.6}$$

where $\Delta_n f_n = f_{n+1} - f_n$. For $N=2$, we have the simplest 2×2 case of Lie group $SU(2)$, for which the matrix U_n is expressed as $U_n = U_n^a \sigma_a$, where σ_a are the familiar Pauli matrices and the constraint on the matrix U_n becomes $U_n^2 = I$. I is the 2×2 identity matrix. We substitute $2(U_n + U_{n-1})^{-1} = (U_n + U_{n-1})/(1 + U_n U_{n-1})$ in (2.6) and are able to express the equation of motion in vector notation as,

$$\frac{d}{dt} U_n = \Delta_n \left[a \frac{U_n \times U_{n-1}}{1 + U_n \cdot U_{n-1}} + \frac{U_n + U_{n-1}}{1 + U_n \cdot U_{n-1}} \right], \quad U_n^2 = 1. \tag{2.7}$$

3. Discrete Darboux transformation

Darboux transformation is an important tool to find solutions of integrable systems represented by differential equations, partial differential equations and differential-difference equations (for details see [20–29]). We then define the Darboux transformation on the Lax pair (2.1) by using the $N \times N$ Darboux matrix $D_n(\lambda)$ to calculate the soliton solutions. The Darboux matrix transforms the matrix solution from the space W to new space \tilde{W} , i.e.

$$\begin{aligned} D_n(\lambda): W &\longrightarrow \tilde{W} \\ : \Psi_n &\longrightarrow \tilde{\Psi}_n. \end{aligned} \tag{3.1}$$

The one-fold Darboux transformation on matrix solution Ψ_n is defined as,

$$\Psi_n[1] = D_n(\lambda)\Psi_n, \tag{3.2}$$

where $D_n(\lambda)$ is the Darboux matrix. The new transformed solution $\Psi_n[1]$ satisfies the following Lax pair (2.1) as,

$$\begin{aligned} \Psi_{n+1}[1] &= A_n[1]\Psi_n[1], \\ \frac{d}{dt} \Psi_n[1] &= B_n[1]\Psi_n[1], \end{aligned} \tag{3.3}$$

having $A_n[1]$ and $B_n[1]$ as,

$$\begin{aligned} A_n[1] &= I + \lambda U_n[1], \\ B_n[1] &= \frac{\lambda}{1 - \lambda^2} J_n[1] + \frac{\lambda^2}{1 - \lambda^2} J_n[1] U_n[1], \end{aligned} \tag{3.4}$$

where I is the identity matrix. In order to obtain the Darboux transformation on matrix solution $U_n[1]$, we define the Darboux matrix as,

$$D_n(\lambda) = \lambda^{-1} I - Q_n, \tag{3.5}$$

where I is the $N \times N$ identity matrix and Q_n is the auxiliary matrix of $N \times N$ order, which is yet to be found. The choice for Q_n is $Q_n = H_n \Lambda^{-1} H_n^{-1}$, where H_n is the distinct matrix solution of the Lax pair (2.1) having order $N \times N$, which can be obtained by using i -eigenvector functions $\Psi(\lambda_i)|\sigma_i$ evaluated at λ_i , $i = 1, \dots, N$, whereas matrix Λ is a diagonal matrix of order $N \times N$ having eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$. Therefore, matrix H_n can be defined as,,

$$H_n = (\Psi_n(\lambda_1)|\sigma_1, \dots, \Psi_n(\lambda_N)|\sigma_N), \tag{3.6}$$

evaluated at,

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N). \tag{3.7}$$

Using (3.6) and (3.7), the Lax pair (2.1) can be written in matrix form as,

$$H_{n+1} = H_n + U_n H_n \Lambda, \tag{3.8}$$

$$\frac{d}{dt} H_n = J_n H_n \Lambda (I - \Lambda^2)^{-1} + J_n U_n H_n \Lambda^2 (I - \Lambda^2)^{-1}. \tag{3.9}$$

Based upon the above results, we can prove the following theorems.

Theorem 1. Under the action of Darboux transformation (3.5), the new solution (3.4) has the identical form as U_n in equation (2.2), provided that matrix Q_n fulfills the following conditions:

$$U_n[1] = U_n - (Q_{n+1} - Q_n), \tag{3.10}$$

$$(Q_{n+1} - Q_n)Q_n = U_n Q_n - Q_{n+1} U_n. \tag{3.11}$$

Proof. The relation between the Darboux transformed solution $U_n[1]$ and the untransformed solution U_n is developed and defined in equation (3.10). We then have to show that the choice of matrix $Q_n = H_n \Lambda^{-1} H_n^{-1}$ satisfies the

condition (3.11), i.e.

$$\begin{aligned} (Q_{n+1} - Q_n)Q_n &= (H_{n+1}\Lambda^{-1}H_{n+1}^{-1} - H_n\Lambda^{-1}H_n^{-1})H_n\Lambda^{-1}H_n^{-1}, \\ &= H_{n+1}\Lambda^{-1}H_{n+1}^{-1}H_n\Lambda^{-1}H_n^{-1} - H_n\Lambda^{-1}H_n^{-1}H_n\Lambda^{-1}H_n^{-1} \\ &\quad + H_{n+1}\Lambda^{-2}H_n^{-1} - H_{n+1}\Lambda^{-2}H_n^{-1}, \\ &= (H_{n+1} - H_n)\Lambda^{-1}H_n^{-1}H_n\Lambda^{-1}H_n^{-1} \\ &\quad - H_{n+1}\Lambda^{-1}H_{n+1}^{-1}(H_{n+1} - H_n)\Lambda^{-1}H_n^{-1}, \\ &= U_nQ_n - Q_{n+1}U_n, \end{aligned}$$

which is equivalent to (3.11). Therefore, the proof is complete.

Theorem 2. Under the action of Darboux transformation (3.5), the new solution (3.4) has the identical form as J_n in equation (2.2), provided that matrix Q_n fulfills the following conditions:

$$J_n[1] = J_n - \frac{dQ_n}{dt}, \tag{3.12}$$

$$\frac{dQ_n}{dt}(I - Q_n^2) = [Q_n, J_n(Q_n + U_n)]. \tag{3.13}$$

Proof. The relation between the Darboux transformed solution $J_n[1]$ and the untransformed solution J_n is developed and defined in equation (3.12). We then have to show that the choice of matrix $Q_n = H_n\Lambda^{-1}H_n^{-1}$ satisfies the condition (3.13). For this, we operate $\frac{d}{dt}$ on matrix $(I - Q_n^2)$ as,

$$\begin{aligned} \frac{dQ_n}{dt}(I - Q_n^2) &= \left(\frac{d}{dt}H_n\Lambda^{-1}H_n^{-1}\right)(I - H_n\Lambda^{-2}H_n^{-1}), \\ &= \frac{dH_n}{dt}\Lambda^{-1}(I - \Lambda^{-2})H_n^{-1} \\ &\quad - H_n\Lambda^{-1}H_n^{-1}\frac{dH_n}{dt}(I - \Lambda^{-2})H_n^{-1}, \\ &= [J_nH_n\Lambda^{-1}(I - \Lambda^{-2})^{-1} + J_nU_nH_n\Lambda^2(I - \Lambda^{-2})^{-1}] \\ &\quad \Lambda^{-1}(I - \Lambda^{-2})H_n^{-1} - \\ &\quad H_n\Lambda^{-1}H_n^{-1}[J_nH_n\Lambda(I - \Lambda^{-2})^{-1} \\ &\quad + J_nU_nH_n\Lambda^2(I - \Lambda^{-2})^{-1}](I - \Lambda^{-2})H_n^{-1}, \\ &= [Q_n, J_n(Q_n + U_n)], \end{aligned}$$

which is equivalent to (3.13). Therefore, the proof is complete.

Remark 1. Thus, the matrix $Q_n = H_n\Lambda^{-1}H_n^{-1}$ is a good choice, which satisfies the conditions imposed by the Darboux transformation. Thus, the Darboux transformation preserves the system, i.e. if Ψ_n , U_n and J_n , respectively, are the solutions of the linear system (2.1) and (2.2) and the equation of motion (2.4), then $\Psi_n[1]$, $U_n[1]$ and $W_n[1]$ are also the solutions of the same equations.

