 Communications in Theoretical Physics

Commun. Theor. Phys. **76** (2024) 055001 (15pp) https://doi.org/10.1088/1572-9494/ad35b1

Cauchy matrix approach to three non-isospectral nonlinear Schrödinger equations

Alemu Yilma Tefera[1], **Shangshuai Li**[1,2,3] **and Da-jun Zhang**[1,2]

[1] Department of Mathematics, Shanghai University, Shanghai 200444, China
[2] Newtouch Center for Mathematics of Shanghai University, Shanghai 200444, China
[3] Department of Applied Mathematics, Faculty of Science and Engineering, Waseda University, Tokyo 169-8555, Japan

E-mail: djzhang@staff.shu.edu.cn

Received 19 February 2024, revised 18 March 2024
Accepted for publication 20 March 2024
Published 22 April 2024

Abstract

This paper aims to develop a direct approach, namely, the Cauchy matrix approach, to non-isospectral integrable systems. In the Cauchy matrix approach, the Sylvester equation plays a central role, which defines a dressed Cauchy matrix to provide τ functions for the investigated equations. In this paper, using the Cauchy matrix approach, we derive three non-isospectral nonlinear Schrödinger equations and their explicit solutions. These equations are generically related to the time-dependent spectral parameter in the Zakharov–Shabat–Ablowitz–Kaup–Newell–Segur spectral problem. Their solutions are obtained from the solutions of unreduced non-isospectral nonlinear Schrödinger equations through complex reduction. These solutions are analyzed and illustrated to show the non-isospectral effects in dynamics of solitons.

Keywords: Cauchy matrix approach, Sylvester equation, nonlinear Schrödinger equation, non-isospectral integrable system, explicit solution

(Some figures may appear in colour only in the online journal)

1. Introduction

The nonlinear Schrödinger (NLS) equation,

$$\mathrm{i}q_t - q_{xx} - 2\epsilon|q|^2 q = 0, \tag{1.1a}$$

as one of the most famous integrable systems, is known as a 'universal' model [1], which means it appears as a governing model in various physical phenomena. Here 'i' is the imaginary unit, $\epsilon = \pm 1$, $|q|^2 = qq^*$ and $*$ stands for complex conjugate. It emerges in describing wave packages in deep water [2, 3], plasma physics [4], optical fiber [5, 6], etc. In addition, the NLS equation with various external potentials (known also as the Gross–Pitaevskii (GP) equation [7–9]) is also the governing model in nonlinear optics and Bose–Einstein condensates (BEC) [10]. One can refer to [11] for more references and applications of the NLS equation and its extensions.

The NLS equation with x-coefficient can describe nonlinear waves in non-uniformity media [12–15]. Such equations have been shown to be integrable in the sense of having Lax pairs, with the spectral parameter η satisfying $\eta_t \neq 0$, which are referred to non-isospectral nonlinear Schrödinger equations (NNLSEs). In this paper, we will investigate the following three NNLSEs:

$$\mathrm{i}q_{1,t} - q_{1,xx} - 2|q_1|^2 q_1 + 2\alpha x q_1 = 0, \tag{1.2a}$$

$$\mathrm{i}q_{2,t} - q_{2,xx} - 2|q_2|^2 q_2 + \mathrm{i}\beta(xq_2)_x = 0, \tag{1.2b}$$

$$\mathrm{i}q_{3,t} - x(q_{3,xx} + 2|q_3|^2 q_3) - 2q_{3,x} - 2q_3\partial^{-1}|q_3|^2 = 0, \tag{1.2c}$$

where α and β are real constants, and ∂^{-1} stands for the integration operator with respect to x. We denote these equations NNLSE-I, NNLSE-II, and NNLSE-III for short, respectively. They correspond to time-dependant spectral parameter η with time evolutions $\eta_t = \alpha$, $\eta_t = -\mathrm{i}\beta\eta$ and $\eta_t = -2\eta^2$, respectively, where $\alpha, \beta \in \mathbb{R}$, e.g. [16]. Although the NNLSE-I and NNLSE-II can be converted to the NLS equation (1.1a) via gauge transformations, e.g. [16], they are physically useful in BEC: the NNLSE-I is the GP equation with a linear potential [17], while the NNLSE-II can provide solutions to the GP equation with a parabolic potential and a gain term e.g. [18]. The NNLSE-III can provide space-time localized soliton waves on zero background [16, 19]. So far, integrable methods, such as the inverse scattering transform [12–14] and bilinear method [16], have been applied to obtain explicit solutions of the above equations. Yet in this paper, we construct their solutions by means of a completely direct method, namely, the Cauchy matrix approach.

The Cauchy matrix approach is a method to construct and study integrable equations by means of the Sylvester-type equations. It is first systematically introduced in [20] to investigate integrable quadrilateral equations and later developed in [21, 22] to more general cases. It has also been applied to the Zakharov–Shabat–Ablowitz–Kaup–Newell–Segur (ZS-AKNS) system [23], equations with self-consistent sources [24], and the self-dual Yang-Mills equation [25, 26], etc. The purpose of this paper is not only to construct solutions to the three NNLSEs in (1.2), but also to extend the Cauchy matrix approach to the non-isospectral case, as the Sylvester-type equation in the Cauchy matrix scheme of the ZS-AKNS system is a typical type (see [22, 27] for the Korteweg–de Vries (KdV) and Kadomtsev–Petviashvili (KP) type equations). One will see that the non-isospectral extension of the Cauchy matrix scheme is quite non-trivial compared with the isospectral case [23].

This paper is organized as follows. Our plan is in the first step to derive three unreduced non-isospectral Schrödinger systems using the Cauchy matrix approach. This will be described in section 2. Then in section 3, we present solution formulae of these unreduced systems. These formulae guide us to implement reduction so that solutions of the NNLSEs are obtained, which will be done in section 4. Dynamics of these solutions are illustrated also in this section. Finally, conclusions are given in section 5. There is an appendix section where solutions of the Sylvester equation with lower triangular Toeplitz matrices are presented.

2. Cauchy matrix approach to unreduced NNLSEs

In this section, we describe the Cauchy matrix approach for unreduced NNLSEs.

2.1. Sylvester equation and master functions

We start from the Sylvester equation of the following type (see [23, 25]):

$$\boldsymbol{K}\boldsymbol{M}-\boldsymbol{M}\boldsymbol{K}=\boldsymbol{r}\boldsymbol{s}^{\mathrm{T}}, \tag{2.1}$$

in which the involved elements are block matrices in the form of

$$\boldsymbol{K}=\begin{pmatrix}\boldsymbol{K}_1 & 0\\ 0 & \boldsymbol{K}_2\end{pmatrix}, \boldsymbol{M}=\begin{pmatrix}0 & \boldsymbol{M}_1\\ \boldsymbol{M}_2 & 0\end{pmatrix},$$
$$\boldsymbol{r}=\begin{pmatrix}\boldsymbol{r}_1 & 0\\ 0 & \boldsymbol{r}_2\end{pmatrix}, \boldsymbol{s}=\begin{pmatrix}0 & \boldsymbol{s}_1\\ \boldsymbol{s}_2 & 0\end{pmatrix}, \tag{2.2}$$

where $\boldsymbol{K}_i\in\mathbb{C}_{N\times N}[t]$, $\boldsymbol{M}_i\in\mathbb{C}_{N\times N}[x,t]$, $\boldsymbol{r}_i$, $\boldsymbol{s}_i\in\mathbb{C}_{N\times 1}[x,t]$, for $i=1,2$. An equivalent form of (2.1) is given by

$$\boldsymbol{K}_1\boldsymbol{M}_1-\boldsymbol{M}_1\boldsymbol{K}_2=\boldsymbol{r}_1\boldsymbol{s}_2^{\mathrm{T}}, \tag{2.3a}$$
$$\boldsymbol{K}_2\boldsymbol{M}_2-\boldsymbol{M}_2\boldsymbol{K}_1=\boldsymbol{r}_2\boldsymbol{s}_1^{\mathrm{T}}. \tag{2.3b}$$

We assume matrices $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ do not share any eigenvalues, so that the Sylvester equation (2.1) has a unique solution $\boldsymbol{M}$ for given $\boldsymbol{K}, \boldsymbol{r}, \boldsymbol{s}$ [28]. By these elements we define master functions

$$\boldsymbol{S}^{(i,j)}\doteq\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}^j(\boldsymbol{I}_{2N}+\boldsymbol{M})^{-1}\boldsymbol{K}^i\boldsymbol{r}=\begin{pmatrix}s_1^{(i,j)} & s_2^{(i,j)}\\ s_3^{(i,j)} & s_4^{(i,j)}\end{pmatrix},\quad (i,j\in\mathbb{Z}), \tag{2.4}$$

where $\boldsymbol{I}_{2N}$ is the $2N$ th-order unit matrix and of which the more explicit versions are

$$s_1^{(i,j)}=-\boldsymbol{s}_2^{\mathrm{T}}\boldsymbol{K}_2^j(\boldsymbol{I}_N-\boldsymbol{M}_2\boldsymbol{M}_1)^{-1}\boldsymbol{M}_2\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{2.5a}$$
$$s_2^{(i,j)}=\boldsymbol{s}_2^{\mathrm{T}}\boldsymbol{K}_2^j(\boldsymbol{I}_N-\boldsymbol{M}_2\boldsymbol{M}_1)^{-1}\boldsymbol{K}_2^i\boldsymbol{r}_2, \tag{2.5b}$$
$$s_3^{(i,j)}=\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{K}_1^j(\boldsymbol{I}_N-\boldsymbol{M}_1\boldsymbol{M}_2)^{-1}\boldsymbol{K}_1^i\boldsymbol{r}_1, \tag{2.5c}$$
$$s_4^{(i,j)}=-\boldsymbol{s}_1^{\mathrm{T}}\boldsymbol{K}_1^j(\boldsymbol{I}_N-\boldsymbol{M}_1\boldsymbol{M}_2)^{-1}\boldsymbol{M}_1\boldsymbol{K}_2^i\boldsymbol{r}_2. \tag{2.5d}$$

In addition, a difference-product formula can be formulated from (2.1) as

$$\boldsymbol{S}^{(i+1,j)}-\boldsymbol{S}^{(i,j+1)}=\boldsymbol{S}^{(0,j)}\boldsymbol{S}^{(i,0)}. \tag{2.6}$$

The proof can be found in [23, 25].

2.2. Unreduced NNLSE-I

To derive an unreduced form of the NNLSE-I equation, let us introduce dispersion relations of $\boldsymbol{r}$ and $\boldsymbol{s}$ as follows,

$$\boldsymbol{r}_x=\frac{1}{2}\boldsymbol{K}\boldsymbol{r}\boldsymbol{a},\quad \boldsymbol{s}_x=-\frac{1}{2}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{s}\boldsymbol{a}, \tag{2.7a}$$
$$\boldsymbol{r}_t=\left(-\frac{1}{2}\boldsymbol{K}^2+\alpha x\right)\boldsymbol{r}\boldsymbol{a},\quad \boldsymbol{s}_t=\left(\frac{1}{2}(\boldsymbol{K}^{\mathrm{T}})^2-\alpha x\right)\boldsymbol{s}\boldsymbol{a}, \tag{2.7b}$$

where

$$\boldsymbol{a}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.8}$$

and matrix $\boldsymbol{K}$ obeys the evolution

$$\boldsymbol{K}_t=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}(t)=2\alpha,\quad \alpha\in\mathbb{R}. \tag{2.9}$$

In addition, we assume $\boldsymbol{K}_t$ and $\boldsymbol{K}$ commute. Then one can derive the evolution of $\boldsymbol{M}$ by taking a derivative of the Sylvester equation (2.1), which gives rise to

$$\boldsymbol{K}_t\boldsymbol{M}+\boldsymbol{K}\boldsymbol{M}_t-\boldsymbol{M}_t\boldsymbol{K}-\boldsymbol{M}\boldsymbol{K}_t=\boldsymbol{r}_t\boldsymbol{s}^{\mathrm{T}}+\boldsymbol{r}\boldsymbol{s}_t^{\mathrm{T}}.$$

Substituting (2.7*b*) and (2.9) into it yields

$$\boldsymbol{K}\boldsymbol{M}_t-\boldsymbol{M}_t\boldsymbol{K}=\frac{1}{2}(-\boldsymbol{K}^2\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}+\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}^2),$$

which leads us to a Sylvester equation,

$$\boldsymbol{K}\left(\boldsymbol{M}_t+\frac{1}{2}(\boldsymbol{K}\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}+\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K})\right)-\left(\boldsymbol{M}_t+\frac{1}{2}(\boldsymbol{K}\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}+\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K})\right)\boldsymbol{K}=0.$$

