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Fermions dynamic equation with Lorentz invariance violation and the corrected Hawking temperature in arbitrarily accelerating black hole[*]

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Abstract

Containing Lorentz invariance violation (LIV), a new form of the fermions dynamic equation under the background of the curved space-time of the arbitrarily accelerating black hole, is studied. Firstly, we consider the new form of the fermions dynamic equation with arbitrary spin containing LIV in curved space-time, and research the fermions dynamic equation with spin $-\frac{1}{2}$ containing LIV. On this basis, according to the semi-classical theory and black hole quantum tunneling radiation theory, the quantum tunneling radiation of the arbitrarily accelerating Kinnersly black hole is modified correctly, and the corrected physical quantities such as black hole temperature and quantum tunneling rate are deeply discussed. The fermions dynamic equation with arbitrary spin in the arbitrarily accelerating black hole space-time and its solution are explained in detail. In order to further obtain the correction effect of the Planck scale, this article considers beyond the semi-classical theory and further obtains new expressions of the black hole temperature and tunneling radiation rate.

Keywords: arbitrarily accelerating black hole, tunneling radiation, Lorentz invariance violation

1. Introduction

Black holes are a very important research object of astrophysics. In the gravitational theory, the study of black holes is divided into astronomical observation and black hole physics. The new progress in astronomical observation is that LIGO detected the gravitational wave generated by the merger of two black holes in 2016 and the photo of the M87 black hole obtained by the Event Horizon Telescope in 2021. These observations not only prove the existence of black holes but also promote people's research on black holes. In the research of black holes, Hawking proved that under the quantum effect, black holes produce radiation, which is called Hawking radiation [1]. Therefore, people have done a series of research on all kinds of black hole radiation. The quantum tunneling theory is used to explain the Hawking radiation of a black hole, that is, there are a large number of virtual particles in the event horizon of a black hole, particles pass through the event horizon by the quantum tunneling effect and change into real particles to form Hawking radiation [2–13]. However, the correct study of Hawking radiation of black holes with the real quantum tunneling radiation theory is carried out through the research method proposed by Kraus and others [14, 15]. This research result is a meaningful correction of the Hawking thermal radiation spectrum [16–23]. On the basis of this research work, people have carried out a series of studies on the quantum tunneling radiation of black holes. In the process of further research, people use the semi-classical method to study the quantum tunneling radiation of black holes [24–30]. Yang and Lin researched the dynamic equations of bosons and

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fermions in curved space-time by using the semi-classical theory, and proved that the dynamic equations of bosons and fermions are unified in the study of quantum tunneling radiation. Hamilton–Jacobi equation in curved space-time can be applied to the study of tunneling radiation of various black holes [31–34]. This is of great significance for the study of quantum tunneling radiation of black holes.

The development of physics is a process of continuous development and innovation. For the four forces in the Universe, researchers have been trying to establish a grand unification theory. However, because it can not be renormalized, it inspires researchers to study the theory of quantum gravity. So far, Einstein–Aether's quantum gravity and other gravitational theories have been studied one after another. According to the research on the theory of quantum gravity, Lorentz invariance violation (LIV) will appear in the field of high energy. Both general relativity and quantum field theory are based on Lorentz dispersion relation. Therefore, it is very important to study the dynamic equations of bosons and fermions and their related quantum tunneling radiation by LIV in curved space-time. The dynamic equations of bosons and fermions in the curved space-time of black holes are modified in reference [31–33, 35–43], and some meaningful studies on the quantum tunneling radiation of black holes are carried out by solving these equations. The purpose of this paper is to make a more accurate correction to the fermions tunneling radiation of arbitrarily accelerating black holes. Firstly, the universal dynamic equation with spin $-\frac{1}{2}$ in arbitrarily accelerating black holes is explained in detail. Secondly, the important characteristic physical quantities such as Hawking temperature and black hole entropy of arbitrarily accelerating black holes are accurately corrected. Thirdly, we aim to explain the application of arbitrary spin fermions with spin $-\frac{3}{2},\cdots$. The second section below considers the correct correction of the fermions dynamic equation of spin $-\frac{1}{2}$ in the arbitrarily accelerating black hole by LIV. The third section is the study of quantum tunneling radiation of arbitrarily accelerating Kinnersly black hole. The last section below is the discussion and explanation for the research methods and results in this paper.