In order to study the solutions we use the technique known as quasideterminants given by,

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & \boxed{P_{22}} \end{vmatrix} = P_{22} - P_{21}P_{11}^{-1}P_{12}.$$

For details see [39, 40]. Thus, we can write the Darboux transformation on matrix solution Ψ_n as,

$$\begin{aligned} \Psi_n[1] &\equiv D_n(\lambda)\Psi_n = (\lambda^{-1}I - H_n\Lambda^{-1}H_n^{-1})\Psi_n, \\ &= \begin{vmatrix} H_n & \Psi_n \\ H_n\Lambda^{-1} & \boxed{\lambda^{-1}I} \end{vmatrix}. \end{aligned} \tag{3.14}$$

For the next iteration of Darboux transformation, take $Q_{n,1}$ and $Q_{n,2}$ as the two particular solutions of the Lax pair (3.3) and (3.4) at $\Lambda = \Lambda_1^{-1}$ and $\Lambda = \Lambda_2^{-1}$, respectively. The two-fold Darboux transformation on $\Psi_n[1]$ is defined as,

$$\begin{aligned} \Psi_n[2] &= (\lambda^{-1}I - Q_n[2])(\lambda^{-1}I - Q_n[1])\Psi_n, \\ &= (\lambda^{-1}I - Q_n[2])\Psi_n[1], \end{aligned} \tag{3.15}$$

where $Q_n[1] = H_{n,1}\Lambda_1^{-1}H_{n,1}^{-1}$, $Q_n[2] = H_n[2]\Lambda_2^{-1}(H_n[2])^{-1}$. Also, $H_n[2]$ is written as,

$$\begin{aligned} H_n[2] &= (H_{n,2}\Lambda_2^{-1} - Q_n[1]H_{n,2}), \\ &= \begin{vmatrix} H_{n,1} & H_{n,2} \\ H_{n,1}\Lambda_1^{-1} & \boxed{H_{n,2}\Lambda_2^{-1}} \end{vmatrix}. \end{aligned} \tag{3.16}$$

Using (3.14) and (3.16) in (3.15), we obtain the following:

$$\begin{aligned} \Psi_n[2] &= \lambda^{-1} \begin{vmatrix} H_{n,1} & \Psi_n \\ H_{n,1}\Lambda_1^{-1} & \boxed{\lambda^{-1}\Psi_n} \end{vmatrix} - \begin{vmatrix} H_{n,1} & H_{n,2} \\ H_{n,1}\Lambda_1^{-1} & \boxed{H_{n,2}\Lambda_2^{-1}} \end{vmatrix} \Lambda_2^{-1} \\ &\quad \times \begin{vmatrix} H_{n,1} & H_{n,2} \\ H_{n,1}\Lambda_1^{-1} & \boxed{H_{n,2}\Lambda_2^{-1}} \end{vmatrix}^{-1} \begin{vmatrix} H_{n,1} & \Psi_n \\ H_{n,1}\Lambda_1^{-1} & \boxed{\lambda^{-1}\Psi_n} \end{vmatrix}, \\ &= \begin{vmatrix} H_{n,1}\Lambda_1^{-1} & \lambda^{-1}\Psi_n \\ H_{n,1}\Lambda_1^{-2} & \boxed{\lambda^{-2}\Psi_n} \end{vmatrix} - \begin{vmatrix} H_{n,1}\Lambda_1^{-1} & H_{n,2}\Lambda_2^{-1} \\ H_{n,1}\Lambda_1^{-2} & \boxed{H_{n,2}\Lambda_2^{-2}} \end{vmatrix} \\ &\quad \times \begin{vmatrix} H_{n,1}\Lambda_1^{-1} & H_{n,2}\Lambda_2^{-1} \\ H_{n,1} & \boxed{H_{n,2}} \end{vmatrix}^{-1} \begin{vmatrix} H_{n,1}\Lambda_1^{-1} & \lambda^{-1}\Psi_n \\ H_{n,1} & \boxed{\Psi_n} \end{vmatrix}, \\ &= \begin{vmatrix} H_{n,1} & H_{n,2} & \Psi_n \\ H_{n,1}\Lambda_1^{-1} & H_{n,2}\Lambda_2^{-1} & \lambda^{-1}\Psi_n \\ H_{n,1}\Lambda_1^{-2} & H_{n,2}\Lambda_2^{-2} & \boxed{\lambda^{-2}\Psi_n} \end{vmatrix}, \end{aligned}$$

where we have used a homological relation in the second step and a noncommutative Jacobi identity in the last step.¹ Similarly, the K -fold Darboux transformation is given by,

$$\Psi_n[K] = \begin{vmatrix} H_{n,1} & H_{n,2} & \dots & H_{n,K} & \Psi_n \\ H_{n,1}\Lambda_1^{-1} & H_{n,2}\Lambda_2^{-1} & \dots & H_{n,K}\Lambda_K^{-1} & \lambda^{-1}\Psi_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n,1}\Lambda_1^{-(K-1)} & H_{n,2}\Lambda_2^{-(K-1)} & \dots & H_{n,K}\Lambda_K^{-(K-1)} & \lambda^{-(K-1)}\Psi_n \\ H_{n,1}\Lambda_1^{-K} & H_{n,2}\Lambda_2^{-K} & \dots & H_{n,K}\Lambda_K^{-K} & \boxed{\lambda^{-K}\Psi_n} \end{vmatrix}. \tag{3.17}$$

¹ For a general quasideterminant expanded about $N \times N$ matrix D , we have

$$\begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} = \begin{vmatrix} E & G \\ J & \boxed{D} \end{vmatrix} - \begin{vmatrix} E & F \\ J & \boxed{C} \end{vmatrix} \left| \begin{vmatrix} E & F \\ H & A \end{vmatrix}^{-1} \begin{vmatrix} E & G \\ H & B \end{vmatrix} \right|.$$

From the noncommutative Jacobi identity, we obtain the homological relation

$$\begin{vmatrix} E & F & G \\ H & A & \boxed{B} \\ J & C & D \end{vmatrix} = \begin{vmatrix} E & F & O \\ H & A & \boxed{C} \\ J & C & I \end{vmatrix} \left| \begin{vmatrix} E & F & G \\ H & A & B \\ J & C & \boxed{D} \end{vmatrix} \right|, \text{ where } O$$

and I denote the null and identity matrices, respectively.

The expression (3.10) can then be expressed as,

$$U_n[1] = H_{n+1}\Lambda^{-1}H_{n+1}^{-1}U_nH_n\Lambda^{-1}H_n^{-1},$$

$$= \begin{vmatrix} H_{n+1} & I \\ H_{n+1}\Lambda^{-1} & \mathcal{O} \end{vmatrix} U_n \begin{vmatrix} H_n & I \\ H_n\Lambda^{-1} & \mathcal{O} \end{vmatrix}^{-1}. \quad (3.18)$$

The result can be generalized to K -times Darboux transformation as,

$$U_n[K] = \begin{vmatrix} H_{n+1,1} & H_{n+1,2} & \cdots & H_{n+1,K} & I \\ H_{n+1,1}\Lambda_1^{-1} & H_{n+1,2}\Lambda_2^{-1} & \cdots & H_{n+1,K}\Lambda_K^{-1} & O \\ H_{n+1,1}\Lambda_1^{-2} & H_{n+1,2}\Lambda_2^{-2} & \cdots & H_{n+1,K}\Lambda_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n+1,1}\Lambda_1^{-K} & H_{n+1,2}\Lambda_2^{-K} & \cdots & H_{n+1,K}\Lambda_K^{-K} & \mathcal{O} \end{vmatrix} \times$$

$$\times U_n \times \begin{vmatrix} H_{n,1} & H_{n,2} & \cdots & H_{n,K} & I \\ H_{n,1}\Lambda_1^{-1} & H_{n,2}\Lambda_2^{-1} & \cdots & H_{n,K}\Lambda_K^{-1} & O \\ H_{n,1}\Lambda_1^{-2} & H_{n,2}\Lambda_2^{-2} & \cdots & H_{n,K}\Lambda_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{n,1}\Lambda_1^{-K} & H_{n,2}\Lambda_2^{-K} & \cdots & H_{n,K}\Lambda_K^{-K} & \mathcal{O} \end{vmatrix}^{-1}. \quad (3.19)$$

The expressions given by equations (3.17) and (3.19) are the K th solutions of the GLHM model and these results can easily be derived through induction. We then construct the adjoint Darboux transformation. The adjoint Lax pair is obtained by taking the formal adjoint of the linear equations (2.1) written as,

$$\Phi_{n+1} = -A_n^\dagger \Phi_n,$$

$$\frac{d}{dt}\Phi_n = -B_n^\dagger \Phi_n, \quad (3.20)$$

with A_n^\dagger and B_n^\dagger given by,

$$A_n^\dagger = I + \eta U_n^\dagger,$$

$$B_n^\dagger = \frac{\eta}{1 - \eta^2} J_n^\dagger + \frac{\eta^2}{1 - \eta^2} U_n^\dagger J_n^\dagger, \quad (3.21)$$

where η is a spectral parameter and Φ_n is an invertible $N \times N$ matrix field in the adjoint space W^\dagger . The Darboux matrix $D_n(\eta)$ transforms the matrix solution Φ_n in space W^\dagger to a new matrix solution $\tilde{\Phi}_n$ in \tilde{W}^\dagger i.e.