It has a unique zero solution in the light of assumption that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ do not share any eigenvalues. Thus we have

$$M_t = -\frac{1}{2}(Kras^{\mathrm{T}} + ras^{\mathrm{T}}K). \tag{2.10}$$

One can also find [23]

$$M_x = \frac{1}{2}ras^{\mathrm{T}}. \tag{2.11}$$

Next we are going to derive the derivative of $S^{(i,j)}$. Let us define the auxiliary vector,

$$u^{(i)} = (I_{2N} + M)^{-1}K^i r, \tag{2.12}$$

which is connected with $S^{(i,j)}$ by

$$S^{(i,j)} = s^{\mathrm{T}}K^j u^{(i)}. \tag{2.13}$$

The derivative of $u^{(i)}$ with respect to x reads [23]

$$u_x^{(i)} = \frac{1}{2}(u^{(i+1)}a - u^{(0)}aS^{(i,0)}). \tag{2.14}$$

To derive the derivative of $u^{(i)}$ with respect to t, taking t-derivative in (2.12) yields

$$M_t u^{(i)} + (I_{2N} + M)u_t^{(i)} = iK^{i-1}K_t r + K^i r_t, \tag{2.15}$$

and we substitute(2.7*b*), (2.9) and (2.10) into (2.15), and then left-multiplied by $(I_{2N} + M)^{-1}$, we get

$$u_t^{(i)} = \frac{1}{2}(4i\alpha u^{(i-1)} - u^{(i+2)}a + u^{(1)}aS^{(i,0)} + u^{(0)}aS^{(i,1)}) + \alpha x u^{(i)}a. \tag{2.16}$$

Using the relation (2.13), it is easy to get x-derivative of $S^{(i,j)}$, which reads [23]

$$S_x^{(i,j)} = \frac{1}{2}(S^{(i+1,j)}a - aS^{(i,j+1)} - S^{(0,j)}aS^{(i,0)}). \tag{2.17}$$

For the t-derivative of $S^{(i,j)}$, from (2.13) we have

$$S_t^{(i,j)} = s_t^{\mathrm{T}}K^j u^{(i)} + js^{\mathrm{T}}K^{j-1}K_t u^{(i)} + s^{\mathrm{T}}K^j u_t^{(i)}. \tag{2.18}$$

Then, substituting (2.7*b*), (2.9) and (2.16) into (2.18), we have

$$S_t^{(i,j)} = \frac{1}{2}(aS^{(i,j+2)} - S^{(i+2,j)}a + S^{(1,j)}aS^{(i,0)} + S^{(0,j)}aS^{(i,1)}) + \alpha(x(S^{(i,j)}a - aS^{(i,j)}) + 2(jS^{(i,j-1)} + iS^{(i-1,j)})). \tag{2.19a}$$

One can repeatedly use (2.17) and get second-order derivatives of $S^{(i,j)}$ with respect to x, which which reads

$$\begin{aligned} S_{xx}^{(i,j)} = \frac{1}{4}(&S^{(i,j+2)} + S^{(i+2,j)} - 2aS^{(i+1,j+1)}a \\ &+2aS^{(0,j+1)}aS^{(i,0)} - 2S^{(0,j)}aS^{(i+1,0)}a \\ &-S^{(1,j)}S^{(i,0)} + S^{(0,j)}S^{(i,1)} \\ &+2S^{(0,j)}aS^{(0,0)}aS^{(i,0)}). \end{aligned} \tag{2.19b}$$

Next, let us define

$$U \doteq S^{(0,0)}, \quad u_i = s_i^{(0,0)}, \quad (i = 1, 2, 3, 4), \tag{2.20}$$

one has the following by taking $i = j = 0$ in (2.17) and (2.19):

$$\begin{aligned} U_x &= \frac{1}{2}(S^{(1,0)}a - aS^{(0,1)} - UaU), \\ U_{xx} &= \frac{1}{4}(S^{(2,0)} + S^{(0,2)} - 2aS^{(1,1)}a + 2aS^{(0,1)}aU - 2UaS^{(1,0)}a \\ &\quad -S^{(1,0)}U + US^{(0,1)} + 2UaUaU), \\ U_t &= \frac{1}{2}(aS^{(0,2)} - S^{(2,0)}a \\ &\quad +S^{(1,0)}aU + UaS^{(0,1)}) + \alpha x[U, a], \end{aligned}$$

where $[A, B] = AB - BA$. Besides, by the difference-product formula (2.6), we obtain the following relations by choosing $(i, j) = (0, 1), (1, 0), (0, 0)$, respectively:

$$\begin{aligned} S^{(0,2)} &= S^{(1,1)} - S^{(0,1)}U, \\ ,S^{(2,0)} &= S^{(1,1)} + US^{(1,0)}, \\ S^{(0,1)} &= S^{(1,0)} - U^2. \end{aligned}$$

Then by direct calculation we find

$$aU_t - U_{xx} = -\frac{1}{2}[U, a]([U, a]U - [S^{(1,0)}, a]) + \alpha x a[U, a]. \tag{2.21}$$

Unfolding (2.21) we obtain a closed system of u_2 and u_3 as the unreduced form of the NNLSE-I:

$$u_{2,t} - u_{2,xx} - 2u_2^2 u_3 + 2\alpha x u_2 = 0, \tag{2.22a}$$

$$u_{3,t} + u_{3,xx} + 2u_2 u_3^2 - 2\alpha x u_3 = 0. \tag{2.22b}$$

2.3. Unreduced NNLSE-II

To obtain an unreduced form of the NNLSE-II, we introduce the following dispersion relations:

$$r_x = \frac{1}{2}Kra, \quad s_x = -\frac{1}{2}K^{\mathrm{T}}sa, \tag{2.23a}$$

$$\begin{aligned} r_t &= \frac{1}{2}(-K^2 + \mathrm{i}\beta xK)ra + \frac{1}{2}\mathrm{i}\beta r, \\ s_t &= -\frac{1}{2}(-(K^{\mathrm{T}})^2 + \mathrm{i}\beta xK^{\mathrm{T}})sa + \frac{1}{2}\mathrm{i}\beta s, \end{aligned} \tag{2.23b}$$

where a is defined as (2.8) and

$$K_t = \mathrm{i}\beta K(t), \quad \beta \in \mathbb{R}. \tag{2.24}$$

Again, we assume K_t and K commute. The evolutions of M and $S^{(i,j)}$ with respect to x are the same as (2.11) and (2.17), so we only consider the t-derivatives. Using (2.23*b*) and (2.24) and employing a similar procedure as we derived (2.10), we have

$$M_t = \frac{1}{2}(\mathrm{i}\beta xras^{\mathrm{T}} - Kras^{\mathrm{T}} - ras^{\mathrm{T}}K). \tag{2.25}$$

Note that (2.15) and (2.18) are generic, and can use them for this case as well. We substitute (2.25), (2.23*b*) and (2.24) into (2.15), similar to the treatment in section 2.2, we have

$$\boldsymbol{u}_t^{(i)} = \frac{1}{2}(2\mathrm{i}\beta \boldsymbol{i}\boldsymbol{u}^{(i)} - \boldsymbol{u}^{(i+2)}a + \mathrm{i}\beta \boldsymbol{u}^{(i)} + \boldsymbol{u}^{(1)}a\boldsymbol{S}^{(i,0)} + \boldsymbol{u}^{(0)}a\boldsymbol{S}^{(i,1)}) + \frac{1}{2}\mathrm{i}\beta x(\boldsymbol{u}^{(i+1)}a - \boldsymbol{u}^{(0)}a\boldsymbol{S}^{(i,0)}). \tag{2.26}$$

Then, substituting (2.23*b*), (2.24) and (2.26) into (2.18), we get the derivative of $\boldsymbol{S}^{(i,j)}$ with respect to t:

$$\boldsymbol{S}_t^{(i,j)} = \frac{1}{2}(a\boldsymbol{S}^{(i,j+2)} - \boldsymbol{S}^{(i+2,j)}a + 2\mathrm{i}\beta(1+i+j)\boldsymbol{S}^{(i,j)} + \boldsymbol{S}^{(1,j)}a\boldsymbol{S}^{(i,0)} + \boldsymbol{S}^{(0,j)}a\boldsymbol{S}^{(i,1)}) + \frac{1}{2}\mathrm{i}\beta x(\boldsymbol{S}^{(i+1,j)}a - a\boldsymbol{S}^{(i,j+1)} - \boldsymbol{S}^{(0,j)}a\boldsymbol{S}^{(i,0)}). \tag{2.27}$$

Thus we have

$$\boldsymbol{U}_t = \frac{1}{2}(\boldsymbol{a}\boldsymbol{S}^{(0,2)} - \boldsymbol{S}^{(2,0)}\boldsymbol{a} + 2\mathrm{i}\beta\boldsymbol{U} + \boldsymbol{S}^{(1,0)}\boldsymbol{a}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{a}\boldsymbol{S}^{(0,1)} + \mathrm{i}\beta x(\boldsymbol{S}^{(1,0)}\boldsymbol{a} - \boldsymbol{a}\boldsymbol{S}^{(0,1)} - \boldsymbol{U}\boldsymbol{a}\boldsymbol{U})).$$

Through a direct calculation, we find

$$\boldsymbol{a}\boldsymbol{U}_t - \boldsymbol{U}_{xx} - \mathrm{i}\beta\boldsymbol{a}(x\boldsymbol{U})_x = -\frac{1}{2}[\boldsymbol{U}, \boldsymbol{a}]([\boldsymbol{U}, \boldsymbol{a}]\boldsymbol{U} - [\boldsymbol{S}^{(1,0)}, \boldsymbol{a}]). \tag{2.28}$$

which reveals an equation set of u_2 and u_3 as the unreduced NNLSE-II:

$$u_{2,t} - u_{2,xx} - 2u_3u_2^2 - \mathrm{i}\beta(xu_2)_x = 0, \tag{2.29a}$$

$$u_{3,t} + u_{3,xx} + 2u_2u_3^2 - \mathrm{i}\beta(xu_3)_x = 0. \tag{2.29b}$$

2.4. Unreduced NNLSE-III

In this case, the dispersion relations are:

$$\boldsymbol{r}_x = \frac{1}{2}\boldsymbol{K}\boldsymbol{r}\boldsymbol{a}, \quad \boldsymbol{s}_x = -\frac{1}{2}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{s}\boldsymbol{a}, \tag{2.30a}$$

$$\boldsymbol{r}_t = -\frac{1}{2}\boldsymbol{K}^2\boldsymbol{r}\boldsymbol{a}x - \boldsymbol{K}\boldsymbol{r}, \quad \boldsymbol{s}_t = \frac{1}{2}(\boldsymbol{K}^{\mathrm{T}})^2\boldsymbol{s}\boldsymbol{a}x - \boldsymbol{K}^{\mathrm{T}}\boldsymbol{s}, \tag{2.30b}$$

where

$$\boldsymbol{K}_t = -\boldsymbol{K}^2 \tag{2.31}$$

and we assume $\boldsymbol{K}\boldsymbol{K}_t = \boldsymbol{K}_t\boldsymbol{K}$. The evolution relations of $\boldsymbol{M}$, $\boldsymbol{u}_t^{(i)}$ and $\boldsymbol{S}^{(i,j)}$ with respect to t are presented as below:

$$\boldsymbol{M}_t = -\frac{1}{2}x(\boldsymbol{K}\boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}} + \boldsymbol{r}\boldsymbol{a}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}), \tag{2.32a}$$

$$\boldsymbol{u}_t^{(i)} = \frac{1}{2}x(\boldsymbol{u}^{(1)}a\boldsymbol{S}^{(i,0)} + \boldsymbol{u}^{(0)}a\boldsymbol{S}^{(i,1)} - \boldsymbol{u}^{(i+2)}a) - (i+1)\boldsymbol{u}^{(i+1)}, \tag{2.32b}$$

$$\boldsymbol{S}_t^{(i,j)} = \frac{1}{2}x(\boldsymbol{a}\boldsymbol{S}^{(i,j+2)} - \boldsymbol{S}^{(i+2,j)}\boldsymbol{a} + \boldsymbol{S}^{(1,j)}\boldsymbol{a}\boldsymbol{S}^{(i,0)} + \boldsymbol{S}^{(0,j)}\boldsymbol{a}\boldsymbol{S}^{(i,1)}) - (j+1)\boldsymbol{S}^{(i,j+1)} - (i+1)\boldsymbol{S}^{(i+1,j)}. \tag{2.32c}$$

Let $i = j = 0$ in (2.32*c*) we find

$$\boldsymbol{U}_t = \frac{1}{2}x(\boldsymbol{a}\boldsymbol{S}^{(0,2)} - \boldsymbol{S}^{(2,0)}\boldsymbol{a} + \boldsymbol{S}^{(1,0)}\boldsymbol{a}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{a}\boldsymbol{S}^{(0,1)}) - \boldsymbol{S}^{(0,1)} - \boldsymbol{S}^{(1,0)}.$$

By a direct calculation, we have

$$\boldsymbol{a}\boldsymbol{U}_t - x\boldsymbol{U}_{xx} - 2\boldsymbol{U}_x = -\frac{1}{2}x[\boldsymbol{U}, \boldsymbol{a}]([\boldsymbol{U}, \boldsymbol{a}]\boldsymbol{U} - [\boldsymbol{S}^{(1,0)}, \boldsymbol{a}]) - \boldsymbol{a}\boldsymbol{S}^{(1,0)} - \boldsymbol{S}^{(1,0)}\boldsymbol{a} + \boldsymbol{U}\boldsymbol{a}\boldsymbol{U}, \tag{2.33}$$

which leads to

$$u_{2,t} - x(u_{2,xx} + 2u_3u_2^2) - 2u_{2,x} - 2u_2\partial^{-1}(u_2u_3) = 0, \tag{2.34a}$$

$$u_{3,t} + x(u_{3,xx} + 2u_2u_3^2) + 2u_{3,x} + 2u_3\partial^{-1}(u_2u_3) = 0, \tag{2.34b}$$

where the relation

$$u_4 - u_1 = -2\partial^{-1}(u_2u_3) \tag{2.35}$$

has been utilized, which has been proved in [23].