2. A new form of fermions dynamic equation containing LIV and the semi-classical theory in curved space-time of the arbitrarily accelerating black hole

Lorentz dispersion relation is a basic physical relation in general relativity and quantum field theory. However, the study of quantum gravity theory shows that in the case of high energy field, the Lorentz dispersion relationship needs to be modified in the Planck scale. Therefore, considering the LIV theory, the correction of the dispersion relationship is expressed as [44–46]

$$p_0{}^2 = p^2 + m^2 - (Lp_0)^\alpha p^\alpha, \tag{2.1}$$

where p_0 is particle energy, p is particle momentum, and L is minimal length, which is of the order of the Plank length $L_p = M_p^{-1}$. For $\alpha = 2$ in the equation (2.1), the Dirac equation describing spin $-\frac{1}{2}$ in flat space-time is

$$\left(\bar{\gamma}^\mu\partial_\mu + \frac{m}{\hbar} - \mathrm{i}L\bar{\gamma}^t\partial_t\bar{\gamma}^j\partial_j\right)\psi = 0. \tag{2.2}$$

The term added to this equation violates Lorentz invariance, so Lorentz invariance is broken. We can think equation (2.2) as an effective wave equation with LIV introducing preferred frame effects. For fermions with spin $-\frac{3}{2}$, spin $-\frac{5}{2}\cdots$, it should be described by the Rarita–Schwinge equation. And it means that for fermions with arbitrary spin, the generalized equation (2.2) becomes

$$\left(\bar{\gamma}^\mu\partial_\mu + \frac{m}{\hbar} - \mathrm{i}L\bar{\gamma}^t\partial_t\bar{\gamma}^j\partial_j\right)\psi_{\alpha_1\ldots\alpha_k} = 0, \tag{2.3}$$

and the following conditions must be met,

$$\bar{\gamma}^\mu\psi_{\mu\alpha_2\ldots\alpha_k} = \partial_\mu\psi^\mu{}_{\alpha_2\ldots\alpha_k} = \psi^\mu{}_{\mu\alpha_3\ldots\alpha_k} = 0. \tag{2.4}$$

As can be seen from equations (2.4) and (2.3), for $k = 0$, $\psi_{\alpha_1\ldots\alpha_k} = \psi$, equation (2.3) degenerates to equation (2.4). For $k = 1$, equation (2.4) becomes $\partial_\mu\psi^\mu$, and equation (2.3) degenerates to a fermions dynamic equation with spin $-\frac{1}{2}$. The correction term of equation (2.3) is a small correction on the quantum scale. Therefore, it can be considered as the coupling constant $\sigma \ll 1$, and σ is a dimensionless real number. Thus, equation (2.3) can be rewritten as

$$\left(\bar{\gamma}^\mu\partial_\mu + \frac{m}{\hbar} - \mathrm{i}\sigma\hbar\bar{\gamma}^t\partial_t\bar{\gamma}^j\partial_j\right)\psi_{\alpha_1\ldots\alpha_k} = 0. \tag{2.5}$$

According to general relativity and Riemannian geometry, we can extend equation (2.5) to the arbitrarily accelerating black hole space-time. It is noted that the derivative calculation of flat space-time should become a covariant derivative in curved space-time. Therefore, in general, non-stationary curved space-time, containing LIV, the fermions dynamic equation with any spin is

$$\left(\gamma^\mu D_\mu + \frac{m}{\hbar} - \mathrm{i}\sigma\hbar\gamma^\nu D_\nu\gamma^j D_j\right)\psi_{\alpha_1\ldots\alpha_k} = 0, \tag{2.6}$$

where $j = 1, 2, 3$. The condition of this equation is

$$\gamma^\mu\psi_{\mu\alpha_2\ldots\alpha_k} = D_\mu\psi^\mu{}_{\alpha_2\ldots\alpha_k} = \psi^\mu{}_{\mu\alpha_3\ldots\alpha_k} = 0, \tag{2.7}$$

where γ^μ is the gamma matrix in general curved space-time, which is defined $\gamma^\mu = g^{\mu\nu}\gamma_\nu$. There are four matrices corresponding to gamma matrix γ^μ, namely γ^0, γ^1, γ^2, γ^3. γ^μ and $g^{\mu\nu}$ satisfy the following relations [47]

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I. \tag{2.8}$$

Here I is the identity matrix. D_μ in equations (2.6) and (2.7) is defined as

$$D_\mu = \partial_\mu + \frac{\mathrm{i}}{\hbar}eA_\mu + \frac{\mathrm{i}}{2}\Gamma^{\alpha\beta}_\mu\Pi_{\alpha\beta}, \tag{2.9}$$

and

$$\Pi_{\alpha\beta} = \frac{\mathrm{i}}{4}[\gamma^\alpha, \gamma^\beta] = \mathrm{i}\tilde{\Pi}_{\alpha\beta}, \tag{2.10}$$

$\frac{\mathrm{i}}{2}\Gamma^{\alpha\beta}_\mu\Pi_{\alpha\beta} = \Omega_\mu$ in equation (2.9) is a spin connection term, which will be nonzero in curved space-time. We can construct