$$D_n(\eta) : W^\dagger \longrightarrow \tilde{W}^\dagger$$

$$: \Phi_n \longrightarrow \tilde{\Phi}_n. \quad (3.22)$$

Based upon the above facts, we can write Darboux transformation Φ_n as,

$$\Phi_n[1] \equiv D_n(\eta)\Phi_n = -(\eta^{-1}I - S_n)\Phi_n, \quad (3.23)$$

where S_n is the $N \times N$ matrix that is to be determined and I is $N \times N$ identity matrix. The covariance of the Lax pair under the Darboux transformation requires that the new solution $\tilde{\Phi}_n$

satisfies the Lax pair (3.20) given by,

$$\Phi_{n+1}[1] = -A_n^\dagger[1]\Phi_n[1],$$

$$A_n^\dagger[1] = I + \eta U_n^\dagger[1],$$

$$\frac{d}{dt}\Phi_n[1] = -B_n^\dagger[1]\Phi_n[1],$$

$$B_n^\dagger[1] = \frac{\eta}{1 - \eta^2} J_n^\dagger[1] + \frac{\eta^2}{1 - \eta^2} U_n^\dagger[1] J_n^\dagger[1]. \quad (3.24)$$

By operating the Darboux transformation (3.23) on (3.24), we obtain the Darboux transformed matrix functions U_n^\dagger and J_n^\dagger as,

$$U_n^\dagger[1] = U_n^\dagger - (S_{n+1} - S_n),$$

$$J_n^\dagger[1] = J_n^\dagger - \frac{d}{dt}S_n. \quad (3.25)$$

The matrix S_n can be constructed from the eigen matrices of the Lax pair and we take S_n to be $S_n = M_n \Xi^{-1} M_n^{-1}$, where $\Xi = \text{diag}(\eta_1, \dots, \eta_n)$ is the eigenvalue matrix. The particular matrix solution M_n of the Lax pair (3.20) is an invertible $N \times N$ matrix, which is given by,

$$M_n = (\Phi_n(\eta_1)|1\rangle, \dots, \Phi_n(\eta_N)|N\rangle) = (|m_1\rangle, \dots, |m_N\rangle). \quad (3.26)$$

Each column $|\Phi_i\rangle_n = \Phi_n(\eta_i)|e_i\rangle$ in M_n is a column solution of the Lax pair (3.20). The K -fold Darboux transformation on the matrix solution and matrix function Φ_n, U_n^\dagger can be written as,

$$\Phi_n[K] = \begin{vmatrix} M_{n,1} & M_{n,2} & \cdots & M_{n,K} & \Phi_n \\ M_{n,1}\Xi_1^{-1} & M_{n,2}\Xi_2^{-1} & \cdots & M_{n,K}\Xi_K^{-1} & \eta^{-1}\Phi_n \\ M_{n,1}\Xi_1^{-2} & M_{n,2}\Xi_2^{-2} & \cdots & M_{n,K}\Xi_K^{-2} & \eta^{-2}\Phi_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,1}\Xi_1^{-K} & M_{n,2}\Xi_2^{-K} & \cdots & M_{n,K}\Xi_K^{-K} & \boxed{\eta^{-K}\Phi_n} \end{vmatrix}. \quad (3.27)$$

Similarly, the quasideterminant of $U_n^\dagger[K]$ is,

$$U_n^\dagger[K] = \begin{vmatrix} M_{n+1,1} & M_{n+1,2} & \cdots & M_{n+1,K} & I \\ M_{n+1,1}\Xi_1^{-1} & M_{n+1,2}\Xi_2^{-1} & \cdots & M_{n+1,K}\Xi_K^{-1} & O \\ M_{n+1,1}\Xi_1^{-2} & M_{n+1,2}\Xi_2^{-2} & \cdots & M_{n+1,K}\Xi_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n+1,1}\Xi_1^{-K} & M_{n+1,2}\Xi_2^{-K} & \cdots & M_{n+1,K}\Xi_K^{-K} & \mathcal{O} \end{vmatrix} \times$$

$$\times U_n^\dagger \times \begin{vmatrix} M_{n,1} & M_{n,2} & \cdots & M_{n,K} & I \\ M_{n,1}\Xi_1^{-1} & M_{n,2}\Xi_2^{-1} & \cdots & M_{n,K}\Xi_K^{-1} & O \\ M_{n,1}\Xi_1^{-2} & M_{n,2}\Xi_2^{-2} & \cdots & M_{n,K}\Xi_K^{-2} & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,1}\Xi_1^{-K} & M_{n,2}\Xi_2^{-K} & \cdots & M_{n,K}\Xi_K^{-K} & \mathcal{O} \end{vmatrix}^{-1}. \quad (3.28)$$

Equations (3.27) and (3.28) are the K th quasideterminant solutions of the GLHM model for the adjoint space.

4. Standard binary Darboux transformation

In order to define the binary Darboux transformation, we consider the hat space \hat{W} , which is a copied version of direct

space W , so the corresponding solutions are $\hat{\Phi}_n \in \hat{W}$. Therefore, the equation of motion and the compatibility condition will have the identical form as that for the direct space given by,

$$\begin{aligned} \hat{\Psi}_{n+1} &= \hat{A}_n \hat{\Psi}_n, \\ \frac{d}{dt} \hat{\Psi}_n &= \hat{B}_n \hat{\Psi}_n, \end{aligned} \tag{4.1}$$

having \hat{A}_n and \hat{B}_n are,

$$\begin{aligned} \hat{A}_n &= I + \lambda \hat{U}_n, \\ \hat{B}_n &= \frac{\lambda}{1 - \lambda^2} \hat{J}_n + \frac{\lambda^2}{1 - \lambda^2} \hat{J}_n \hat{U}_n. \end{aligned} \tag{4.2}$$

The specific solutions for the direct and adjoint spaces are H_n and S_n , respectively. Thus, the corresponding solutions for \hat{W} are $\hat{H}_n \in \hat{W}$ and $\hat{\Phi}_n \in \hat{W}_n^\dagger$. Also assuming that $i(\hat{H}_n) \in \hat{W}^\dagger$, we can then write the transformation as,

$$D_n^{(-1)\dagger}(\lambda): W_n^\dagger \rightarrow \hat{W}_n^\dagger. \tag{4.3}$$

Since $\Phi_n \in W_n^\dagger$, we have,

$$i(\hat{H}_n) = D_n^{(-1)\dagger}(\lambda) \Phi_n. \tag{4.4}$$

Also from $D_n^\dagger(\lambda)(i(H_n)) = 0$, we obtain $i(H_n) = M_n^{(-1)\dagger}$ and similarly $i(\hat{M}_n) = \hat{M}_n^{(-1)\dagger}$. Therefore, from the above equations we can write,

$$\hat{H}_n^{(-1)\dagger} = D_n^{(-1)\dagger}(\lambda) \Phi_n,$$

and

$$\hat{H}_n = (D_n^{(-1)\dagger}(\lambda) \Phi_n)^{(-1)\dagger}, \tag{4.5}$$

where $D_n(\lambda) = \lambda^{-1}I - H_n \Lambda^{-1} H_n^{-1}$. By substituting the expression of $D_n(\lambda)$ in equation (4.5), we obtain,

$$\begin{aligned} \hat{H}_n &= ((\lambda^{-1}I - H_n \Lambda^{-1} H_n^{-1})^{(-1)\dagger} \Phi_n)^{(-1)\dagger} \\ &= (\lambda^{-1}I - H_n \Lambda^{-1} H_n^{-1}) \Phi_n^{(-1)\dagger} \\ &= H_n (\lambda^{-1}I - \Lambda^{-1}) H_n^{-1} \Phi_n^{(-1)\dagger} \\ &= H_n (\lambda^{-1}I - \Lambda^{-1}) (\Phi_n^\dagger H_n)^{-1} \\ &= H_n \Delta_n(H_n, \Phi_n)^{-1}, \end{aligned} \tag{4.6}$$

where the algebraic potential Δ_n is defined as,

$$\Delta_n(H_n, \Phi_n) = (\Phi_n^\dagger H_n) (\lambda^{-1}I - \Lambda^{-1})^{-1}. \tag{4.7}$$

Similarly, for the adjoint space matrix \hat{M}_n is written as,

$$\hat{M}_n = M_n \Delta_n(\Psi_n, M_n)^{(-1)\dagger}, \tag{4.8}$$

where,

$$\Delta_n(\Psi_n, M_n) = -(\lambda^{-1}I - \Xi^{(-1)\dagger})^{-1} (M_n^\dagger \Psi_n). \tag{4.9}$$

By writing equations (4.7) and (4.9) in matrix form for the solutions H_n and M_n , we obtain the condition on Δ_n which is given by,

$$\Xi^{(-1)\dagger} \Delta_n(H_n, M_n) - \Delta_n(H_n, M_n) \Lambda^{-1} = M_n^\dagger H_n, \tag{4.10}$$

where the Δ_n matrix is given by,

$$\Delta_n(H_n, M_n)_{ij} = \frac{\langle M_n | H_n \rangle_{(i,j)}}{\Xi^{(-1)\dagger} - \Lambda^{-1}}. \tag{4.11}$$