3. Explicit solutions for the unreduced NNLSEs

We have derived the unreduced NNLSEs. Their solutions are given by $\boldsymbol{S}^{(0,0)}$ which is determined by $\boldsymbol{K}$, $\boldsymbol{M}$, $\boldsymbol{r}$ and $\boldsymbol{s}$ through the formula (2.4). In isospectral case (see [23]) $\boldsymbol{K}$ is a constant matrix and one can equivalently consider its canonical form, i.e. diagonal or Jordan forms or their combinations, and the resulted solutions can be classified by the canonical forms of $\boldsymbol{K}$. However, it is much different in non-isospectral case as $\boldsymbol{K}$ is no longer a constant matrix and must obey evolutions such as (2.9), (2.24) and (2.31). That means, in principle, we can not classify solutions by considering the canonical forms of $\boldsymbol{K}$.

In this section, for convenience, we only consider the case of $\boldsymbol{K}$ being diagonal. There will be a case of $\boldsymbol{K}$ composed by lower Toeplitz matrices to be presented in appendix A.

In the following, let us take the unreduced NNLSE-I (2.22) as an example. Consider $\boldsymbol{K} = \mathrm{diag}(\boldsymbol{K}_1, \boldsymbol{K}_2)$ with $\boldsymbol{K}_1$, $\boldsymbol{K}_2$ being the following diagonal forms

$$\boldsymbol{K}_1 = \mathrm{diag}(k_1, k_2, \cdots, k_N), \quad \boldsymbol{K}_2 = \mathrm{diag}(l_1, l_2, \cdots, l_N), \tag{3.1}$$

where

$$k_j(t) = 2\alpha t + c_j, \quad l_j(t) = 2\alpha t + d_j, \quad c_j, d_j \in \mathbb{C}, \tag{3.2}$$

such that $\boldsymbol{K}$ satisfies (2.9). The dispersion relation (2.7) yields

$$\boldsymbol{r}_{1,x} = \frac{1}{2}\boldsymbol{K}_1\boldsymbol{r}_1, \quad \boldsymbol{r}_{2,x} = -\frac{1}{2}\boldsymbol{K}_2\boldsymbol{r}_2, \quad \boldsymbol{r}_{1,t} = \left(-\frac{1}{2}\boldsymbol{K}_1^2 + \alpha x\right)\boldsymbol{r}_1, \quad \boldsymbol{r}_{2,t} = \left(\frac{1}{2}\boldsymbol{K}_2^2 - \alpha x\right)\boldsymbol{r}_2,$$
$$\boldsymbol{s}_{1,x} = \frac{1}{2}\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{s}_1, \quad \boldsymbol{s}_{2,x} = -\frac{1}{2}\boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{s}_2, \quad \boldsymbol{s}_{1,t} = \left(-\frac{1}{2}(\boldsymbol{K}_1^{\mathrm{T}})^2 + \alpha x\right)\boldsymbol{s}_1, \quad \boldsymbol{s}_{2,t} = \left(\frac{1}{2}(\boldsymbol{K}_2^{\mathrm{T}})^2 - \alpha x\right)\boldsymbol{s}_2. \tag{3.3a}$$

(3.3*a*) has the following solutions:

$$\boldsymbol{r}_1 = (\rho_1,\, \rho_2,\cdots,\rho_N)^{\mathrm{T}}, \qquad \boldsymbol{s}_1 = (\sigma_1,\, \sigma_2,\cdots,\sigma_N)^{\mathrm{T}}, \tag{3.4a}$$

$$\boldsymbol{r}_2 = (\varrho_1,\, \varrho_2,\cdots,\varrho_N)^{\mathrm{T}}, \qquad \boldsymbol{s}_2 = (\varpi_1,\, \varpi_2,\cdots,\varpi_N)^{\mathrm{T}}, \tag{3.4b}$$

where

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{k_j(t)}{2}x - \frac{k_j^3(t)}{12\alpha} + \xi_j^{(0)}, \tag{3.5a}$$

$$\varrho_j = \mathrm{e}^{\eta_j}, \quad \eta_j = -\frac{l_j(t)}{2}x + \frac{l_j^3(t)}{12\alpha} + \eta_j^{(0)}, \tag{3.5b}$$

$$\sigma_j = \mathrm{e}^{\zeta_j}, \quad \zeta_j = \frac{k_j(t)}{2}x - \frac{k_j^3(t)}{12\alpha} + \zeta_j^{(0)}, \tag{3.5c}$$

$$\varpi_j = \mathrm{e}^{\varsigma_j}, \quad \varsigma_j = -\frac{l_j(t)}{2}x + \frac{l_j^3(t)}{12\alpha} + \varsigma_j^{(0)}, \tag{3.5d}$$

and $\xi_j^{(0)}$, $\eta_j^{(0)}$, $\zeta_j^{(0)}$, $\varsigma_j^{(0)}$ are constants. Then, the set of Sylvester equations (2.3) allow solutions $\boldsymbol{M}_1 = ((\boldsymbol{M}_1)_{i,j})_{N\times N}$ and $\boldsymbol{M}_2 = ((\boldsymbol{M}_2)_{i,j})_{N\times N}$ where

$$(\boldsymbol{M}_1)_{i,j} = \frac{\rho_i \varpi_j}{k_i - l_j}, \qquad (\boldsymbol{M}_2)_{i,j} = \frac{\varrho_i \sigma_j}{l_i - k_j}. \tag{3.6}$$

Finally, we reach to the explicit expressions of u_2 and u_3:

$$u_2 = \boldsymbol{s}_2^{\mathrm{T}}(\boldsymbol{I}_N - \boldsymbol{M}_2\boldsymbol{M}_1)^{-1}\boldsymbol{r}_2, \tag{3.7a}$$

$$u_3 = \boldsymbol{s}_1^{\mathrm{T}}(\boldsymbol{I}_N - \boldsymbol{M}_1\boldsymbol{M}_2)^{-1}\boldsymbol{r}_1. \tag{3.7b}$$

which satisfy the unreduced NNLSE-I (2.22).

For the unreduced NNLSE-II (2.29) and the unreduced NNLSE-III (2.34), there solutions can be expressed through the formulae (3.7) with (3.4) and (3.6) but where ρ_j, ϱ_j, σ_j, ϖ_j and k_j, l_j are defined differently. For the unreduced NNLSE-II (2.29), we have

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{k_j(t)}{2}x + \frac{\mathrm{i}k_j^2(t)}{4\beta} + \frac{1}{2}\mathrm{i}\beta t + \xi_j^{(0)}, \tag{3.8a}$$

$$\varrho_j = \mathrm{e}^{\eta_j}, \quad \eta_j = -\frac{l_j(t)}{2}x - \frac{\mathrm{i}l_j^2(t)}{4\beta} + \frac{1}{2}\mathrm{i}\beta t + \eta_j^{(0)}, \tag{3.8b}$$

$$\sigma_j = \mathrm{e}^{\zeta_j}, \quad \zeta_j = \frac{k_j(t)}{2}x + \frac{\mathrm{i}k_j^2(t)}{4\beta} + \frac{1}{2}\mathrm{i}\beta t + \zeta_j^{(0)}, \tag{3.8c}$$

$$\varpi_j = \mathrm{e}^{\varsigma_j}, \quad \varsigma_j = -\frac{l_j(t)}{2}x - \frac{\mathrm{i}l_j^2(t)}{4\beta} + \frac{1}{2}\mathrm{i}\beta t + \varsigma_j^{(0)}, \tag{3.8d}$$

where

$$k_j(t) = c_j \mathrm{e}^{\mathrm{i}\beta t}, \quad l_j(t) = d_j \mathrm{e}^{\mathrm{i}\beta t}. \tag{3.9}$$

For the unreduced NNLSE-III (2.34), we have

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{k_j(t)}{2}x + \ln(k_j(t)) + \xi_j^{(0)}, \tag{3.10a}$$

$$\varrho_j = \mathrm{e}^{\eta_j}, \quad \eta_j = -\frac{l_j(t)}{2}x + \ln(l_j(t)) + \eta_j^{(0)}, \tag{3.10b}$$

$$\sigma_j = \mathrm{e}^{\zeta_j}, \quad \zeta_j = \frac{k_j(t)}{2}x + \ln(k_j(t)) + \zeta_j^{(0)}, \tag{3.10c}$$

$$\varpi_j = \mathrm{e}^{\varsigma_j}, \quad \varsigma_j = -\frac{l_j(t)}{2}x + \ln(l_j(t)) + \varsigma_j^{(0)}, \tag{3.10d}$$

where

$$k_j(t) = \frac{1}{t - c_j}, \quad l_j(t) = \frac{1}{t - d_j}. \tag{3.11}$$

4. Reduction to three NNLSEs and their solutions

4.1. General case

The reduction from the unreduced NNLSEs (i.e. (2.22), (2.29), (2.34)) to the three NNLSEs in (1.2) is

$$u_2 = u_3^*, \tag{4.1}$$

together with replacing $t \to \mathrm{i}t$. To achieve the above relation, we introduce constraint on $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ such that

$$\boldsymbol{K}_2 = -\boldsymbol{K}_1^*. \tag{4.2}$$

Then, from the dispersion relations (2.7), (2.23) and (2.30), one can always get[4]

$$\boldsymbol{r}_2 = \boldsymbol{r}_1^*, \quad \boldsymbol{s}_2 = \boldsymbol{s}_1^*. \tag{4.3}$$

Next, the original Sylvester equations (2.3) yield

$$\boldsymbol{K}_1\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{K}_1^* = \boldsymbol{r}_1\boldsymbol{s}_1^\dagger, \quad -\boldsymbol{K}_1^*\boldsymbol{M}_2 - \boldsymbol{M}_2\boldsymbol{K}_1 = \boldsymbol{r}_1^*\boldsymbol{s}_1^{\mathrm{T}},$$

and hence we have

$$\boldsymbol{M}_2 = -\boldsymbol{M}_1^* \tag{4.4}$$

thanks to the uniqueness of the solutions of the Sylvester equation. Here $\boldsymbol{s}_1^\dagger = (\boldsymbol{s}_1^{\mathrm{T}})^*$. It then follows that

$$u_2 = \boldsymbol{s}_2^{\mathrm{T}}(\boldsymbol{I}_N - \boldsymbol{M}_2\boldsymbol{M}_1)^{-1}\boldsymbol{r}_2 = \boldsymbol{s}_1^\dagger(\boldsymbol{I}_N - \boldsymbol{M}_1^*\boldsymbol{M}_2^*)\boldsymbol{r}_1^* = u_3^*.$$

In conclusion, for the three NNLSEs in (1.2), their solutions can be expressed in the form

$$q_j = u_3 = \boldsymbol{s}_1^{\mathrm{T}}(\boldsymbol{I}_N + \boldsymbol{M}_1\boldsymbol{M}_1^*)^{-1}\boldsymbol{r}_1, \quad (j = 1, 2, 3), \tag{4.5}$$

with $\boldsymbol{r}_1$, $\boldsymbol{s}_1$ and $\boldsymbol{M}_1$ accordingly.

In the following, for the three NNLSEs, we will look at their explicit solutions and illustrate their dynamics.