γ^μ according to different types of curved space-time, and then study the solution method of equation (2.6). By solving equation (2.6) in the arbitrarily accelerating black hole space-time, the modified correction in the case of LIV is made for the physical quantities such as fermions tunneling rate and Hawking temperature. According to the semi-classical Wentzel–Kramers–Brillouin (WKB) theory, the probability of tunneling at the event horizon of arbitrarily accelerating black holes is related to the particle action S. Therefore, we can rewrite the fermion wave function of any spin as

$$\psi_{\alpha_1\ldots\alpha_k} = \xi_{\alpha_1\ldots\alpha_k} e^{\frac{i}{\hbar}s}, \tag{2.11}$$

where $\xi_{\alpha_1\ldots\alpha_k}$ is a column matrix. By substituting equations (2.11) into (2.6), we can get a matrix equation. Before obtaining the matrix equation, it should be noted that for arbitrarily accelerating black holes in curved space-time, the advanced Eddington coordinate v is used to represent its dynamic characteristics. Therefore, the metric in the arbitrarily accelerating black hole space-time $g_{\mu\nu}$ is a function of v, r, θ, φ. Here v corresponds to the time coordinate. The particle action S in equation (2.11) should be $S(v, r, \theta, \varphi)$, therefore, equation (2.11) is substituted into the following dynamic equation

$$\left(\gamma^\mu D_\mu + \frac{m}{\hbar} - i\sigma\hbar\gamma^\nu D_\nu \gamma^j D_j\right)\psi_{\alpha_1\ldots\alpha_k} = 0. \tag{2.12}$$

The following matrix equation is obtained

$$\begin{aligned}&[i\gamma^\mu(\partial_\mu S + eA_\mu) + m\\ &\quad + i\sigma\gamma^\nu\gamma^j(\partial_\nu S + eA_\nu)(\partial_j S + eA_j)\\ &\quad \left.- \frac{\hbar}{2}(\bar{\Gamma} + \sigma\tilde{\Gamma})\right]\xi_{\alpha_1\ldots\alpha_k} = 0,\end{aligned} \tag{2.13}$$

where

$$\bar{\Gamma} = \gamma^\mu \Gamma_\mu^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}, \tag{2.14}$$

$$\begin{aligned}\tilde{\Gamma} &= \gamma^\nu\gamma^j[(\partial_\nu S + eA_\nu)\Gamma_j^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}\\ &\quad + (\partial_j S + eA_j)\Gamma_\nu^{\alpha\beta}\tilde{\Pi}_{\alpha\beta}].\end{aligned} \tag{2.15}$$

In order to solve the matrix equation (2.13), we need to make

$$\begin{aligned}\eta^\mu &= [1 - i\sigma(\partial_\nu S + eA_\nu)\gamma^\nu - i\sigma(\partial_j S + eA_j)\gamma^j]\gamma^\mu\\ &= \gamma^\mu - i\sigma[(\partial_\nu S + eA_\nu)\gamma^\nu + (\partial_j S + eA_j)\gamma^j]\gamma^\mu.\end{aligned} \tag{2.16}$$

So here is

$$\begin{aligned}i\eta^\mu(\partial_\mu S + eA_\mu) &= i\gamma^\mu(\partial_\mu S + eA_\mu)\\ &\quad + \sigma[(\partial_\nu S + eA_\nu)\gamma^\nu + (\partial_j S + eA_j)\gamma^j]\\ &\quad \gamma^\mu(\partial_\mu S + eA_\mu).\end{aligned} \tag{2.17}$$

Substitute equation (2.17) into matrix equation (2.13), we can obtain

$$\begin{aligned}&\{i\eta^\mu(\partial_\mu S + eA_\mu)\\ &\quad - \sigma[(\partial_\nu S + eA_\nu)\gamma^\nu + (\partial_j S + eA_j)\gamma^j]\\ &\quad \gamma^\mu(\partial_\mu S + eA_\mu) + m - \frac{\hbar}{2}(\bar{\Gamma} + \sigma\tilde{\Gamma})\\ &\quad + i\sigma g^{\nu j}(\partial_\nu S + eA_\nu)(\partial_j S + eA_j)\}\\ &\quad \xi_{\alpha_1\ldots\alpha_k} = 0.\end{aligned} \tag{2.18}$$

Multiplying both sides of the equation with $i\eta^\nu(\partial_\nu S + eA_\nu)$, and using the relationship between gamma matrix γ^μ and γ^ν in equation (2.8), we can get

$$\begin{aligned}&\{g^{\mu\nu}(\partial_\mu S + eA_\mu)(\partial_\nu S + eA_\nu)\\ &\quad + 2\sigma m[(\partial_\nu S + eA_\nu)g^{\nu\mu} + (\partial_j S + eA_j)g^{j\mu}]\\ &\quad (\partial_\mu S + eA_\mu) + m^2 - m\hbar\bar{\Gamma} - i2\sigma y_0\}\xi_{\alpha_1\ldots\alpha_k} = 0,\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}y_0 &= g^{\nu j}(\partial_\nu S + eA_\nu)(\partial_j S + eA_j)\left(m - \frac{\hbar}{2}\bar{\Gamma}\right)\\ &\quad + (\partial_\rho S + eA_\rho)\gamma^\rho g^{\mu\nu}(\partial_\mu S + eA_\mu)(\partial_\nu S + eA_\nu).\end{aligned} \tag{2.20}$$