Therefore, the required potential is expressed in terms of particular matrix solutions to the Lax pair as well as to the adjoint Lax pair of the GLHM model. We then define the Darboux matrix in hat space:

$$\hat{D}_n(\lambda) \equiv (\lambda^{-1}I - \hat{Q}_n) = (\lambda^{-1}I - \hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1}), \tag{4.12}$$

where,

$$\hat{D}_n(\lambda) \hat{\Psi}_n = \Psi_n[1]. \tag{4.13}$$

We may write above formalism as the following Darboux maps:

$$\begin{aligned} D_n(\lambda) : W_n &\rightarrow \hat{W}_n, \\ \hat{D}_n(\lambda) : \hat{W}_n &\rightarrow \hat{W}_n, \\ D_n(\eta) : W_n^\dagger &\rightarrow \hat{W}_n^\dagger. \end{aligned} \tag{4.14}$$

When we apply $\hat{D}_n(\lambda)$ on equation (4.1), the equation must be covariant, i.e.

$$\begin{aligned} \hat{\Psi}_{n+1}[1] &= \hat{A}_n[1] \hat{\Psi}_n[1], \\ \frac{d}{dt} \hat{\Psi}_n[1] &= \hat{B}_n[1] \hat{\Psi}_n[1], \end{aligned} \tag{4.15}$$

Matrix \hat{Q}_n is defined as,

$$\hat{Q}_n = \hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1}.$$

The Darboux transformation on matrix field $\hat{\Psi}_n$ and \hat{U}_n in hat space \hat{W}_n is,

$$\begin{aligned} \hat{\Psi}_n[1] &= (\lambda^{-1}I - \hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1}) \Psi_n, \\ \hat{U}_n[1] &= \hat{H}_{n+1} \Xi^{(-1)\dagger} \hat{H}_{n+1}^{-1} U (\hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1})^{-1}. \end{aligned} \tag{4.16}$$

We then define the standard binary Darboux transformation from equation (4.14), which relates $\hat{\Psi}_n$ and Ψ_n as,

$$\hat{D}_n(\lambda) \hat{\Psi}_n = D_n(\lambda) \Psi_n, \tag{4.17}$$

which implies that,

$$\hat{\Psi}_n = \hat{D}_n^{-1}(\lambda) D_n(\lambda) \Psi_n. \tag{4.18}$$

We then operate the standard binary Darboux transformation on matrix solution Ψ_n as,

$$\begin{aligned} \hat{\Psi}_n &= (\lambda^{-1}I - \hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1})^{-1} (\lambda^{-1}I - H_n \Lambda^{-1} H_n^{-1}) \Psi_n, \\ &= \hat{H}_n (\lambda^{-1}I - \Xi^{(-1)\dagger})^{-1} \hat{H}_n^{-1} H_n (\lambda^{-1}I - \Lambda^{-1}) H_n^{-1} \Psi_n. \end{aligned}$$

By using the expression (4.6) in the above equation, we obtain,

$$\begin{aligned} \hat{\Psi}_n &= H_n \Delta_n(H_n, \Phi_n)^{-1} (\lambda^{-1}I - \Xi^{(-1)\dagger})^{-1} \\ &\quad \times \Delta_n(H_n, \Phi_n) H_n^{-1} H_n (\lambda^{-1}I - \Lambda^{-1}) H_n^{-1} \Psi_n, \\ &= H_n \Delta_n(H_n, \Phi_n)^{-1} (\lambda^{-1}I - \Xi^{(-1)\dagger})^{-1} \\ &\quad \times (\lambda^{-1} \Delta_n(H_n, \Phi_n) - \Delta_n(H_n, \Phi_n) \Lambda^{-1}) H_n^{-1} \Psi_n. \end{aligned} \tag{4.19}$$

By substituting the expression of $\Delta_n(H_n, \Phi_n) \Lambda^{-1}$ from

equation (4.10) in the equation (4.19), we obtain,

$$\begin{aligned} \hat{\Psi}_n &= H_n \Delta_n(H_n, \Phi_n)^{-1} (\lambda^{-1} I - \Xi^{(-1)\dagger})^{-1} \\ &\times (\lambda^{-1} \Delta_n(H_n, \Phi_n) H_n^{-1} - \Xi^{(-1)\dagger} \Delta_n(H_n, \Phi_n) H_n^{-1} + M_n^\dagger) \Psi \\ &= (\lambda^{-1} I - \Xi^{(-1)\dagger})^{-1} (\lambda^{-1} I - \Xi^{(-1)\dagger}) \left(I + \frac{H_n \Delta_n(H_n, \Phi_n)^{-1} M_n^\dagger}{\lambda^{-1} I - \Xi^{(-1)\dagger}} \right) \Psi_n \\ &= \Psi_n + H_n \Delta_n(H_n, \Phi_n)^{-1} (\lambda^{-1} I - \Xi^{(-1)\dagger})^{-1} M_n^\dagger \Psi_n. \end{aligned}$$

By using equation (4.9), the above expression becomes,

$$\hat{\Psi}_n = \Psi_n - H_n \Delta_n(H_n, M_n)^{-1} \Delta_n(\Psi_n, M_n), \tag{4.20}$$

which is the standard form of binary Darboux transformation on matrix Ψ_n . In terms of quasideterminant, the above expression can be expressed as,

$$\hat{\Psi}_n = \begin{vmatrix} \Delta_n(H_n, M_n) & \Delta_n(\Psi_n, M_n) \\ H_n & \boxed{\Psi_n} \end{vmatrix}. \tag{4.21}$$

This is known as the quasi-Grammian solution of the system. Similarly, for the adjoint space $\hat{\Phi} \in \hat{W}^\dagger$ we obtain,

$$\begin{aligned} \hat{\Phi}_n &= \Phi_n - M_n \Delta_n(H_n, M_n)^{(-1)\dagger} \Delta_n(H_n, \Phi_n)^\dagger \\ &= \begin{vmatrix} \Delta_n(H_n, M_n)^\dagger & \Delta_n(H_n, \Phi_n)^\dagger \\ M_n & \boxed{\Phi_n} \end{vmatrix}. \end{aligned} \tag{4.22}$$

The standard binary Darboux transformation on the solution of GLHM model U_n , is given by,

$$\hat{U}_n = \hat{Q}_{n+1}^{-1} Q_{n+1} U_n Q_n^{-1} \hat{Q}_n, \tag{4.23}$$

where $\hat{Q}_{n+1}^{-1} = \hat{H}_{n+1} \Xi^\dagger \hat{H}_{n+1}^{-1}$, $\hat{Q}_n = \hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1}$, $Q_{n+1} = H_{n+1} \Lambda^{-1} H_{n+1}^{-1}$, $Q_n^{-1} = H_n \Lambda H_n^{-1}$, and so equation (4.23) becomes,

$$\hat{U}_n = (\hat{H}_{n+1} \Xi^\dagger \hat{H}_{n+1}^{-1}) (H_{n+1} \Lambda^{-1} H_{n+1}^{-1}) U_n (H_n \Lambda H_n^{-1}) (\hat{H}_n \Xi^{(-1)\dagger} \hat{H}_n^{-1}). \tag{4.24}$$

This is in fact a product of quasideterminants, i.e.

$$\begin{aligned} \hat{U}_n &= \begin{vmatrix} \hat{H}_{n+1} \Xi^{(-1)\dagger} & I \\ \hat{H}_{n+1} & \boxed{Q} \end{vmatrix} \begin{vmatrix} \hat{H}_{n+1} \Lambda & I \\ \hat{H}_{n+1} & \boxed{Q} \end{vmatrix} \\ U_n &= \begin{vmatrix} \hat{H}_n & I \\ \hat{H}_n \Lambda & \boxed{Q} \end{vmatrix} \begin{vmatrix} \hat{H}_n \Xi^\dagger & I \\ \hat{H}_n & \boxed{Q} \end{vmatrix}. \end{aligned} \tag{4.25}$$

This expression can be further simplified when we introduce potential Δ_n instead of matrices in the hat space. By using equation (4.6), the above equation becomes,

$$\begin{aligned} \hat{U}_n &= (H_{n+1} \Delta_{n+1}(H_{n+1}, M_{n+1})^{-1} \Xi^\dagger \Delta_{n+1}(H_{n+1}, M_{n+1}) H_{n+1}^{-1}) \\ &\quad (H_{n+1} \Lambda^{-1} H_{n+1}^{-1}) \\ &\quad \times U_n (H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger \Delta_n(H_n, M_n) H_n^{-1}) \\ &\quad (H_n \Lambda^{-1} H_n^{-1})^{-1} \\ &= H_{n+1} \Delta_{n+1}(H_{n+1}, M_{n+1})^{-1} \Xi^\dagger \Delta_{n+1} \\ &\quad (H_{n+1}, M_{n+1}) \Lambda^{-1} H_{n+1}^{-1} \\ &\quad \times U_n (H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger \Delta_n(H_n, M_n) \Lambda^{-1} H_n^{-1})^{-1}. \end{aligned}$$