4.2. Explicit solutions of the NNLSE-I and dynamics

4.2.1. Formulation of solitons. With the above results, we rewrite the Sylvester equation and dispersion relations as

$$\boldsymbol{K}_1\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{K}_1^* = \boldsymbol{r}_1\boldsymbol{s}_1^\dagger, \tag{4.6a}$$

$$\boldsymbol{r}_{1,x} = \frac{1}{2}\boldsymbol{K}_1\boldsymbol{r}_1, \qquad \boldsymbol{s}_{1,x} = \frac{1}{2}\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{s}_1, \tag{4.6b}$$

$$\boldsymbol{r}_{1,t} = \mathrm{i}\left(-\frac{1}{2}\boldsymbol{K}_1^2 + \alpha x\right)\boldsymbol{r}_1, \qquad \boldsymbol{s}_{1,t} = \mathrm{i}\left(-\frac{1}{2}(\boldsymbol{K}_1^{\mathrm{T}})^2 + \alpha x\right)\boldsymbol{s}_1, \tag{4.6c}$$

where

$$\boldsymbol{K}_{1,t} = 2\mathrm{i}\ \alpha\boldsymbol{I}_N, \quad \alpha \in \mathbb{R}. \tag{4.7}$$

[4] In the diagonal case one should suitably take constants $\xi_j^{(0)}$, $\eta_j^{(0)}$, $\zeta_j^{(0)}$, $\varsigma_j^{(0)}$ such that $\eta_j = \xi_j^*$, $\varsigma_j = \zeta_j^*$.

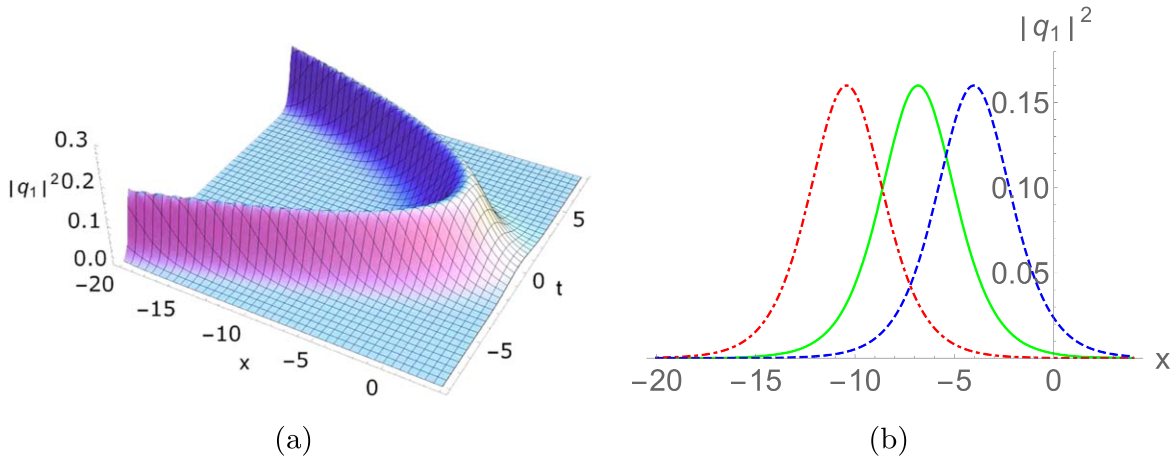


Figure 1. The shape and motion of the envelope of one-soliton solution given by (4.13) for $c_1 = 0.4$, $\alpha = 0.2$, $\xi_1^{(0)} = \zeta_1^{(0)} = 0$. (a) 3D plot. (b) 2D plot of (a) at $t = 5$ (red dot-dashed curve), $t = -3$ (blue dashed curve) and $t = -4$ (green solid curve).

Then, solutions of the NNLSE-1 (1.2*a*) are given by (4.5) where $\boldsymbol{M}_1, \boldsymbol{r}_1, \boldsymbol{s}_1$ satisfy the above settings. In particular, when

$$\boldsymbol{K}_1 = \mathrm{diag}(k_1, k_2, \cdots, k_N) \tag{4.8}$$

we have

$$\boldsymbol{r}_1 = (\rho_1, \rho_2, \cdots, \rho_N)^{\mathrm{T}}, \quad \boldsymbol{s}_1 = (\sigma_1, \sigma_2, \cdots, \sigma_N)^{\mathrm{T}}, \tag{4.9a}$$

$$(\boldsymbol{M}_1)_{i,j} = \frac{\rho_i \sigma_j^*}{k_i + k_j^*}, \tag{4.9b}$$

where

$$k_j(t) = 2\mathrm{i}\alpha t + c_j, \tag{4.10a}$$

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{k_j(t)}{2}x - \frac{k_j^3(t)}{12\alpha} + \xi_j^{(0)}, \tag{4.10b}$$

$$\sigma_j = \mathrm{e}^{\zeta_j}, \quad \zeta_j = \frac{k_j(t)}{2}x - \frac{k_j^3(t)}{12\alpha} + \zeta_j^{(0)}. \tag{4.10c}$$

Note that $c_j, \xi_j^{(0)}, \zeta_j^{(0)} \in \mathbb{C}$, and throughout this section we write

$$c_j = a_j + \mathrm{i}b_j, \quad \xi_j^{(0)} = d_j + \mathrm{i}e_j,$$
$$\zeta_j^{(0)} = f_j + \mathrm{i}h_j,$$
$$a_j, b_j, d_j, e_j, f_j, h_j \in \mathbb{R}.$$

4.2.2. One-soliton solution. For $N = 1$ case, we have the following:

$$\boldsymbol{K}_1 = k_1, \quad \boldsymbol{r}_1 = \rho_1 = \mathrm{e}^{\xi_1}, \quad \boldsymbol{s}_1 = \sigma_1 = \mathrm{e}^{\zeta_1}, \quad \boldsymbol{M}_1 = \frac{\rho_1\sigma_1^*}{k_1 + k_1^*}, \tag{4.11}$$

where ρ_1, σ_1 are defined as in (4.10). Substitute them into (4.5) one obtains

$$q_1 = \frac{4a_1^2\,\mathrm{e}^{\zeta_1+\xi_1}}{4a_1^2 + \mathrm{e}^{\mathrm{Re}[\zeta_1+\xi_1]}}, \tag{4.12}$$

where for complex number $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, $\mathrm{Re}[z] = x$. Then the square module of one-soliton solution, namely, the envelope of the soliton, becomes:

$$|q_1|^2 = a_1^2\,\mathrm{sech}^2 \times \left(d_1 + f_1 + a_1 x - \frac{a_1^3 - 3a_1(b_1 + 2t\alpha)^2}{6\alpha} - \ln(2a_1)\right). \tag{4.13}$$

For the shape and motion of $|q_1|^2$ given by (4.13), we illustrate them in figure 1.

The soliton $|q_1|^2$ travels with a fixed amplitude $A = a_1^2$. The top trace of $|q_1|^2$ is a parabola, which can be derived as

$$x(t) = \frac{a_1^3 - 3a_1(b_1 + 2\alpha t)^2 - 6\alpha(d_1 + f_1) + 6\alpha\ln(2a_1)}{6\alpha a_1}, \tag{4.14}$$

of which the vertex point is

$$(x, t) = \left(\frac{a_1^3 - 6\alpha(d_1 + f_1) + 6\alpha\ln(2a_1))}{6\alpha a_1}, -\frac{b_1}{2\alpha}\right),$$

and the velocity of the soliton is

$$x'(t) = -2(2\alpha t + b_1).$$

4.2.3. Two-soliton solution. For $N = 2$ case, we have

$$\boldsymbol{K}_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad \boldsymbol{r}_1 = (\rho_1, \rho_2)^{\mathrm{T}}, \quad \boldsymbol{s}_1 = (\sigma_1, \sigma_2)^{\mathrm{T}}, \tag{4.15}$$

and the dressed Cauchy matrix comes to be

$$\boldsymbol{M}_1 = \begin{pmatrix} \frac{\rho_1\sigma_1^*}{k_1 + k_1^*} & \frac{\rho_1\sigma_2^*}{k_1 + k_2^*} \\ \frac{\rho_2\sigma_1^*}{k_2 + k_1^*} & \frac{\rho_2\sigma_2^*}{k_2 + k_2^*} \end{pmatrix}.$$

Then the explicit formula of the two-soliton solution (4.5)

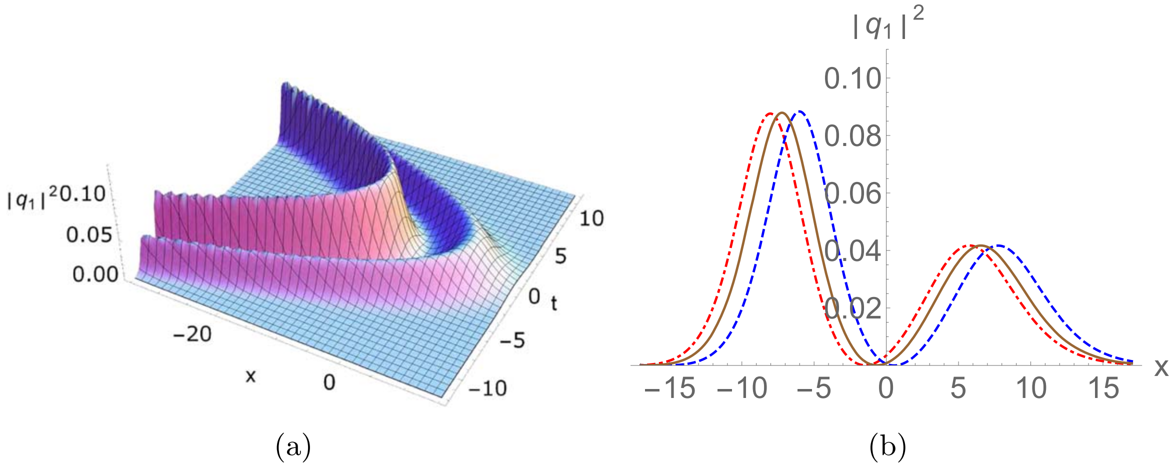


Figure 2. The shape and motion of the two-soliton solution given by $|q_1|^2$ with (4.16) for $c_1 = 0.2 + 0.2\mathrm{i}$, $c_2 = 0.2$, $\alpha = 0.2$, $\xi_1^{(0)} = \xi_2^{(0)} = \zeta_1^{(0)} = \zeta_2^{(0)} = 0$. (a) 3D plot. (b) 2D plot of (a) at $t = 2$ (red dot-dashed curve), $t = 0$ (blue dashed curve) and $t = -2$ (brown solid curve).

becomes:

$$q_1 = \frac{(1+M_{11})\mathrm{e}^{\zeta_2+\xi_2} - M_{12}\mathrm{e}^{\zeta_1+\xi_2} - M_{21}\mathrm{e}^{\zeta_2+\xi_1} + (1+M_{22})\mathrm{e}^{\zeta_1+\xi_1}}{1 + M_{11} - M_{12}M_{21} + M_{22}(1+M_{11})}, \tag{4.16}$$

where

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{e}^{2\,\mathrm{Re}[\xi_1+\zeta_1]}}{4a_1^2} + \frac{\mathrm{e}^{\xi_1+\zeta_1+\xi_2^*+\zeta_2^*}}{B^2} & \frac{1}{2}\mathrm{e}^{\zeta_2+\xi_1}\left(\frac{\mathrm{e}^{\xi_1^*+\zeta_1^*}}{a_1 B^*} + \frac{\mathrm{e}^{\xi_2^*+\zeta_2^*}}{a_2 B}\right) \\ \frac{1}{2}\mathrm{e}^{\zeta_1+\xi_2}\left(\frac{\mathrm{e}^{\xi_1^*+\zeta_1^*}}{a_1 B^*} + \frac{\mathrm{e}^{\xi_2^*+\zeta_2^*}}{a_2 B}\right) & \frac{\mathrm{e}^{2\,\mathrm{Re}[\xi_2+\zeta_2]}}{4a_2^2} + \frac{\mathrm{e}^{\xi_2+\zeta_2+\xi_1^*+\zeta_1^*}}{(B^*)^2} \end{pmatrix},$$

$$B = c_1 + c_2^* = a_1 + a_2 + \mathrm{i}(b_1 - b_2).$$

For the shape and motion of $|q_1|^2$, we illustrate them in figure 2.