From this equation, it can be seen that the particularity of the term containing imaginary unit i is that the expression of ν, j and μ, after changing positions remains unchanged. It can be obtained from equation (2.19)

$$\begin{aligned}&\{g^{\mu\nu}(1 + 2\sigma m)(\partial_\mu S + eA_\mu)(\partial_\nu S + eA_\nu)\\ &\quad + m^2 - m\hbar\bar{\Gamma} - i2\sigma y_0\}\xi_{\alpha_1\ldots\alpha_k} = 0.\end{aligned} \tag{2.21}$$

This is the fermions dynamic equation with arbitrary spin derived from the arbitrarily accelerating black hole space-time. For the sake of clarity, for fermions with spin $-\frac{1}{2}$, and $\psi = \xi e^{\frac{i}{\hbar}s}$, there is

$$\xi = \begin{pmatrix} A \\ B \end{pmatrix}. \tag{2.22}$$

This is a matrix equation of 2×1, for fermions with spin $-\frac{3}{2}$, there is

$$\xi_\lambda = \begin{pmatrix} A_\lambda \\ B_\lambda \end{pmatrix}, \tag{2.23}$$

where $A_\lambda = (a_\lambda \;\; c_\lambda)^{Tm}$, $B_\lambda = \left(b_\lambda \;\; d_\lambda\right)^{Tm}.a_\lambda$, b_λ, c_λ and d_λ represent the corresponding matrix respectively. Therefore, the matrix equation (2.21) is an eigenmatrix equation. The condition for the solution of this eigenmatrix equation is that the value of the determinant corresponding to its matrix must be zero. Therefore, the modified dynamic fermions equation by the action S is expressed as

$$g^{\mu\nu}(1 + 2\sigma m)(\partial_\mu S + eA_\mu)(\partial_\nu S + eA_\nu) + m^2 - m\hbar\bar{\Gamma} = 0. \tag{2.24}$$

In the process of obtaining this equation, $\hbar^2$, $\sigma^2\hbar$, $\sigma\hbar$ is ignored. In the semi-classical situation, $S(v, r, \theta, \varphi)$ can be obtained without considering the $\hbar$ term. If we want to make a more accurate correction of quantum properties, we need to consider the $\hbar$ term in equation (2.24), and we can conduct more in-depth research by adopting the way beyond semi-classical theory. From equation (2.24), for a special class of arbitrarily accelerating black holes, $g^{vv} = g^{00} = 0$, Starting from equation (2.24), the quantum tunneling radiation can still be corrected. According to the specific arbitrarily accelerating black hole space-time metric $g_{\mu\nu}$ or $g^{\mu\nu}$, we can make necessary corrections to the black hole temperature and tunneling radiation rate from equation (2.24).

3. Correction of quantum tunneling radiation of the arbitrarily accelerating Kinnersly black hole

The fermions dynamic equation with any spin of mass m and charge e is shown in equation (2.24). In the different arbitrarily accelerating black hole space-time, the method of solving equation (2.24) is different. For an arbitrarily accelerating Kinnersly black hole, the curved space-time line element representing the dynamic characteristics with the advanced Eddington–Finkelstein coordinate v is expressed as [48]

$$\begin{aligned} \mathrm{d}s^2 = g_{00}\mathrm{d}v^2 - 2\mathrm{d}v\mathrm{d}r - 2r^2 f\mathrm{d}v\mathrm{d}\theta \\ - 2r^2 G\sin^2\theta\mathrm{d}v\mathrm{d}\phi - r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}\varphi^2, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} g_{00} = g_{vv} &= 1 - 2M(v)r^{-1} \\ &\quad - 2a(v)r\cos\theta - r^2 f^2 - G^2 r^2\sin^2\theta, \\ f &= -a(v)\sin\theta + b(v)\sin\phi + c(v)\cos\varphi, \\ G &= ctg\theta[b(v)\cos\varphi - c(v)\sin\varphi], \end{aligned} \tag{3.2}$$

where $a(\upsilon)$ is the acceleration value and its direction always pointing towards the north pole. $b(\upsilon)$, $c(\upsilon)$ represents the change rate in the direction. It can be seen from equations (3.1) and (3.2) that the determinant of curved space-time metric and the non-zero inverse metric tensor are

$$g = -r^4\sin^2\theta \tag{3.3}$$

$$\begin{aligned} g^{01} &= g^{vr} = g^{rv} = -1, \\ g^{22} &= g^{\theta\theta} = -\frac{1}{r^2}, \\ g^{12} &= g^{r\theta} = g^{\theta r} = -f, \\ g^{33} &= g^{\varphi\varphi} = -\frac{1}{r^2\sin^2\theta}, \\ g^{11} &= g^{rr} = -1 + \frac{2M}{r} + 2a(v)\cos\theta \\ &= -\left(1 - \frac{2M}{r} - 2a(v)\cos\theta\right) = -\tilde{g}^{11}, \\ g^{13} &= g^{r\varphi} = g^{\varphi r} = G, \end{aligned} \tag{3.4}$$