By substituting equation (4.10) in the above expression, we obtain,

$$\begin{aligned} \hat{U}_n &= (H_{n+1} \Delta_{n+1}(H_{n+1}, M_{n+1})^{-1} \Xi^\dagger \Delta_{n+1} \\ &\quad (H_{n+1}, M_{n+1}) H_{n+1}^{-1} \\ &\quad - H_{n+1} \Delta_{n+1}(H_{n+1}, M_{n+1})^{-1} \Xi^\dagger S_{n+1}^{-1} H_{n+1} H_{n+1}^{-1} U_n \\ &\quad (H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger \Delta_n(H_n, M_n) H_n^{-1} \\ &\quad - H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger M_n^\dagger H_n H_n^{-1})^{-1} \\ &= (I - H_{n+1} \Delta_{n+1}(H_{n+1}, M_{n+1})^{-1} \Xi^\dagger M_{n+1}^\dagger) U_n (I - H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger M_n^\dagger)^{-1} \\ &= \hat{F}_{n+1} U_n \hat{F}_n^{-1} \\ &= U_n - \hat{F}_{n+1} + \hat{F}_n^{-1}. \end{aligned}$$

In terms of quasideterminants, the above expression can be written as,

$$\hat{U}_n = \begin{vmatrix} \Delta_{n+1}(H_{n+1}, M_{n+1}) & \Xi^\dagger M_{n+1}^\dagger \\ H_{n+1} & \boxed{U_n} \end{vmatrix} \begin{vmatrix} \Delta_n(H_n, M_n) \\ H_n \end{vmatrix} \tag{4.26}$$

Similarly, we can calculate the K th iteration of Ψ_n through the iteration of binary Darboux transformation given by,

$$\Psi_n[K+1] = \begin{vmatrix} \Delta_n(H_{n,1}, M_{n,1}) & \cdots & \Delta_n(H_{n,K}, M_{n,1}) & \Delta_n(\Psi_n, M_{n,1}) \\ \vdots & \cdots & \vdots & \vdots \\ \Delta_n(H_{n,1}, M_{n,K}) & \cdots & \Delta_n(H_{n,K}, M_{n,K}) & \Delta_n(\Psi_n, M_{n,K}) \\ H_{n,1} & \cdots & H_{n,K} & \boxed{\Psi_n} \end{vmatrix}. \tag{4.27}$$

The K th iteration for the adjoint binary Darboux transformation is given by,

$$\Phi_n[K+1] = \begin{vmatrix} \Delta_n(H_{n,1}, M_{n,1})^\dagger & \Delta_n(H_{n,2}, M_{n,1})^\dagger & \cdots & \Delta_n(H_{n,K}, M_{n,1})^\dagger & \Delta_n(H_{n,1}, \Phi_n)^\dagger \\ \Delta_n(H_{n,1}, M_{n,2})^\dagger & \Delta_n(H_{n,2}, M_{n,2})^\dagger & \cdots & \Delta_n(H_{n,K}, M_{n,2})^\dagger & \Delta_n(H_{n,2}, \Phi_n)^\dagger \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Delta_n(H_{n,1}, M_{n,K})^\dagger & \Delta_n(H_{n,2}, M_{n,K})^\dagger & \cdots & \Delta_n(H_{n,K}, M_{n,K})^\dagger & \Delta_n(H_{n,K}, \Phi_n)^\dagger \\ S_{n,1} & M_{n,2} & \cdots & M_{n,K} & \boxed{\Phi_n} \end{vmatrix}. \tag{4.28}$$

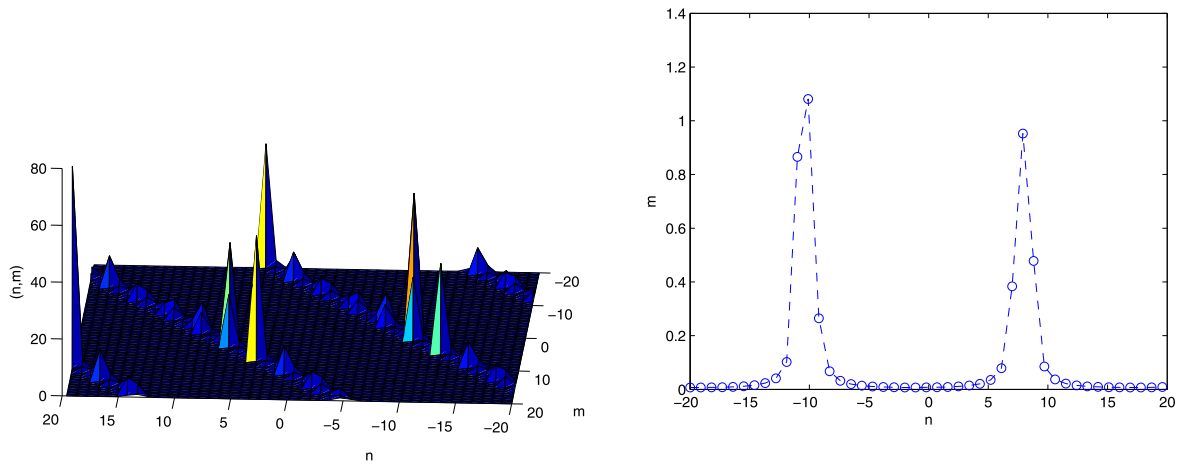


Figure 1. Dynamics of $\hat{U}_{n,11}$.

$$\hat{U}_n[K + 1] = \begin{pmatrix} \Delta_{n+1}(H_{n+1,1}, M_{n+1,1}) & \cdots & \Delta_{n+1}(H_{n+1,K}, M_{n+1,1}) & \Xi_1^\dagger M_{n+1,1}^\dagger \\ \Delta_{n+1}(H_{n+1,1}, M_{n+1,2}) & \cdots & \Delta_{n+1}(H_{n+1,K}, M_{n+1,2}) & \Xi_2^\dagger M_{n+1,2}^\dagger \\ \vdots & \vdots & \cdots & \vdots \\ \Delta_{n+1}(H_{n+1,1}, M_{n+1,K}) & \cdots & \Delta_{n+1}(H_{n+1,K}, M_{n+1,K}) & \Xi_K^\dagger M_{n+1,K}^\dagger \\ H_{n+1,1} & \cdots & H_{n+1,K} & \boxed{\mathbb{I}} \end{pmatrix} \\ \times U_n \times \begin{pmatrix} \Delta_n(H_{n,1}, M_{n,1}) & \cdots & \Delta_n(H_{n,K}, M_{n,1}) & \Xi_1^\dagger M_{n,1}^\dagger \\ \Delta_n(H_{n,1}, M_{n,2}) & \cdots & \Delta_n(H_{n,K}, M_{n,2}) & \Xi_2^\dagger M_{n,2}^\dagger \\ \vdots & \cdots & \vdots & \vdots \\ \Delta_n(H_{n,1}, M_{n,K}) & \cdots & \Delta_n(H_{n,K}, M_{n,K}) & \Xi_K^\dagger M_{n,K}^\dagger \\ H_{n,1} & \cdots & H_{n,K} & \boxed{\mathbb{I}} \end{pmatrix}^{-1}.$$

Similarly, $\hat{U}_n[K]$ can be written as,

Similarly, using the iteration process we can calculate the quasideterminant solutions for U_n^\dagger .

Remark 2. Therefore, we can calculate the Grammian-type solutions for the GLHM model by using standard binary Darboux transformation. In addition, the potential can be expressed in the form of quasideterminants. Thus, by developing the binary Darboux transformation in terms of spectral parameters, we can obtain expressions of matrix solutions in the form of Grammian-type quasideterminants that have a different form as calculated using elementary Darboux transformation.

5. Explicit solutions

In this section, we consider the GLHM model based on the Lie group $SU(2)$ and obtain the soliton solutions by using the binary Darboux transformation. To obtain an explicit expression for the soliton solution in the general $N \times N$ case,

we take the seed solution,

$$U_0 \equiv U_n = i \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & -c_N \end{pmatrix}, \quad a = 0, \quad b = 1,$$

where c_i are real constants and $\text{Tr } U_n = 0$. A trivial calculation then yields a matrix solution Ψ_n of the Lax pair (2.1) with the form,

$$\Psi_n = \begin{pmatrix} \Pi_n(\lambda)_p & O \\ O & \Pi_n(\lambda)_{N-p} \end{pmatrix}$$

where,

$$\Pi_n(\lambda)_p = \begin{pmatrix} \xi_n(\lambda)_1 & & \\ & \ddots & \\ & & \xi_n(\lambda)_p \end{pmatrix}, \\ \Pi_n(\lambda)_{N-p} = \begin{pmatrix} \xi_n(\lambda)_{p+1} & & \\ & \ddots & \\ & & \xi_n(\lambda)_N \end{pmatrix}$$

are respectively $p \times p$ and $(N - p) \times (N - p)$ matrices. Here, n

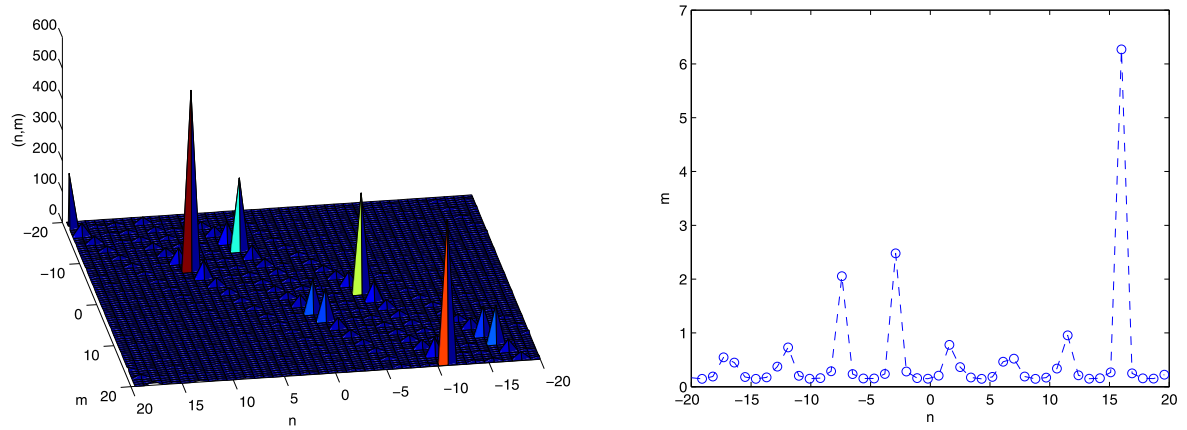


Figure 2. Dynamics of $\hat{U}_{n,22}$.