4.2.4. Double-pole solution. The matrix $\boldsymbol{K}_1$ can also be a triangle Toeplitz matrix (see appendix A), which will lead to the so-called multiple-pole solutions. We will present solution formula in appendix A for the Sylvester equations in (2.3) where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are triangular Toeplitz matrices. For the double-pole solution, it can be obtained by setting

$$\boldsymbol{K}_1 = \begin{pmatrix} k_1 & 0 \\ \partial_{c_1}k_1 & k_1 \end{pmatrix}, \quad \boldsymbol{r}_1 = (\rho_1, \partial_{c_1}\rho_1)^{\mathrm{T}}, \quad \boldsymbol{s}_1 = (\sigma_1, \partial_{c_1}\sigma_1)^{\mathrm{T}}, \tag{4.17}$$

where k_1, ρ_1, σ_1 are defined as in (4.10). Then the dressed Cauchy matrix can be constructed via appendix A as

$$\boldsymbol{M}_1 = \begin{pmatrix} \rho_1 & 0 \\ \partial_{c_1}\rho_1 & \rho_1 \end{pmatrix} \begin{pmatrix} \frac{1}{k_1 + k_1^*} & -\frac{(\partial_{c_1}k_1)^*}{(k_1+k_1^*)^2} \\ -\frac{\partial_{c_1}k_1}{(k_1+k_1^*)^2} & \frac{2(\partial_{c_1}k_1)(\partial_{c_1}k_1)^*}{(k_1+k_1^*)^3} \end{pmatrix} \begin{pmatrix} \sigma_1^* & (\partial_{c_1}\sigma_1)^* \\ (\partial_{c_1}\sigma_1)^* & 0 \end{pmatrix}. \tag{4.18}$$

The explicit formula of the double-pole solution (4.5) reads

$$q_1 = \frac{16a_1^3\mathrm{e}^{\xi_1+\zeta_1}(16a_1^5(1+D^2) - (a_1 - 2D^* + a_1(D^2)^*)D^2\mathrm{e}^{2\,\mathrm{Re}[\xi_1+\zeta_1]})}{256a_1^8 + 32A_1a_1^4\mathrm{e}^{2\,\mathrm{Re}[\xi_1+\zeta_1]} + |D|^4\mathrm{e}^{4\,\mathrm{Re}[\xi_1+\zeta_1]}}, \tag{4.19}$$

where

$$A_1 = (3|D|^2 - 4a_1\,\mathrm{Re[D]}(1 + |D|^2) + 2a_1^2(1 + D^2)(1 + (D^2)^*)),$$
$$D = \frac{x}{2} - \frac{k_1^2}{4\alpha}.$$

The shape and motion of $|q_1|^2$ are illustrated in figure 3.

4.3. Explicit solutions of the NNLSE-II and dynamics

4.3.1. Formulation of solitons. Solution q_2 of the NNLSE-II equation (1.2*b*) can be expressed through (4.5) where $\boldsymbol{M}_1$, $\boldsymbol{r}_1$, $\boldsymbol{s}_1$ are determined by

$$\boldsymbol{K}_1\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{K}_1^* = \boldsymbol{r}_1\boldsymbol{s}_1^{\dagger}, \tag{4.20a}$$

$$\boldsymbol{r}_{1,x} = \frac{1}{2}\boldsymbol{K}_1\boldsymbol{r}_1, \quad \boldsymbol{s}_{1,x} = \frac{1}{2}\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{s}_1, \tag{4.20b}$$

$$\boldsymbol{r}_{1,t} = -\frac{1}{2}(\mathrm{i}\boldsymbol{K}_1^2 + \beta x\boldsymbol{K}_1 + \beta)\boldsymbol{r}_1,$$
$$\boldsymbol{s}_{1,t} = -\frac{1}{2}(\mathrm{i}(\boldsymbol{K}_1^{\mathrm{T}})^2 + \beta x\boldsymbol{K}_1^{\mathrm{T}} + \beta)\boldsymbol{s}_1, \tag{4.20c}$$

and

$$\boldsymbol{K}_{1,t} = -\beta\boldsymbol{K}_1(t), \quad \beta \in \mathbb{R}. \tag{4.21}$$

In the case of $\boldsymbol{K}_1$ being a diagonal matrix (4.8), $\boldsymbol{r}_1$, $\boldsymbol{s}_1$, $\boldsymbol{M}_1$ are given as (4.9) where

$$k_j(t) = c_j\mathrm{e}^{-\beta t}, \tag{4.21a}$$

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{1}{2}xk_j(t) + \frac{\mathrm{i}k_j^2(t)}{4\beta} - \frac{\beta}{2}t + \xi_j^{(0)}, \tag{4.21b}$$

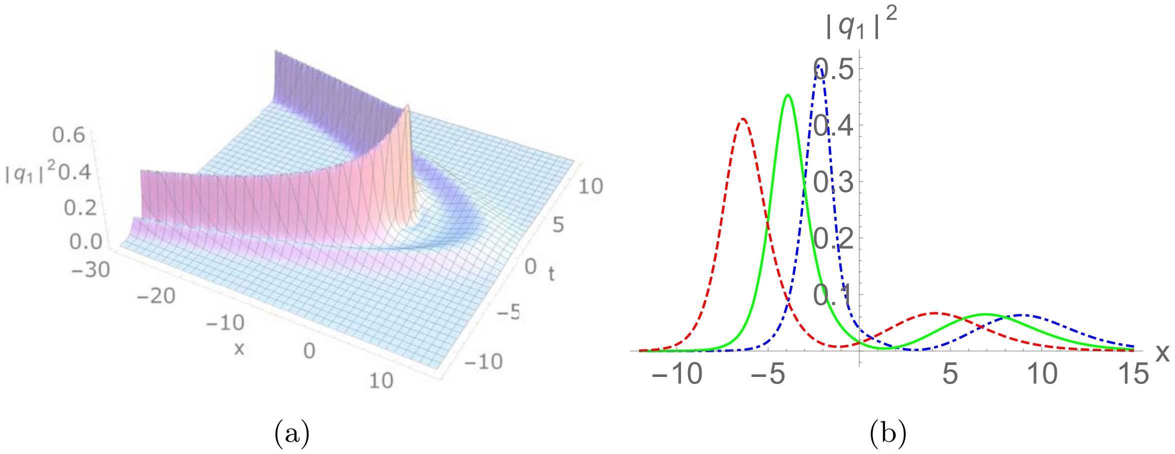


Figure 3. The shape and motion of the double-pole soliton $|q_1|^2$ with (4.19) for $c_1 = -0.4$, $\alpha = 0.2$, $\xi_1^{(0)} = \zeta_1^{(0)} = 0$. (a) 3D plot. (b) 2D plot (a) at $t = 4$ (red dashed curve), $t = 2$ (green solid curve) and $t = -2$ (blue dot-dashed curve).

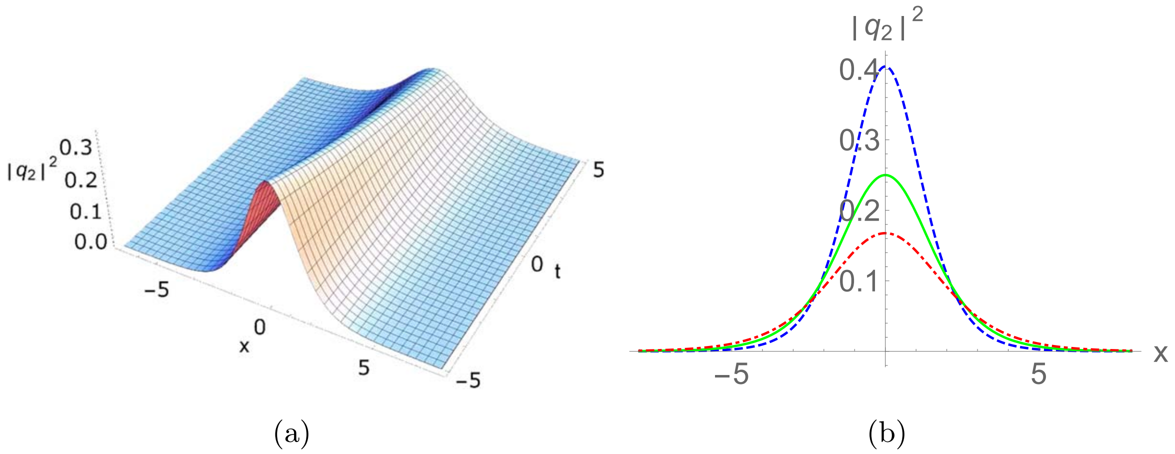


Figure 4. The shape and motion of stationary one-soliton solution given by (4.25) for $c_1 = 0.5$, $\beta = 0.04$, $\xi_1^{(0)} = \zeta_1^{(0)} = 0$. (a) 3D plot. (b) 2D plot of (a) at $t = -6$ (blue dashed curve), $t = 0$ (green solid curve) and $t = 5$ (red dot-dashed curve).

$$\sigma_j = \mathrm{e}^{\zeta_j},\quad \zeta_j = \frac{1}{2}xk_j(t) + \frac{\mathrm{i}k_j^2(t)}{4\beta} - \frac{\beta}{2}t + \zeta_j^{(0)}. \tag{4.21c}$$

4.3.2. One-soliton solution. The one-soliton solution corresponds to the $N = 1$ case, in which we have

$$\boldsymbol{K}_1 = k_1,\quad \boldsymbol{r}_1 = \rho_1 = \mathrm{e}^{\xi_1},\quad \boldsymbol{s}_1 = \sigma_1 = \mathrm{e}^{\zeta_1},\quad \boldsymbol{M}_1 = \frac{\rho_1\sigma_1^*}{k_1 + k_1^*}, \tag{4.23}$$

where k_1, ρ_1, σ_1 are defined as in (4.21). Substituting them into (4.5) one obtains

$$q_2 = \frac{4a_1^2\mathrm{e}^{\xi_1+\zeta_1}}{4a_1^2 + \mathrm{e}^{2(\beta t+\mathrm{Re}[\xi_1+\zeta_1])}}, \tag{4.24}$$

and then the envelope reads

$$|q_2|^2 = a_1^2\mathrm{e}^{-2t\beta}\,\mathrm{sech}^2\left[a_1\mathrm{e}^{-\beta t}\left(x - \frac{b_1}{\beta}\right) - \ln(2a_1) + d_1 + f_1\right]. \tag{4.25}$$

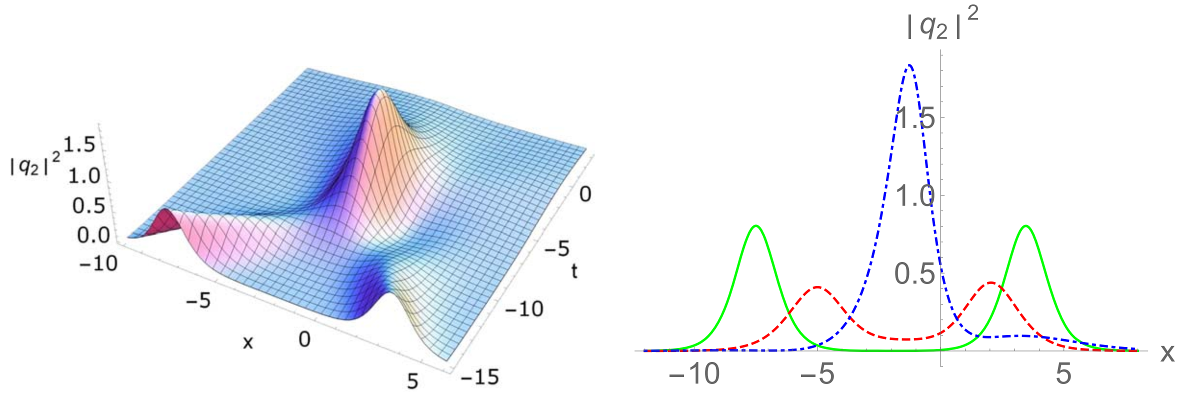
equation (4.25) describes a solitary wave traveling with an amplitude $a_1^2\mathrm{e}^{-2\beta t}$, top trace

$$x(t) = \frac{(\ln(2a_1) - (d_1 + f_1))\mathrm{e}^{\beta t}}{a_1} + \frac{b_1\mathrm{e}^{-\beta t}}{\beta} \tag{4.26}$$

and velocity

$$x'(t) = \frac{\beta(\ln(2a_1) - (d_1 + f_1))\mathrm{e}^{\beta t}}{a_1} - b_1\mathrm{e}^{-\beta t}.$$

When $\frac{(\ln(2a_1) - (d_1+f_1))}{a_1}\frac{b_1}{\beta} > 0$, the top trace has a similar shape to $\mathrm{sgn}[\frac{b_1}{\beta}]\cosh\beta t$; while when $\frac{(\ln(2a_1) - (d_1+f_1))}{a_1}\frac{b_1}{\beta} < 0$, the top trace has a similar shape to $-\mathrm{sgn}[\frac{b_1}{\beta}]\sinh\beta t$. From (4.26) we can also find that when $(\ln(2a_1) - (d_1 + f_1)) = b_1 = 0$, there is $x(t) = 0$, which corresponds to a stationary soliton as



(a) (b)

Figure 5. The shape and motion of the envelope of two-soliton solution (4.28) for $c_1 = -0.2 - 0.2\mathrm{i}$, $c_2 = 0.2 + 0.1\mathrm{i}$, $\beta = 0.1$, $\xi_1^{(0)} = \zeta_1^{(0)} = \xi_2^{(0)} = \zeta_2^{(0)} = 0$. (a) 3D plot. (b) 2D plot of (a) at $t = -15$ (green solid curve), $t = -12$ (red dashed curve) and $t = -6$ (blue dashed curve).

shown in figure 4. Note that the non-isospectral effects affect amplitude, velocity and shape of (4.25).