where $g^{vv} = g^{00} = 0$. Ignoring the $\hbar$ term in equation (2.24), we can get the following semi-classical equation

$$g^{\mu\nu}(1 + 2\sigma m)(\partial_\mu S + eA_\mu)(\partial_\nu S + eA_\nu) + m^2 = 0, \tag{3.5}$$

where m can be the mass of fermion with spin $-\frac{1}{2}$, or the mass of fermion with spin $-\frac{3}{2},\cdots$. The e in equation (3.5) is the charge of fermion. Since the arbitrarily accelerating black hole represented by (3.1) is not charged, so $\mathrm{A}_\mu = 0$ in equation (3.5). For the convenience of narration, we let $m_{\frac{1}{2}} = m$, which denotes the fermion quality with spin $-\frac{1}{2}$. Then, according to equations (3.4), (3.5) is simplified to

$$\begin{aligned} &(1 + 2\sigma m)g^{11}(\partial_v S)^2 \\ &\quad - 2(1 + 2\sigma m)(\partial_v S\partial_r S + f\partial_r S\partial_\theta S - G\partial_r S\partial_\varphi S) \\ &\quad - (1 + 2\sigma m)\left[\frac{1}{r^2}(\partial_\theta S)^2 + \frac{1}{r^2\sin^2\theta}(\partial_\varphi S)^2\right] + m^2 = 0. \end{aligned} \tag{3.6}$$

In order to solve this equation, we must make the following generalized tortoise coordinate transformation for equation (3.6) [12]

$$\begin{aligned} r_* &= r - \frac{1}{2k}\ln\frac{r - r_H(v,\theta,\varphi)}{\tilde{r}_H(v_0,\theta_0,\varphi_0)}, \\ v_* &= v - v_0, \\ \theta_* &= \theta - \theta_0, \\ \varphi_* &= \varphi - \varphi_0. \end{aligned} \tag{3.7}$$

From this transformation, several partial derivatives closely related to equation (3.6) can be obtained, and the operations are as follows

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{1 - 2k(r - r_H)}{2k(r - r_H)}\frac{\partial}{\partial r_*}, \\ \frac{\partial}{\partial v} &= \frac{\partial}{\partial v_*} + \frac{r_{H,v}}{2k(r - r_H)}\frac{\partial}{\partial r_*}, \\ \frac{\partial}{\partial\theta} &= \frac{\partial}{\partial\theta_*} + \frac{r_{H,\theta}}{2k(r - r_H)}\frac{\partial}{\partial r_*}, \\ \frac{\partial}{\partial\varphi} &= \frac{\partial}{\partial\varphi_*} + \frac{r_{H,\varphi}}{2k(r - r_H)}\frac{\partial}{\partial r_*}, \end{aligned} \tag{3.8}$$

where the advanced Eddington coordinate is $v = \mathrm{t} + r_*$. Bring equations (3.8) into (3.6), we can get

$$\begin{aligned} &(1 + 2\sigma m)\tilde{g}^{11}\left[\frac{1 - 2k(r - r_H)}{2k(r - r_H)}\right]^2\left(\frac{\partial S}{\partial r_*}\right)^2 + 2(1 + 2\sigma m) \\ &\times\left\{\left[\frac{\partial S}{\partial v_*} + \frac{r_{H,v}}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\left[\frac{1 - 2k(r - r_H)}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\right. \\ &\quad + f\left[\frac{1 - 2k(r - r_H)}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\left[\frac{\partial S}{\partial\theta_*} + \frac{r_{H,\theta}}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right] \\ &\quad \left. - G\left[\frac{1 - 2k(r - r_H)}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\left[\frac{\partial S}{\partial\varphi_*} + \frac{r_{H,\varphi}}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\right\} \\ &\quad + (1 + 2\sigma m) \\ &\times\left\{\frac{1}{r^2}\left[\frac{\partial S}{\partial\theta_*} + \frac{r_{H,\theta}}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]^2\right. \\ &\quad \left. + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial S}{\partial\varphi_*} + \frac{r_{H,\varphi}}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]^2\right\} - m^2 = 0. \end{aligned} \tag{3.9}$$

In this equation $r_H = r_H(v, r, \theta, \varphi)$ is the event horizon of the black hole, and r_H is determined by the following null hypersurface equation, that is

$$g^{\mu\nu}\frac{\partial F}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} = 0. \tag{3.10}$$

As a hypersurface equation $F = F(v, r, \theta, \varphi)$ or $r = r(v, \theta, \varphi)$, from equations (3.4)–(3.10), we can get the following equation

$$\begin{aligned} &\tilde{g}^{11} + 2r_{H,v} + 2fr_{H,\theta} - 2Gr_{H,\varphi} \\ &+\frac{1}{r^2}(r_{H,\theta})^2 + \frac{1}{r^2\sin^2\theta}(r_{H,\varphi})^2 = 0. \end{aligned} \tag{3.11}$$