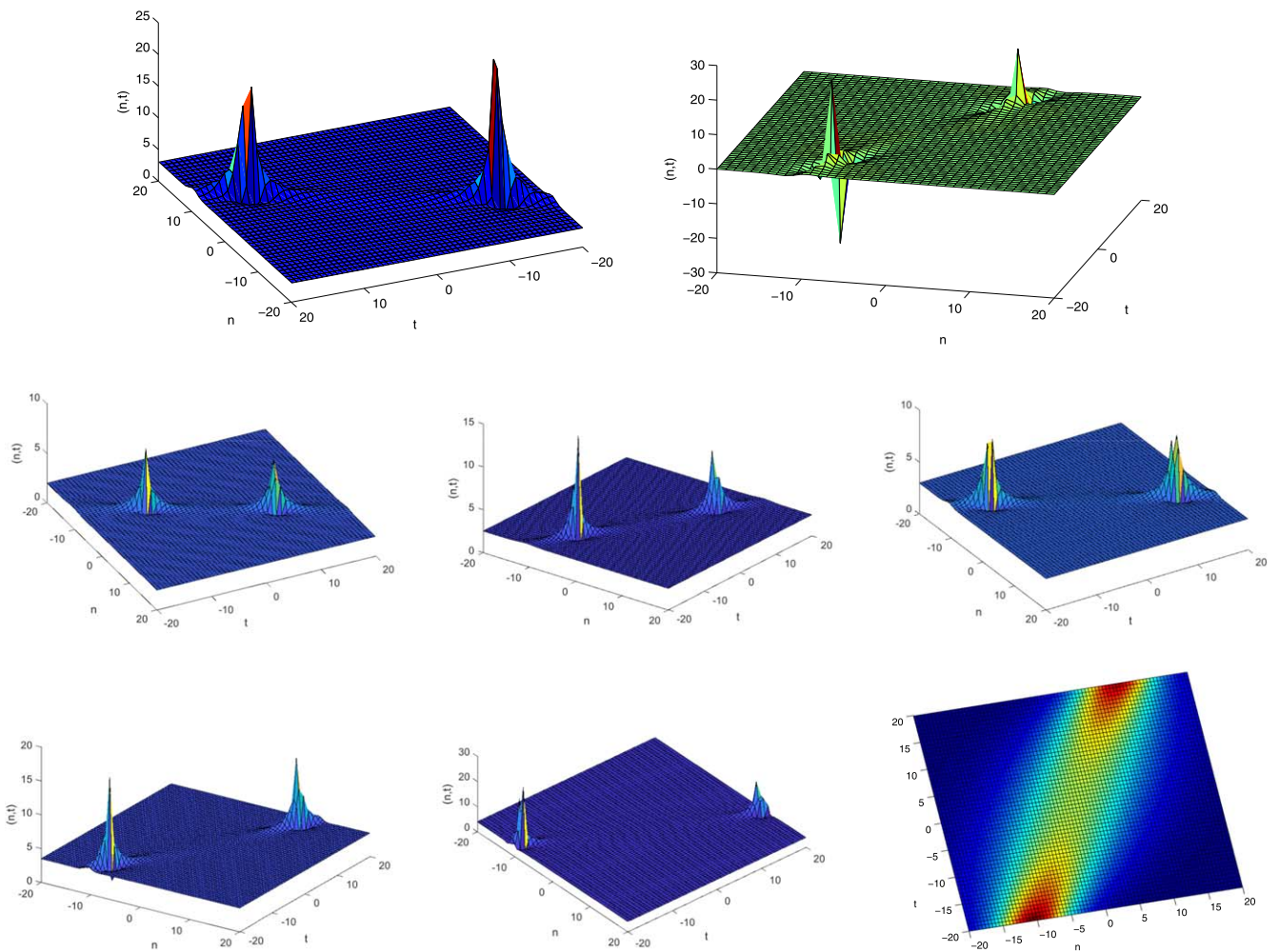


Figure 3. Dynamics of U_n^+ for different values of c .

in the subscript is a discrete index. We then take the seed solution for the case $N = 2$, which is given as,

$$U_0 \equiv U_n = i \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}. \tag{5.1}$$

Thus, the solution of the linear system (2.1) can be expressed as,

$$\Psi_n = \begin{pmatrix} \xi_n(\lambda) \\ \bar{\xi}_n(\lambda) \end{pmatrix}, \tag{5.2}$$

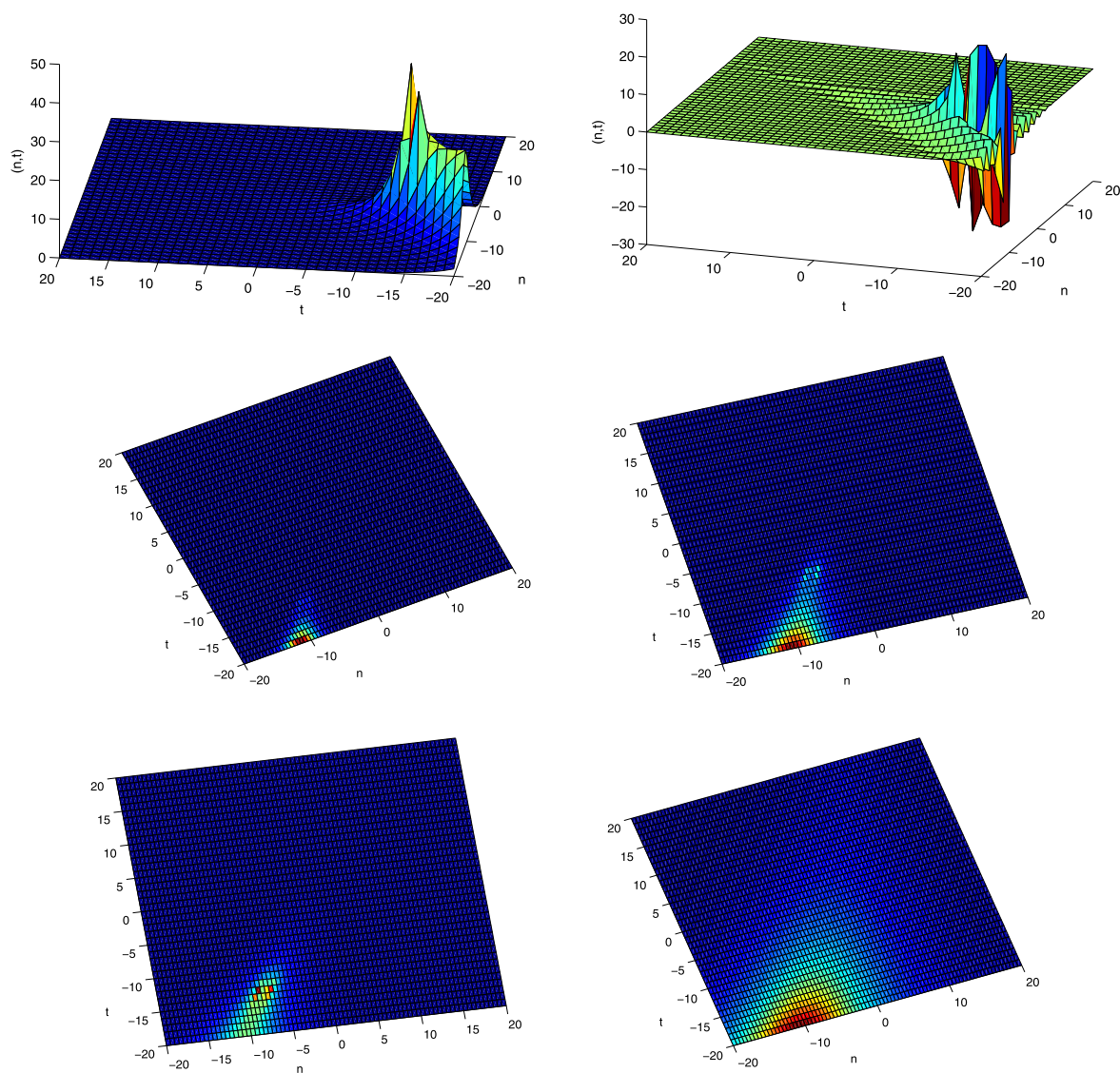


Figure 4. Dynamics of U_n^- : for different values of c .