4.3.3. Two-soliton solution. When $N = 2$ we have

$$\boldsymbol{K}_1 = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \ \boldsymbol{r}_1 = (\rho_1, \rho_2)^{\mathrm{T}},$$
$$\boldsymbol{s}_1 = (\sigma_1, \sigma_2)^{\mathrm{T}}, \quad \boldsymbol{M}_1 = \begin{pmatrix} \dfrac{\rho_1\sigma_1^*}{k_1 + k_1^*} & \dfrac{\rho_1\sigma_2^*}{k_1 + k_2^*} \\ \dfrac{\rho_2\sigma_1^*}{k_2 + k_1^*} & \dfrac{\rho_2\sigma_2^*}{k_2 + k_2^*} \end{pmatrix}, \tag{4.27}$$

where k_j, ρ_j, σ_j are defined as in (4.21). Then, the explicit formula the two-soliton solution (4.5) becomes:

$$q_2 = \frac{(1 + H_{11})\mathrm{e}^{\zeta_2+\xi_2} - H_{12}\mathrm{e}^{\zeta_1+\xi_2} - H_{21}\mathrm{e}^{\zeta_2+\xi_1} + (1 + H_{22})\mathrm{e}^{\zeta_1+\xi_1}}{1 + H_{11} - H_{12}H_{21} + H_{22}(1 + H_{11})}, \tag{4.28}$$

where

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\mathrm{e}^{2t\beta+\zeta_1+\xi_1}\left(\frac{\mathrm{e}^{\zeta_1^*+\xi_1^*}}{a_1^2} + \frac{4\mathrm{e}^{\xi_2^*+\zeta_2^*}}{B^2}\right) & \frac{1}{2}\mathrm{e}^{2t\beta+\zeta_2+\xi_1}\left(\frac{\mathrm{e}^{\zeta_1^*+\xi_1^*}}{a_1B^*} + \frac{\mathrm{e}^{\xi_2^*+\zeta_2^*}}{a_2B}\right) \\ \frac{1}{2}\mathrm{e}^{2t\beta+\zeta_1+\xi_2}\left(\frac{\mathrm{e}^{\zeta_1^*+\xi_1^*}}{a_1B^*} + \frac{\mathrm{e}^{\xi_2^*+\zeta_2^*}}{a_2B}\right) & \frac{1}{4}\mathrm{e}^{2t\beta+\zeta_2+\xi_2}\left(\frac{4\mathrm{e}^{\zeta_1^*+\xi_1^*}}{(B^*)^2} + \frac{\mathrm{e}^{\xi_2^*+\zeta_2^*}}{a_2^2}\right) \end{pmatrix},$$
$$B = c_1 + c_2^* = a_1 + a_2 + \mathrm{i}(b_1 - b_2).$$

The shape and motion of $|q_2|^2$ are illustrated in figure 5.

4.3.4. Double-pole solution. Double-pole solution of the NNLSE-II can be given by the formula (4.5) with the setting (4.17) and $\boldsymbol{M}_1$ takes the form (4.18), where k_1, ρ_1, σ_1 are defined as in (4.21). The explicit formula of such a solution reads

$$q_2 = \frac{16a_1^3\mathrm{e}^{\xi_1+\zeta_1}(16a_1^5(1 + D^2) - D^2(a_1 - 2D^* + a_1(D^2)^*)\mathrm{e}^{2(\beta t+\mathrm{Re}[\xi_1+\zeta_1])})}{256a_1^8 + 32A_1a_1^4\mathrm{e}^{2(\beta t+\mathrm{Re}[\xi_1+\zeta_1])} + |D|^4\mathrm{e}^{4(t\beta+\mathrm{Re}[\xi_1+\zeta_1])}}, \tag{4.29}$$

where

$$A_1 = 3|D|^2 - 4a_1\,\mathrm{Re}[\mathrm{D}](1 + |D|^2) + 2a_1^2(1 + D^2)(1 + (D^*)^2),$$
$$D = \frac{1}{2}x\mathrm{e}^{-\beta t} + \frac{\mathrm{i}c_1\mathrm{e}^{-2\beta t}}{2\beta}.$$

Shape and motion of $|q_2|^2$ are illustrated in figure 6.

4.4. Explicit solution of the NNLSE-III and dynamics

4.4.1. Formulation of solitons. For the NNLSE-III equation (1.2c), its solutions are formulated by (4.5) where $\boldsymbol{M}_1$, $\boldsymbol{r}_1$, $\boldsymbol{s}_1$ are determined by

$$\boldsymbol{K}_1\boldsymbol{M}_1 + \boldsymbol{M}_1\boldsymbol{K}_1^* = \boldsymbol{r}_1\boldsymbol{s}_1^{\dagger}, \tag{4.30a}$$

$$\boldsymbol{r}_{1,x} = \frac{1}{2}\boldsymbol{K}_1\boldsymbol{r}_1, \quad \boldsymbol{s}_{1,x} = \frac{1}{2}\boldsymbol{K}_1^{\mathrm{T}}\boldsymbol{s}_1, \tag{4.30b}$$

$$\boldsymbol{r}_{1,t} = -\mathrm{i}\left(\frac{1}{2}x\boldsymbol{K}_1^2 + \boldsymbol{K}_1\right)\boldsymbol{r}_1, \quad \boldsymbol{s}_{1,t} = -\mathrm{i}\left(\frac{1}{2}x(\boldsymbol{K}_1^{\mathrm{T}})^2 + \boldsymbol{K}_1\right)\boldsymbol{s}_1, \tag{4.30c}$$

and

$$\boldsymbol{K}_{1,t} = -\mathrm{i}\boldsymbol{K}_1^2. \tag{4.31}$$

In the case of $\boldsymbol{K}_1$ being a diagonal matrix (4.8), $\boldsymbol{r}_1$, $\boldsymbol{s}_1$, $\boldsymbol{M}_1$ are given as (4.9) where

$$k_j(t) = \frac{1}{\mathrm{i}t - c_j}, \tag{4.32a}$$

$$\rho_j = \mathrm{e}^{\xi_j}, \quad \xi_j = \frac{1}{2}xk_j(t) + \ln(k_j(t)) + \xi_j^{(0)}, \tag{4.32b}$$

$$\sigma_j = \mathrm{e}^{\zeta_j}, \quad \zeta_j = \frac{1}{2}xk_j(t) + \ln(k_j(t)) + \zeta_j^{(0)}. \tag{4.32c}$$

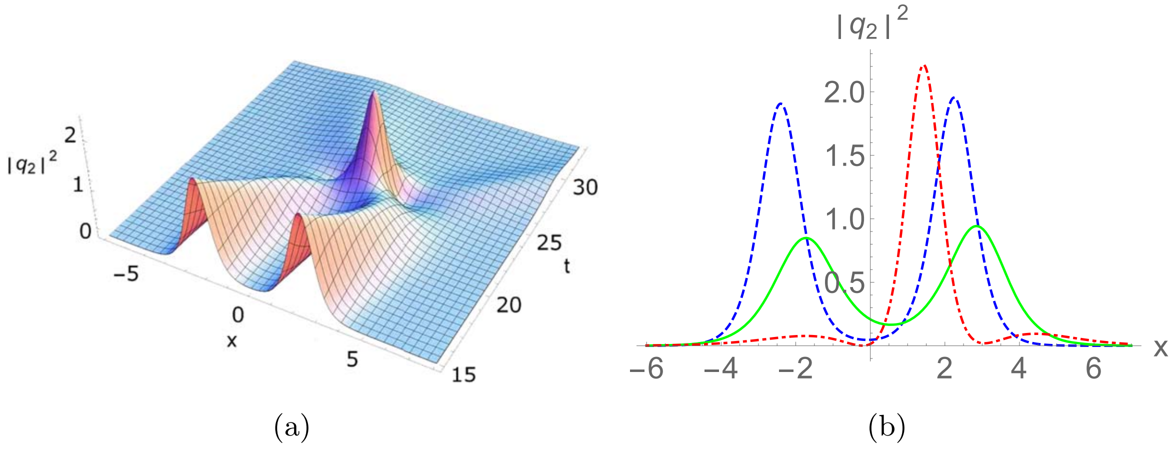


Figure 6. The shape and motion of the envelope of the double-pole soliton (4.29) for $c_1 = 4$, $\beta = 0.07$, $\xi_1^{(0)} = 3$, $\zeta_1^{(0)} = 0$. (a) 3D plot. (b) 2D plot (a) at $t = 24$ (red dot-dashed curve), $t = 20$ (green solid curve) and $t = 15$ (blue dashed curve).

4.4.2. One-soliton solution. For one-soliton, we have

$$\boldsymbol{K}_1 = k_1, \quad \boldsymbol{r}_1 = \rho_1 = e^{\xi_1}, \quad \boldsymbol{s}_1 = \sigma_1 = e^{\zeta_1}, \quad \boldsymbol{M}_1 = \frac{\rho_1\sigma_1^*}{k_1 + k_1^*}, \tag{4.33}$$

where k_1, ρ_1, σ_1 are defined as in (4.32). From (4.5) we have

$$q_3 = \frac{4a_1^2 e^{\xi_1+\zeta_1}}{4a_1^2 + (a_1^2 + (b_1 - t)^2)^2 e^{2(Re[\xi_1+\zeta_1])}}, \tag{4.34}$$

which yields

$$|q_3|^2 = \frac{a_1^2}{(a_1^2 + (b_1 - t)^2)^2}\,\mathrm{sech}^2 \left[d_1 + f_1 - \frac{a_1 x}{a_1^2 + (b_1 - t)^2} - \ln(2a_1)\right]. \tag{4.35}$$

The envelope is depicted in figure 7. It is interesting that $|q_3|^2$ has a time-dependent amplitude $\frac{a_1^2}{(a_1^2 + (b_1 - t)^2)^2}$, which indicates that the soliton is a localized wave with respect to both space and time. In addition, the top trace for (4.35) reads

$$x(t) = \frac{A(a_1^2 + (b_1 - t)^2)}{a_1}, \quad \text{where } A = d_1 + f_1 - \ln(2a_1),$$

which, in general, is a parabola curve, and along which the soliton travels and changes its direction at the vertex $(t, x) = (b_1, Aa_1)$ where $|q_3|^2$ takes maximum value $\frac{1}{a_1^2}$, see, e.g. Figure 7(c). A special case takes place when $A = 0$, which yields a stationary soliton, as depicted in figure 7(a).

4.4.3. Two-soliton solution. The two-soliton solution is given by (4.5) where $\boldsymbol{K}_1$, $\boldsymbol{r}_1$, $\boldsymbol{s}_1$, $\boldsymbol{M}_1$ are given as in (4.27) but k_j, ρ_j, σ_j are defined as in (4.32). The solution is written as

$$q_3 = \frac{(1 + F_{11})e^{\zeta_2+\xi_2} - F_{12}e^{\zeta_1+\xi_2} - F_{21}e^{\zeta_2+\xi_1} + (1 + F_{22})e^{\zeta_1+\xi_1}}{1 + F_{11} - F_{12}F_{21} + F_{22}(1 + F_{11})}, \tag{4.36}$$

where

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \frac{e^{2\,\mathrm{Re}[\xi_1+\zeta_1]}}{(k_1 + k_1^*)^2} + \frac{e^{\xi_1+\zeta_1+\zeta_2^*+\xi_2^*}}{(k_1 + k_2^*)^2} & \frac{e^{2\,\mathrm{Re}[\xi_1]+\zeta_2+\zeta_1^*}}{(k_1 + k_1^*)(k_1^* + k_2)} + \frac{e^{2\,\mathrm{Re}[\zeta_2]+\xi_1+\xi_2^*}}{(k_1 + k_2^*)(k_2 + k_2^*)} \\ \frac{e^{2\,\mathrm{Re}[\zeta_1]+\xi_2+\xi_1^*}}{(k_1 + k_1^*)(k_1^* + k_2)} + \frac{e^{2\,\mathrm{Re}[\xi_2]+\zeta_1+\zeta_2^*}}{(k_1 + k_2^*)(k_2 + k_2^*)} & \frac{e^{2\,\mathrm{Re}[\xi_2+\zeta_2]}}{(k_2 + k_2^*)^2} + \frac{e^{\xi_1^*+\zeta_1^*+\zeta_2+\xi_2}}{(k_1^* + k_2)^2} \end{pmatrix}.$$

For the shape and motion of the envelope $|q_3|^2$, we illustrate them in figure 8, where (a) shows the scattering of two solitons and (b) describes interactions of two different stationary solitons.