This is the specific form of the equation satisfied by the event horizon r_H of the arbitrarily accelerating black hole. In order to solve equation (3.9), we treat the left edge of equation (3.9) as a combined congener and consider the case when $r \to r_H$, $[1 - 2k(r - r_H)]|_{r\to r_H} = 1$ and $2k(r - r_H)|_{r\to r_H} = 0$, then equation (3.9) is rewritten as

$$\begin{aligned} &\left(\frac{\partial S}{\partial r_*}\right)^2 [2k(r - r_H)]^{-1}|_{r\to r_H} \\ &(1 + 2\sigma m)[\tilde{g}^{11} + 2r_{H,v} + 2fr_{H,\theta} - 2Gr_{H,\varphi} \\ &+\frac{1}{r^2}(r_{H,\theta})^2 + \frac{1}{r^2\sin^2\theta}(r_{H,\varphi})^2\Bigg]|_{r\to r_H} \\ &+2(1 + 2\sigma m)\frac{\partial S}{\partial v_*}\frac{\partial S}{\partial r_*} + 2(1 + 2\sigma m) \\ &\times \frac{\partial S}{\partial r_*}\Bigg[\left(f\frac{\partial S}{\partial \theta_*} - G\frac{\partial S}{\partial \varphi_*}\right) \\ &+ r_H^{-2}\left(\frac{\partial S}{\partial \theta_*} + \sin^{-2}\theta r_{H,\varphi}\frac{\partial S}{\partial \varphi_*}\right)\Bigg]|_{r\to r_H} = 0. \end{aligned} \tag{3.12}$$

Considering that the tunneling radiation will cause the event horizon of the black hole to shrink in fact, r_H will become $r_H - \varepsilon$, here $\varepsilon \ll 1$. Therefore, there is a small required item in equation (3.11) which can be expressed as

$$\begin{aligned} &\tilde{g}^{11}(r_H) + 2r_{H,v} + 2fr_{H,\theta} - 2Gr_{H,\varphi} \\ &+\frac{1}{{r_H}^2}(r_{H,\theta})^2 + \frac{1}{{r_H}^2\sin^2\theta}(r_{H,\varphi})^2 + Y(\varepsilon) = 0, \end{aligned} \tag{3.13}$$

where $Y(\varepsilon)$ is a small correction term. Obviously, in equation (3.12), the coefficient of the term of $\left(\frac{\partial S}{\partial r_*}\right)^2$ has $\frac{0}{0}$ type. Therefore, the following limit holds

$$\begin{aligned} &\lim_{r\to r_H}\frac{(1 + 2\sigma m)}{2k(r - r_H)} \\ &\times[\tilde{g}^{11} + 2r_{H,v} + 2fr_{H,\theta} - 2Gr_{H,\varphi} \\ &+\frac{1}{r^2}(r_{H,\theta})^2 + \frac{1}{r^2\sin^2\theta}(r_{H,\varphi})^2\Bigg] = 1, \end{aligned} \tag{3.14}$$

and we presume

$$\frac{\partial S}{\partial v_*} = -\omega, \tag{3.15}$$

$$\omega_0 = fp_\theta - Gp_\varphi + {r_H}^2(p_\theta + \sin^{-2}\theta r_{H,\varphi}p_\varphi). \tag{3.16}$$

Equation (3.12) becomes

$$\left(\frac{\partial S}{\partial r_*}\right)^2 - 2(\omega - \omega_0)\frac{\partial S}{\partial r_*} = 0, \tag{3.17}$$

where ω is the radiant energy, ω_0 is related to the chemical potential and is often directly related to the maximum energy of non-thermal radiation [13]. p_θ and p_φ are the components of the generalized momentum of the radiating particles in the θ and φ direction, respectively. Available from equation (3.17), it can be obtained

$$\frac{\partial S_\pm}{\partial r_*} = (\omega - \omega_0) \pm (\omega - \omega_0). \tag{3.18}$$

It can also be seen from equation (3.8)

$$\begin{aligned} \frac{\partial S_\pm}{\partial r} &= \frac{1 - 2k(r - r_H)}{2k(r - r_H)}\frac{\partial S_\pm}{\partial r_*} \\ &= \frac{1 - 2k(r - r_H)}{2k(r - r_H)}[(\omega - \omega_0) \pm (\omega - \omega_0)]. \end{aligned} \tag{3.19}$$

For $r \to r_H$, the integral is obtained by the residue theorem

$$S_\pm = \frac{\mathrm{i}\pi}{2k}[(\omega - \omega_0) \pm (\omega - \omega_0)]. \tag{3.20}$$