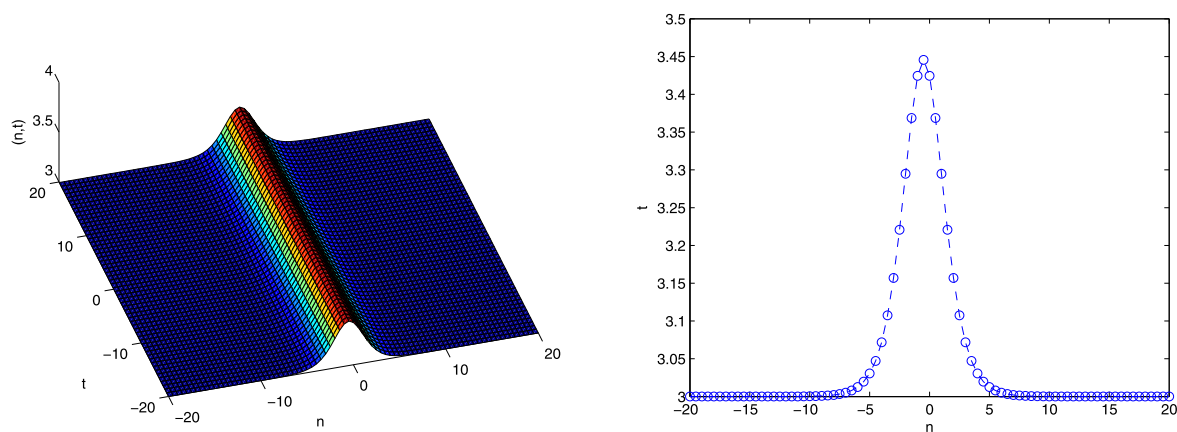


Figure 5. Semi-discrete one-soliton solution U_n^+ .

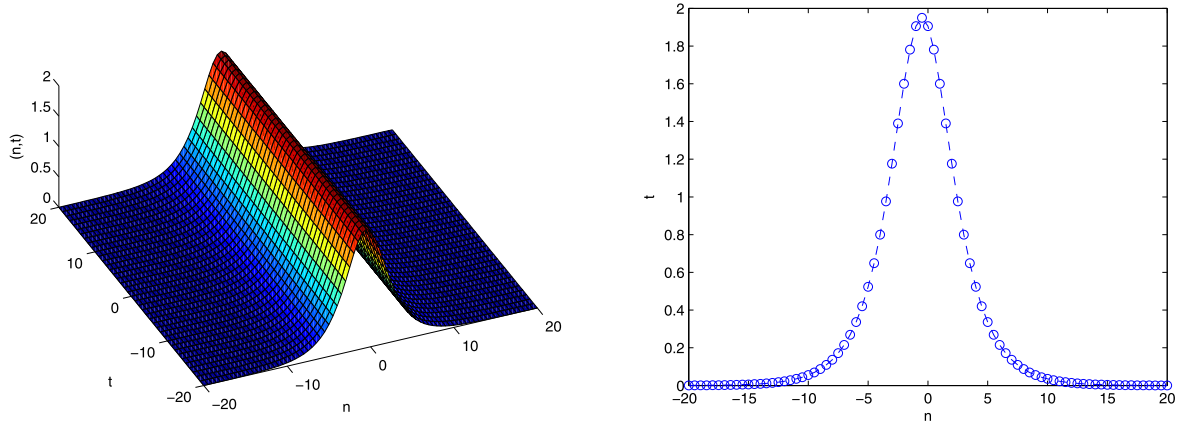


Figure 6. Semi-discrete one-soliton solution U_n^- .

where,

$$\xi_n(\lambda) = (1 + ic\lambda)^n \exp\left[\frac{-ic^{-1}\lambda}{1 - \lambda^2} + \frac{\lambda^2}{1 - \lambda^2}\right]t. \quad (5.3)$$

The particular matrix solution H_n of the direct Lax pair can be written by using the above equation (5.2) as,

$$H_n = (\Psi_n(\lambda)|1\rangle, \Psi_n(\bar{\lambda})|2\rangle) = \begin{pmatrix} \xi_n(\lambda) & -\xi_n(\bar{\lambda}) \\ \bar{\xi}_n(\lambda) & \bar{\xi}_n(\bar{\lambda}) \end{pmatrix}. \quad (5.4)$$

The expression for $Q_n = H_n\Lambda^{-1}H_n^{-1}$ by using $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$ becomes,

$$Q_n = \frac{1}{X_n^+ + X_n^-} \begin{pmatrix} \lambda^{-1}X_n^+ + \bar{\lambda}^{-1}X_n^- & (\lambda^{-1} - \bar{\lambda}^{-1})Y_n^+ \\ (\lambda^{-1} - \bar{\lambda}^{-1})Y_n^+ & \lambda^{-1}X_n^- + \bar{\lambda}^{-1}X_n^+ \end{pmatrix}, \quad (5.5)$$

where,

$$X_n^{(\pm)} = (1 \pm ic\lambda)^n (1 \mp ic\bar{\lambda})^n \exp\left[\mp ic^{-1}\left(\frac{\lambda}{1 - \lambda^2} + \frac{\bar{\lambda}}{1 - \bar{\lambda}^2}\right)t\right],$$

$$Y_n^+ = (1 \pm ic\lambda)^n (1 \pm ic\bar{\lambda})^n \exp\left[\pm ic^{-1}\left(\frac{\lambda}{1 - \lambda^2} + \frac{\bar{\lambda}}{1 - \bar{\lambda}^2}\right)t\right]. \quad (5.6)$$

Similarly, the particular matrix solution M_n for adjoint space can be written as,

$$M_n = (\Phi_n(\eta)|1\rangle, \Phi_n(\bar{\eta})|2\rangle) = \begin{pmatrix} \xi_n(\eta) & -\xi_n(\bar{\eta}) \\ \bar{\xi}_n(\eta) & \bar{\xi}_n(\bar{\eta}) \end{pmatrix}, \quad (5.7)$$

where,

$$\xi_n(\eta) = (1 + ic\eta)^n \exp\left[\frac{-ic^{-1}\eta}{1 - \eta^2} + \frac{\eta^2}{1 - \eta^2}\right]t. \quad (5.8)$$

In order to obtain the expression for \hat{U}_n , we start from the definition of $\Delta_n(H_n, M_n)$ given in (4.11) and by using (5.4),

(5.7), we obtain,

$$\Delta_n(H_n, M_n) = \begin{pmatrix} A_n + \bar{A}_n & \bar{B}_n - B_n \\ \frac{A_n + \bar{A}_n}{\eta^{-1} - \bar{\lambda}^{-1}} & \frac{\bar{B}_n - B_n}{\eta^{-1} - \bar{\lambda}^{-1}} \\ \bar{C}_n - C_n & D_n + \bar{D}_n \\ \frac{\bar{C}_n - C_n}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} & \frac{D_n + \bar{D}_n}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} \end{pmatrix},$$

where,

$$A_n = (1 + ic\eta)^n (1 + ic\lambda)^n \exp\left[ic^{-1}\left(\frac{\eta}{1 - \eta^2} + \frac{\lambda}{1 - \lambda^2}\right)t\right] \exp\left(\frac{\eta^2}{1 - \eta^2} + \frac{\lambda^2}{1 - \lambda^2}\right)t,$$

$$B_n = (1 + ic\eta)^n (1 + ic\bar{\lambda})^n \exp\left[ic^{-1}\left(\frac{\eta}{1 - \eta^2} + \frac{\bar{\lambda}}{1 - \bar{\lambda}^2}\right)t\right] \exp\left(\frac{\eta^2}{1 - \eta^2} + \frac{\bar{\lambda}^2}{1 - \bar{\lambda}^2}\right)t,$$

$$C_n = (1 + ic\bar{\eta})^n (1 + ic\lambda)^n \exp\left[ic^{-1}\left(\frac{\bar{\eta}}{1 - \bar{\eta}^2} + \frac{\lambda}{1 - \lambda^2}\right)t\right] \exp\left(\frac{\bar{\eta}^2}{1 - \bar{\eta}^2} + \frac{\lambda^2}{1 - \lambda^2}\right)t,$$

$$D_n = (1 + ic\bar{\eta})^n (1 + ic\bar{\lambda})^n \exp\left[ic^{-1}\left(\frac{\bar{\eta}}{1 - \bar{\eta}^2} + \frac{\bar{\lambda}}{1 - \bar{\lambda}^2}\right)t\right] \exp\left(\frac{\bar{\eta}^2}{1 - \bar{\eta}^2} + \frac{\bar{\lambda}^2}{1 - \bar{\lambda}^2}\right)t.$$