4.4.4. Double-pole solution. Double-pole solution of the NNLSE-III is given by the formula (4.5) with the setting (4.17) and $\boldsymbol{M}_1$ takes the form (4.18), where k_1, ρ_1, σ_1 are

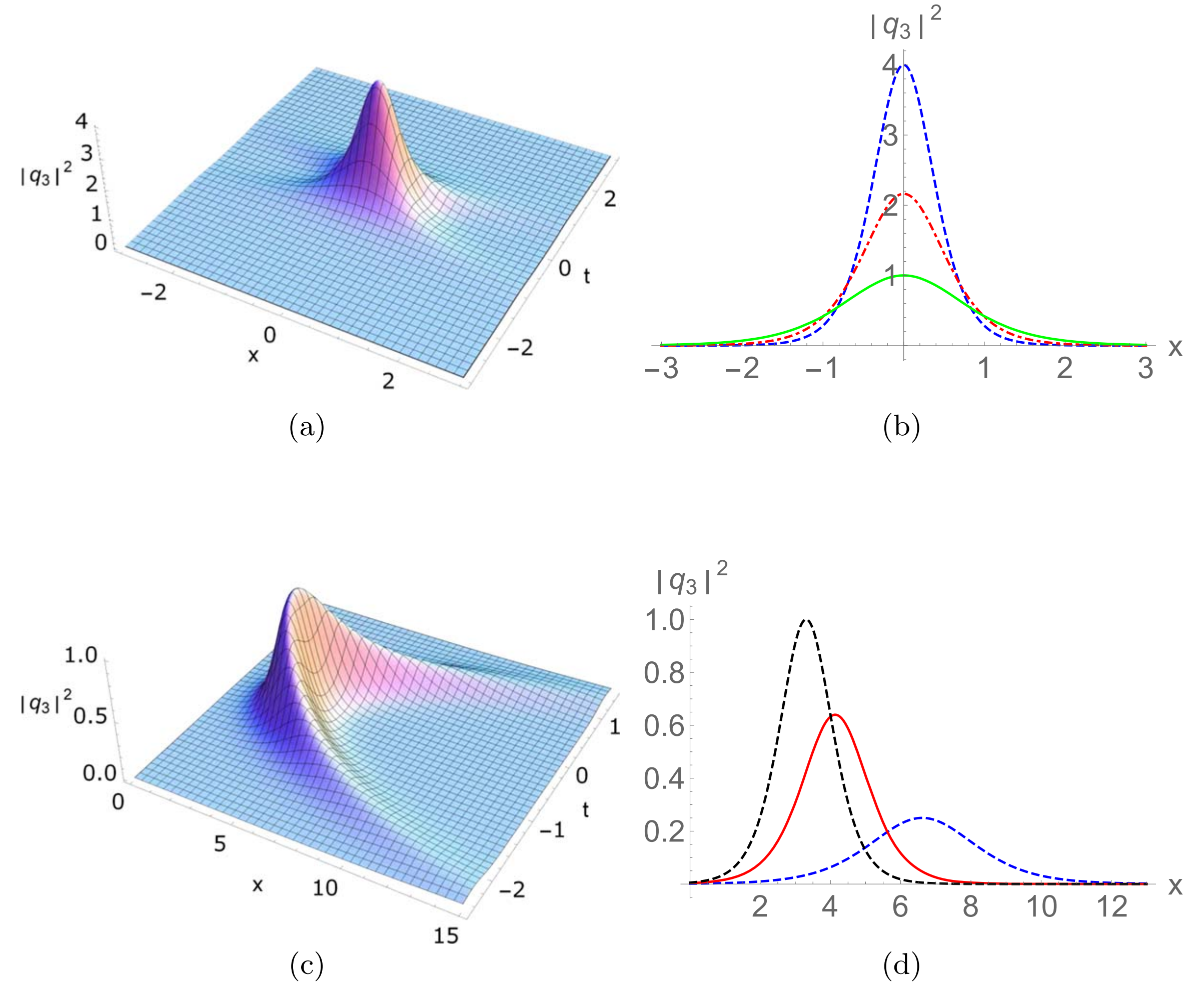


Figure 7. The shape and motion the envelope of one-soliton solution given by (4.35) for (a) $c_1 = 0.5$, $\xi_1^{(0)} = \zeta_1^{(0)} = 0$. (b) 2D plot of (a) at $t = 0$ (blue dashed curve), $t = 0.3$ (red dot-dashed curve) and $t = 0.5$ (green solid curve). (c) $c_1 = 1 - 0.5\mathrm{i}$, $\xi_1^{(0)} = \zeta_1^{(0)} = 2$, (d) 2D plot of (c) at $t = -0.5$ (black dashed curve), $t = 0$ (red solid curve) and $t = 0.5$ (blue dashed curve).

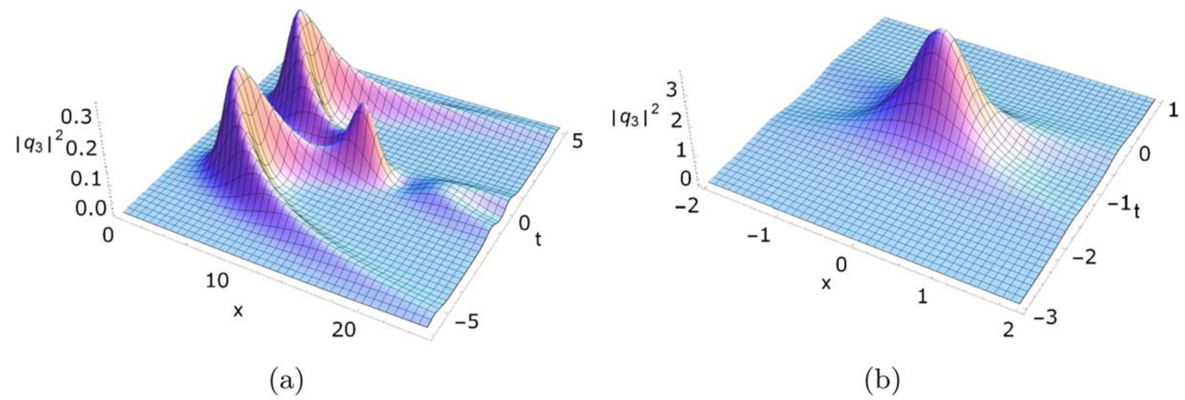


Figure 8. The shape and motion of the envelope of two-soliton solution (4.36) for (a) $c_1 = -1.7 + 1.5\mathrm{i}$, $c_2 = -1.7 - 2.5\mathrm{i}$, $\xi_1^{(0)} = \zeta_1^{(0)} = \xi_2^{(0)} = \zeta_2^{(0)} = -1$. (b) $c_1 = -4 - 0.5\mathrm{i}$, $c_2 = 0.5 - \mathrm{i}$, $\xi_1^{(0)} = \zeta_1^{(0)} = \xi_2^{(0)} = \zeta_2^{(0)} = 0$.

defined as in (4.32). Its explicit formula is

$$q_3 = \frac{(1 + R_{22} - (R_{12} + R_{21})V + (1 + R_{11})V^*)\mathrm{e}^{\xi_1+\zeta_1}}{1 + R_{11} - R_{12}R_{21} + R_{22} + R_{11}R_{22}}, \tag{4.37}$$

where

$$V = \frac{1}{2}k_1^2 x + k_1, \quad \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \boldsymbol{M}_1\boldsymbol{M}_1^*.$$

The shape and motion of $|q_3|^2$ are illustrated in figure 9. When

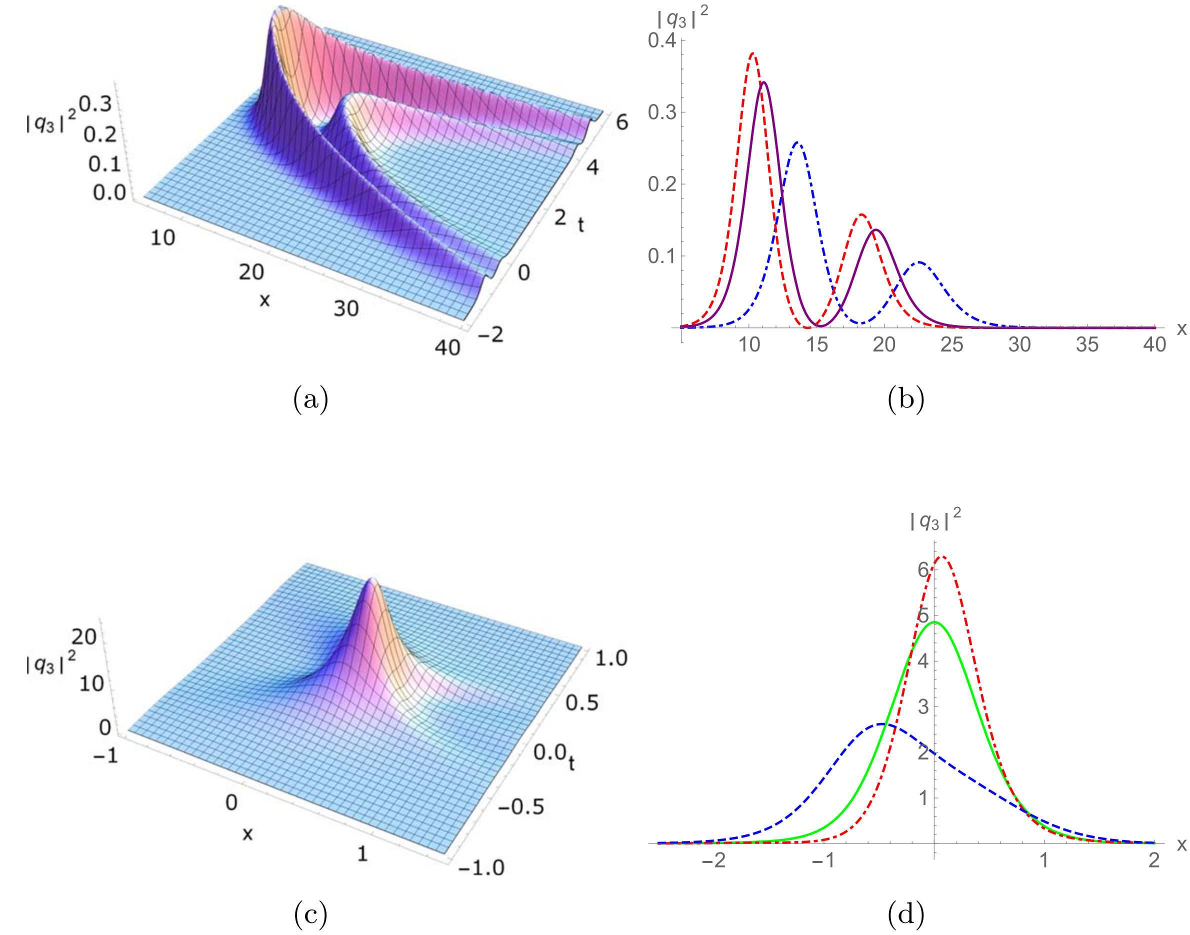


Figure 9. The shape and motion of the envelope of the double-pole solution (4.37) for (a) $c_1 = -2 + 2\mathrm{i}$, $\xi_1^{(0)} = -2$, $\zeta_1^{(0)} = -3$. (b) 2D plot (a) at $t = 3$ (blue dot-dashed curve), $t = 2$ (red dashed curve) and $t = 1.5$ (purple solid curve). (c) $c_1 = 0.4$, $\xi_1^{(0)} = \zeta_1^{(0)} = 0$. (d) 2D plot (c) at $t = 0.2$ (green solid curve), $t = 0.1$ (red dot-dashed curve) and $t = -0.4$ (blue dashed curve).

$\xi_1^{(0)} \neq \zeta_1^{(0)}$ we get two solitons moving along the same parabolic top trace, as shown in figure 9(a). When $\xi_1^{(0)} = \zeta_1^{(0)} = 0$ we get two overlapped solitons as shown in figure 9(c).

5. Conclusions

In this paper we have developed the Cauchy matrix approach to the NNLSEs, which serve as example models in the ZS-AKNS hierarchy. We believe that solutions of other order equations in the non-isospectral ZS-AKNS hierarchy, such as the non-isospectral sine-Gordon equation, the non-isospectral modified KdV (mKdV) equation and the non-isospectral Hirota equation (combination of the NLS and the mKdV) can be obtained along with this line.