Here, k is the same as k in equation (3.14), and k needs to be calculated from equation (3.14). By applying L' Hospital' rule to equation (3.14) to find the limit, we can get

$$k = (1 + 2\sigma m)\frac{1}{{r_H}^2}\left[M - \frac{1}{r_H}(r_{H,\theta})^2 - \frac{1}{r_H\sin^2\theta}(r_{H,\varphi})^2\right] \tag{3.21}$$

Therefore, by ignoring the higher order term of σ, the particle action $S_\pm$ can be expressed as

$$\begin{aligned} S_\pm &= \pm\mathrm{i}\pi\frac{(1 - 2\sigma m){r_H}^2(\omega - \omega_0)}{M - \frac{(r_{H,\theta})^2}{r_H} - \frac{(r_{H,\varphi})^2}{r_H\sin^2\theta}} \\ &= \pm\mathrm{i}\pi(1 - 2\sigma m){r_H}^2(\omega - \omega_0)/\{r_H - M \\ &\quad - 2r_H a(v)\cos\theta + 2r_H r_{H,v} \\ &\quad + 2fr_H r_{H,\theta} - 2Gr_H r_{H,\varphi}\}. \end{aligned} \tag{3.22}$$

According to the WKB theory and quantum tunneling radiation theory, we get the tunneling rate of this black hole at the event horizon r_H, which is

$$\Gamma \sim \exp(-2\,\mathrm{Im}\,S_\pm) = \exp\left(-\frac{\omega - \omega_0}{T_H}\right), \tag{3.23}$$

$$\begin{aligned} T_H = &\frac{1 - 2\sigma m}{2\pi {r_H}^2}\{r_H - M - 2r_H a(v)\cos\theta \\ &+ 2r_H r_{H,v} + 2fr_H r_{H,\theta} - 2Gr_H r_{H,\varphi}\}. \end{aligned} \tag{3.24}$$

T_H is the Hawking temperature at the event horizon of this black hole. According to equation (3.24), T_H is related to $a(v)$, $b(v)$, $c(v)$ and v, θ, φ, *and* $\frac{\partial r_H}{\partial v}$, $\frac{\partial r_H}{\partial \theta}$, $\frac{\partial r_H}{\partial \varphi}$, still has the correction term σ of LIV theory. Equation (3.24) is a new expression of Hawking temperature of Kinnersly black hole with arbitrary acceleration. For tunneling radiation with spin $-\frac{3}{2}$, spin $-\frac{5}{2}$,$\cdots$, the new expressions of the corresponding quantum tunneling rate and Hawking temperature are shown in equations (3.23) and (3.24).

The above formula (2.24) is a transformation formula. From this, we can get the fermions dynamic equation with spin $1/2$ after LIV correction. The above results are obtained by using the semi-classical theory. In order to reflect the higher-order quantum effect, according to equation (3.18), there are

$$E_0 = \omega - \omega_0, \tag{3.25}$$

$$S^{\pm} = S_0{}^{\pm} = S_0. \tag{3.26}$$

So after considering the $\hbar$ perturbation, we can rewrite the energy and action of the tunneling particle as [49, 50]

$$E = E_0 + \sum_i \hbar^i E_i, \tag{3.27}$$

$$S = S_0 + \sum_i \hbar^i S_i. \tag{3.28}$$

By (3.27), (3.28) and (3.17), we can rewrite equation (3.17) as

$$\left(\frac{\partial S_0}{\partial r_*}\right)^2 - 2E_0\frac{\partial S_0}{\partial r_*} = 0. \tag{3.29}$$

It can be concluded that

$$\frac{\partial S_0{}^{\pm}}{\partial r_*} = (\omega - \omega_0) \pm (\omega - \omega_0), \tag{3.30}$$

and

$$\begin{aligned}\frac{\partial S_0{}^{\pm}}{\partial r} &= \frac{1 - 2k(r - r_H)}{2k(r - r_H)}[(\omega - \omega_0) \pm (\omega - \omega_0)]\\ &= \frac{1 - 2k(r - r_H)}{2k(r - r_H)}(E_0 \pm E_0),\end{aligned} \tag{3.31}$$

$$S_0{}^{\pm} = \frac{\mathrm{i}\pi}{2k}[(\omega - \omega_0) \pm (\omega - \omega_0)] = \frac{\mathrm{i}\pi}{2k}(E_0 \pm E_0). \tag{3.32}$$

Equation (3.30) corresponds to the semi-classical result, that is, the result corresponding to $\hbar^0$. For $\hbar^1$, equation (3.29) should be

$$\left[\frac{\partial(S_0 + \hbar S_1)}{\partial r_*}\right]^2 - 2(E_0 + \hbar E_1)\frac{\partial(S_0 + \hbar S_1)}{\partial r_*} = 0. \tag{3.33}$$