Then, we take,

$$\hat{M}_n = H_n \Delta_n(H_n, M_n)^{-1} \Xi^\dagger M_n^\dagger = \begin{pmatrix} \hat{M}_{n,11} & \hat{M}_{n,12} \\ \hat{M}_{n,21} & \hat{M}_{n,22} \end{pmatrix}, \quad (5.9)$$

$$= \frac{1}{K} \begin{pmatrix} \frac{\eta \xi_n(\eta) \xi_n(\lambda)(D_n + \bar{D}_n)}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} - \frac{\bar{\eta} \xi_n(\bar{\eta}) \xi_n(\lambda)(B_n - \bar{B}_n)}{\eta^{-1} - \bar{\lambda}^{-1}} & \frac{\eta \bar{\xi}_n(\eta) \xi_n(\lambda)(D_n + \bar{D}_n)}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} + \frac{\bar{\eta} \bar{\xi}_n(\bar{\eta}) \xi_n(\lambda)(B_n - \bar{B}_n)}{\eta^{-1} - \bar{\lambda}^{-1}} \\ -\frac{\eta \xi_n(\eta) \xi_n(\bar{\lambda})(C_n - \bar{C}_n)}{\bar{\eta}^{-1} - \lambda^{-1}} + \frac{\bar{\eta} \xi_n(\bar{\eta}) \xi_n(\bar{\lambda})(A_n + \bar{A}_n)}{\eta^{-1} - \lambda^{-1}} & -\frac{\eta \bar{\xi}_n(\eta) \xi_n(\bar{\lambda})(C_n - \bar{C}_n)}{\bar{\eta}^{-1} - \lambda^{-1}} - \frac{\bar{\eta} \bar{\xi}_n(\bar{\eta}) \xi_n(\bar{\lambda})(A_n + \bar{A}_n)}{\eta^{-1} - \lambda^{-1}} \\ \frac{\eta \xi_n(\eta) \bar{\xi}_n(\lambda)(D_n + \bar{D}_n)}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} - \frac{\bar{\eta} \xi_n(\bar{\eta}) \bar{\xi}_n(\lambda)(B_n - \bar{B}_n)}{\eta^{-1} - \bar{\lambda}^{-1}} & \frac{\eta \bar{\xi}_n(\eta) \bar{\xi}_n(\lambda)(D_n + \bar{D}_n)}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} + \frac{\bar{\eta} \bar{\xi}_n(\bar{\eta}) \bar{\xi}_n(\lambda)(B_n - \bar{B}_n)}{\eta^{-1} - \bar{\lambda}^{-1}} \\ +\frac{\eta \xi_n(\eta) \bar{\xi}_n(\bar{\lambda})(C_n - \bar{C}_n)}{\bar{\eta}^{-1} - \lambda^{-1}} - \frac{\bar{\eta} \xi_n(\bar{\eta}) \bar{\xi}_n(\bar{\lambda})(A_n + \bar{A}_n)}{\eta^{-1} - \lambda^{-1}} & +\frac{\eta \bar{\xi}_n(\eta) \bar{\xi}_n(\bar{\lambda})(C_n - \bar{C}_n)}{\bar{\eta}^{-1} - \lambda^{-1}} + \frac{\bar{\eta} \bar{\xi}_n(\bar{\eta}) \bar{\xi}_n(\bar{\lambda})(A_n + \bar{A}_n)}{\eta^{-1} - \lambda^{-1}} \end{pmatrix},$$

where,

$$K = \left(\frac{A_n + \bar{A}_n}{\eta^{-1} - \lambda^{-1}} \right) \left(\frac{D_n + \bar{D}_n}{\bar{\eta}^{-1} - \bar{\lambda}^{-1}} \right) - \left(\frac{\bar{B}_n - B_n}{\eta^{-1} - \bar{\lambda}^{-1}} \right) \left(\frac{\bar{C}_n - C_n}{\bar{\eta}^{-1} - \lambda^{-1}} \right). \tag{5.10}$$

The matrix field in hat space is given by,

$$\hat{U}_n = U_n - \hat{F}_{n+1} + \hat{F}_n,$$

where,

$$\begin{aligned} \hat{F}_n &= \begin{pmatrix} I - \hat{M}_{n,11} & -\hat{M}_{n,12} \\ -\hat{M}_{n,21} & I - \hat{M}_{n,22} \end{pmatrix} \\ &= \begin{pmatrix} \hat{U}_{n,11} & \hat{U}_{n,12} \\ \hat{U}_{n,21} & \hat{U}_{n,22} \end{pmatrix}. \end{aligned} \tag{5.11}$$

The expressions (5.11) are presented in figures 1 and 2.

5.1. Reduction

For the reduction we take $\eta = -\bar{\lambda}$ and $\bar{\eta} = -\lambda$, which gives $B_n - \bar{B}_n = 0 = C_n - \bar{C}_n$, then,

$$\begin{aligned} K &= \frac{(X_n^+ + X_n^-)^2 (\exp A)^2}{(\lambda^{-1} + \bar{\lambda}^{-1})^2}, \text{ where} \\ A &= \left(\frac{\lambda^2}{1 - \lambda^2} + \frac{\bar{\lambda}^2}{1 - \bar{\lambda}^2} \right) t. \end{aligned} \tag{5.12}$$

Thus, we can write,

$$\hat{F}_n = \frac{1}{X_n^+ + X_n^-} \begin{pmatrix} -\bar{\lambda} \lambda^{-1} X_n^+ - \lambda \bar{\lambda}^{-1} X_n^- & (\lambda \bar{\lambda}^{-1} - \bar{\lambda} \lambda^{-1}) Y_n^+ \\ (\lambda \bar{\lambda}^{-1} - \bar{\lambda} \lambda^{-1}) Y_n^+ & -\lambda \bar{\lambda}^{-1} X_n^+ - \bar{\lambda} \lambda^{-1} X_n^- \end{pmatrix}. \tag{5.13}$$

Therefore, the solution of matrix function \hat{U}_n by using expressions (5.1) and (4.12) can be written as,

$$\hat{U}_n = \begin{pmatrix} U_n^+ & -U_n^- \\ U_n^- & -U_n^+ \end{pmatrix}, \tag{5.14}$$

where,

$$\begin{aligned} U_n^+ &= ic + (\bar{\lambda} \lambda^{-1} + \lambda \bar{\lambda}^{-1}) \\ &\quad \times \frac{X_{n+1}^+ X_n^- - X_n^+ X_{n+1}^-}{(X_n^+ + X_n^-)(X_{n+1}^+ + X_{n+1}^-)}, \\ U_n^- &= (\bar{\lambda} \lambda^{-1} + \lambda \bar{\lambda}^{-1}) \\ &\quad \times \frac{(X_n^+ + X_n^-) Y_{n+1}^+ - Y_n^+(X_{n+1}^+ + X_{n+1}^-)}{(X_n^+ + X_n^-)(X_{n+1}^+ + X_{n+1}^-)}. \end{aligned}$$

The expressions (5.14) are depicted in figures 3 and 4.

By substituting $\lambda = \bar{\lambda}$, we obtain the soliton solutions shown in figures 5 and 6.

$$\begin{aligned} U_n^+ &= ic + \frac{X_{n+1}^+ X_n^- - X_n^+ X_{n+1}^-}{(X_n^+ + X_n^-)(X_{n+1}^+ + X_{n+1}^-)}, \\ U_n^- &= \frac{(X_n^+ + X_n^-) Y_{n+1}^+ - Y_n^+(X_{n+1}^+ + X_{n+1}^-)}{(X_n^+ + X_n^-)(X_{n+1}^+ + X_{n+1}^-)}. \end{aligned} \tag{5.15}$$

From (5.15), it can be seen that $U^\dagger[1] = -U[1]$ and $\text{Tr}(U[1]) = 0$. Therefore, it can be said that the above expression (5.15) is an explicit equation based on $SU(2)$ one-soliton solutions of the GLHM model.

We can obtain solutions in the form of direct and adjoint space parameters, by using the standard binary Darboux transformation, which are different to those obtained by elementary Darboux transformation. In addition, we reduce the solutions into the elementary Darboux transformation solutions, which is the advantage of binary Darboux transformation.

6. Conclusion

In this paper, we have composed the discrete Darboux transformation of the GLHM model not only for the direct space, but also for the adjoint space, and also calculated the standard binary Darboux transformation of the model. By iterating the standard binary Darboux transformation we have obtained the multi-Grammian solutions in terms of

quasideterminants. We have calculated the explicit solutions for the Grammian solutions of the model. We presented the dynamics of the solutions and by reducing the solutions also obtain the one-soliton solution for the semi-discrete model. This work can be extended in various interesting directions. For example, one can study discrete and semi-discrete versions of the multi-component GLHM model as well as their multi-soliton solutions. It would also be interesting to study discrete rogue and hump wave solutions for the GLHM model.

Declaration of competing interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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