Using equation (3.29) and ignoring the $\hbar^2$ term, equation (3.33) can be simplified, and by the same token, here are

$$\begin{aligned}&\hbar^2:\left(\frac{\partial S_2}{\partial r_*}\right)^2 - 2E_2\frac{\partial S_2}{\partial r_*} = 0,\\ &\hbar^3:\left(\frac{\partial S_3}{\partial r_*}\right)^2 - 2E_3\frac{\partial S_3}{\partial r_*} = 0,\\ &\vdots\cdots\end{aligned} \tag{3.34}$$

Obviously, the equations about S_1, S_2, S_3, $\cdots$ are not independent, and they are intrinsically related to the S_0 equation. There are proportional coefficients between these particle actions

$$\frac{S_{i+1}}{S_i} = \alpha_i. \tag{3.35}$$

So, considering $\hbar$, we can get the particle action with quantum correction meaning as

$$S = S_0 + \hbar^1 S_1 + \hbar^2 S_2 + \cdots = S_0\left(1 + \sum_i \hbar^i \alpha_i\right). \tag{3.36}$$

By substituting equation (3.33), we can get

$$\begin{aligned}S^{\pm} &= S_0^{\pm}\left(1 + \sum_i \hbar^i \alpha_i\right)\\ &= \frac{i\pi}{2k}\left(1 + \sum_i \hbar^i \alpha_i\right)[(\omega - \omega_0) \pm (\omega - \omega_0)].\end{aligned} \tag{3.37}$$

So we get that the tunneling rate and temperature of the black hole beyond the semi-classical theory are

$$\Gamma'^{\sim} \exp(-2\,\mathrm{Im}\,S_{\pm}) = \exp\left(-\frac{\omega - \omega_0}{T_H^{'}}\right) \tag{3.38}$$

and

$$\begin{aligned}T_H' = \frac{1 - 2\sigma m}{2\pi r_H^2}\left(1 + \sum_i \hbar^i \alpha_i\right)\{&r_H - M - 2r_H a(\upsilon)\cos\theta\\ &+ 2r_H r_{H,v} + 2fr_H r_{H,\theta} - 2Gr_H r_{H,\varphi}\}.\end{aligned} \tag{3.39}$$

So the tunneling radiation of this kind of black hole will be corrected more accurately by using the way beyond semi-classical theory. From $\hbar^1$, $\hbar^2$, $\cdots$, it reflected the impact. It should be further explained that for a specific black hole space-time metric, we can construct a specific gamma matrix γ^{μ}. From equation (2.8), it can be seen that γ^{μ} has an inevitable connection with the contravariant metric tensor $g^{\mu\nu}$. Therefore, the fermions dynamic equation (2.24) containing γ^{μ} has an inevitable connection through WKB theory with action S. Equation (2.24) is a new form of the fermions dynamic equation.

4. Discussion

T_H' is a new and more accurate expression of the black hole. For T_H' and Γ', arbitrarily accelerating black holes, Vaidya black holes and other special cases are included in the following equation. In the special case of $a(v) = 0$, $b(v) = 0$, $c(v) = 0$, according to equation (3.23) and equation (3.24), we can get the tunneling rate Γ_V and the relative Hawking temperature T_{VH} of the Vaidya black hole after LIV correction, respectively

$$\Gamma_V = \exp\left(-\frac{\omega}{T_{VH}}\right) \tag{4.1}$$

and

$$\begin{aligned}T_{VH} &= \frac{1 - 2\sigma \mathrm{m}}{2\pi {r_{VH}}^2}(r_{VH} - \mathrm{M})\\ &= \frac{(1 - 2\sigma \mathrm{m})M}{2\pi {r_{VH}}^2}\left(1 - \frac{2r_{VH}\dot{r}_{VH}}{M}\right),\end{aligned} \tag{4.2}$$

where T_{VH} is the Hawking temperature of Vaidya black hole and $\dot{\mathrm{r}}_{\mathrm{VH}} = \frac{\partial \mathrm{r}_{VH}}{\partial v}$. If the items σ and $\hbar$ are ignored, the equation (4.2) will degenerate to $T_{SH} = \frac{1}{8\pi M}$, which is consistent with the known results. Obviously, according to equations (3.39) and (3.38), this is the more accurate physical tunneling rate and Hawking temperature of the black hole. Black hole entropy is an important physical quantity in the process of black hole evolution. The correction of the Hawking temperature $T_H^{'}$ of the arbitrarily accelerating black hole will inevitably lead to the correction of black hole entropy. If we use $S_{BH}^{'}$ to represent the Bekenstein–Hawking entropy of the arbitrarily accelerating Kinnersly black hole and use $\triangle S_{BH}^{'}$ to represent the entropy change of the black hole, then the tunneling rate Γ' can be expressed as

$$\Gamma' \sim \mathrm{e}^{\triangle S_{BH}'}. \tag{4.3}$$

If in the above process of correcting the Hawking temperature of this black hole, we ignore the $\hbar$ in equation (2.24), the meaningful results of the above equations can be obtained in the semi-classical theory. If the $\hbar$ in equation (2.24) is not ignored, then, we need to consider the perturbation effect of the $\hbar$.

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