In the Cauchy matrix approach, the Sylvester equation (e.g. (2.1)) plays an central role, which defines a dressed Cauchy matrix to provide τ functions (i.e. $|\boldsymbol{I}_{2N} + \boldsymbol{M}|$) for the investigated equations. In non-isospectral case, one needs to suitably select dispersion relations of the time part (e.g. (2.7*b*), (2.23*b*) and (2.30*b*)) according to the time-evolution of the spectral parameters. One needs also to formulate special relations (e.g. (2.35)) to figure out the integration term (e.g. in (2.34)). Apparently, compared with the isospectral case [23], the non-isospectral extension of the Cauchy matrix scheme is quite non-trivial. In addition, in the isospectral case (see [23]) $\boldsymbol{K}$ is a constant matrix and it can be proved that $\boldsymbol{K}$ and its any similar form lead to same $\boldsymbol{S}^{(i,j)}$ therefore one only needs to consider its canonical form and the resulted solutions can be classified by the canonical forms of $\boldsymbol{K}$. However, as we have seen that in non-isospectral case $\boldsymbol{K}$ is no longer a constant matrix, usually $\boldsymbol{K}$ and its similar form do not obey the same evolutions (e.g. (2.9), (2.24) and (2.31)). Therefore in the non-isospectral case, we can not classify solutions by considering the canonical forms of $\boldsymbol{K}$. In appendix A we will formulate solutions of the Sylvester equations in (2.3) where $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are triangular Toeplitz matrices, which are used to get multiple-pole solutions. Note that the Sylvester equation (2.1) to formulate the ZS-AKNS system is different from the one for the KdV type and KP type equations (see [22, 27]). Extension of the Cauchy matrix approach to the non-isospectral KdV and KP type equations (as well as the non-isospectral equations with sources, e.g. [30]) will be considered elsewhere.

Acknowledgments

This project is supported by the National Natural Science Foundation of China (No. 12271334).

Appendix A. Solutions to (2.3) with triangular Toeplitz matrices

We sketch a procedure to construct solutions to the Sylvester equations in (2.3) when $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are lower triangular Toeplitz matrices. A lower triangular Toeplitz matrix is a square matrix in the following form and can be considered to be generated by a certain function:

$$\boldsymbol{F}_c^{[N]}[f(c)] = \begin{pmatrix} f & 0 & 0 & \cdots & 0 \\ \frac{\partial_c f}{1!} & f & 0 & \cdots & 0 \\ \frac{\partial_c^2 f}{2!} & \frac{\partial_c f}{1!} & f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial_c^{N-1} f}{(N-1)!} & \frac{\partial_c^{N-2} f}{(N-2)!} & \frac{\partial_c^{N-3} f}{(N-3)!} & \cdots & f \end{pmatrix}_{N\times N}, \tag{A.1}$$

where $f(c)$ is the generating function. Note that the subindex 'c' indicates that the lower triangular Toeplitz matrix is generated by taking derivatives with respect to c. We also introduce a symmetric matrix generated by $f(c)$, denoted as

$$\boldsymbol{H}_c^{[N]}[f(c)] = \begin{pmatrix} f & \frac{\partial_c f}{1!} & \frac{\partial_c^2 f}{2!} & \cdots & \frac{\partial_c^{N-1} f}{(N-1)!} \\ \frac{\partial_c f}{1!} & \frac{\partial_c^2 f}{2!} & \frac{\partial_c^3 f}{3!} & \cdots & 0 \\ \frac{\partial_c^2 f}{2!} & \frac{\partial_c^3 f}{3!} & \frac{\partial_c^4 f}{4!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial_c^{N-1} f}{(N-1)!} & 0 & 0 & \cdots & 0 \end{pmatrix}_{N\times N}. \tag{A.2}$$

As a special property of such two types of matrices, we mention that [26]

$$\begin{aligned} F_c^{[N]}[f(c)g(c)] &= F_c^{[N]}[f(c)]F_c^{[N]}[g(c)], \\ H_c^{[N]}[f(c)g(c)] &= H_c^{[N]}[f(c)]F_c^{[N]}[g(c)]. \end{aligned} \tag{A.3}$$

For more properties, one can refer to [29] and proposition 3 in [26].

In the following we will use the notations k_1, l_1, ρ_1, ϱ_1, σ_1 and ϖ_1 that we introduced in section 3 but we do not specify them since the following description is generic and true for all the three NNLSEs. We consider k_1 and l_1 to be functions of c_1 and d_1, respectively, e.g. (3.2), (3.9) and (3.11). Let

$$\boldsymbol{K}_1 = \boldsymbol{F}_{c_1}^{[N]}[k_1], \quad \boldsymbol{K}_2 = \boldsymbol{F}_{d_1}^{[N]}[l_1]. \tag{A.4a}$$

Then, it can be verified that $\boldsymbol{K} = \mathrm{diag}(\boldsymbol{K}_1, \boldsymbol{K}_2)$ satisfies the evolutions (2.9), (2.24) and (2.31) when k_1, l_1 are defined as (3.2), (3.9) and (3.11), respectively. Next, define

$$\boldsymbol{r}_1 = \boldsymbol{F}_1\,\boldsymbol{e}_N, \quad \boldsymbol{s}_2 = \boldsymbol{H}_2\,\boldsymbol{e}_N, \quad \boldsymbol{F}_1 = \boldsymbol{F}_{c_1}^{[N]}[\rho_1], \quad \boldsymbol{H}_2 = \boldsymbol{H}_{d_1}^{[N]}[\varpi_1], \tag{A.4b}$$

$$\boldsymbol{r}_2 = \boldsymbol{F}_2\,\boldsymbol{e}_N, \quad \boldsymbol{s}_1 = \boldsymbol{H}_1\,\boldsymbol{e}_N, \quad \boldsymbol{F}_2 = \boldsymbol{F}_{d_1}^{[N]}[\varrho_1], \quad \boldsymbol{H}_1 = \boldsymbol{H}_{c_1}^{[N]}[\sigma_1], \tag{A.4c}$$

where

$$\boldsymbol{e}_N = (\underbrace{1,0,0,\cdots,0}_{N-\text{dimensional}})^{\mathrm{T}}.$$

Then, the above defined elements satisfy the dispersion relations (2.7), (2.23) and (2.30) when k_1, l_1 are defined as (3.2), (3.9) and (3.11), respectively.

Next, we look for solution $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of the Sylvester equations in (2.3) in the form

$$\boldsymbol{M}_1 = \boldsymbol{F}_1\boldsymbol{G}_1\boldsymbol{H}_2, \quad \boldsymbol{M}_2 = \boldsymbol{F}_2\boldsymbol{G}_2\boldsymbol{H}_1, \tag{A.5}$$

where $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are unknowns. In these settings, equation (2.3*a*) can be rewritten as

$$\boldsymbol{K}_1\boldsymbol{F}_1\boldsymbol{G}_1\boldsymbol{H}_2 - \boldsymbol{F}_1\boldsymbol{G}_1\boldsymbol{H}_2\boldsymbol{K}_2 = \boldsymbol{F}_1\boldsymbol{e}_N\boldsymbol{e}_N^{\mathrm{T}}\boldsymbol{H}_2. \tag{A.6}$$

Using the relations [26]

$$\boldsymbol{K}_1\boldsymbol{F}_1 = \boldsymbol{F}_1\boldsymbol{K}_1, \quad \boldsymbol{H}_2\boldsymbol{K}_2 = \boldsymbol{K}_2^{\mathrm{T}}\boldsymbol{H}_2.$$

(A.6) reduces to

$$\boldsymbol{K}_1\boldsymbol{G}_1 - \boldsymbol{G}_1\boldsymbol{K}_2^{\mathrm{T}} = \boldsymbol{e}_N\boldsymbol{e}_N^{\mathrm{T}}. \tag{A.7}$$

For convenience, we write $\boldsymbol{G}_1 = (\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_N)$ where $\boldsymbol{g}_j = (g_{1,j}, g_{2,j}, \cdots, g_{N,j})^{\mathrm{T}}$. Then, the first column of (A.7) reads

$$\begin{pmatrix} k_1 g_{1,1} \\ (\partial_{c_1}k_1)g_{1,1} + k_1 g_{2,1} \\ \frac{1}{2!}(\partial_{c_1}^2 k_1)g_{1,1} + (\partial_{c_1}k_1)g_{2,1} + k_1 g_{3,1} \\ \vdots \\ \frac{1}{(N-1)!}(\partial_{c_1}^{N-1}k_1)g_{1,1} + \cdots + k_1 g_{N,1} \end{pmatrix} - l_1\begin{pmatrix} g_{1,1} \\ g_{2,1} \\ g_{3,1} \\ \vdots \\ g_{N,1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which gives rise to

$$g_{1,1} = \frac{1}{k_1 - l_1}, \quad g_{m,1} = -\frac{1}{k_1 - l_1}\left(\sum_{j=1}^{m-1}\frac{\partial_{c_1}^j k_1}{j!}\, g_{m-j,1}\right), \quad m = 2, 3, \cdots, N, \tag{A.8}$$

from which one can successively determine $g_{2,1}, g_{3,1}, \ldots, g_{N,1}$. For example, we have

$$g_{2,1} = -\frac{\partial_{c_1}k_1}{(k_1 - l_1)^2}, \quad g_{3,1} = \frac{(\partial_{c_1}k_1)^2}{(k_1 - l_1)^3} - \frac{\partial_{c_1}^2 k_1}{2!(k_1 - l_1)^2}.$$

Once with $\boldsymbol{g}_1$ in hand, we can look at the second column of

(A.7), which is

$$\begin{pmatrix} k_1 g_{1,2} \\ (\partial_{c_1}k_1)g_{1,2}+k_1 g_{2,2} \\ \frac{1}{2!}(\partial_{c_1}^2 k_1)g_{1,2}+(\partial_{c_1}k_1)g_{2,2}+k_1 g_{3,2} \\ \vdots \\ \frac{1}{(N-1)!}(\partial_{c_1}^{N-1}k_1)g_{1,2}+\cdots+k_1 g_{N,2} \end{pmatrix} - \begin{pmatrix} (\partial_{d_1}l_1)g_{1,1}+l_1 g_{1,2} \\ (\partial_{d_1}l_1)g_{2,1}+l_1 g_{2,2} \\ (\partial_{d_1}l_1)g_{3,1}+l_1 g_{3,2} \\ \vdots \\ (\partial_{d_1}l_1)g_{N,1}+l_1 g_{N,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Element of $\boldsymbol{g}_2$ can be calculated as:

$$g_{1,2}=\frac{1}{k_1-l_1}(\partial_{d_1}l_1)g_{1,1},\quad g_{m,2} = \frac{1}{k_1-l_1}\left((\partial_{d_1}l_1)g_{m,1} - \sum_{j=1}^{m-1}\frac{\partial_{c_1}^j k_1}{j!}g_{m-j,2}\right),\quad m=2,3,\cdots,N. \tag{A.9}$$

The first few elements are

$$g_{1,2}=\frac{\partial_{d_1}l_1}{(k_1-l_1)^2},\quad g_{2,2}=-\frac{2(\partial_{c_1}k_1)(\partial_{d_1}l_1)}{(k_1-l_1)^3},$$
$$g_{3,2}=\frac{3(\partial_{c_1}k_1)^2(\partial_{d_1}l_1)}{(k_1-l_1)^4}-\frac{(\partial_{c_1}^2k_1)(\partial_{d_1}l_1)}{(k_1-l_1)^3}.$$

For the n-th column ($n>1$) of (A.7), we have

$$\boldsymbol{K}_1\boldsymbol{g}_n-\sum_{j=1}^{n-1}\frac{\partial_{d_1}^j l_1}{j!}\boldsymbol{g}_{n-j}-l_1\boldsymbol{g}_n=0,$$

which indicates that

$$\boldsymbol{g}_n=\sum_{j=1}^{n-1}\frac{\partial_{d_1}^j l_1}{j!}(\boldsymbol{K}_1-l_1\boldsymbol{I}_N)^{-1}\boldsymbol{g}_{n-l}.$$

Finally, $\boldsymbol{G}_1=(\boldsymbol{g}_1,\boldsymbol{g}_2,\ldots,\boldsymbol{g}_N)$ can be derived.

$\boldsymbol{G}_2$ can be solved from equation (2.3*b*) in a similar way. Here we skip the details.

References

[1] Zakharov V E (ed) 1991 *What Is Integrablity?* (Springer) (https://link.springer.com/book/10.1007/978-3-642-88703-1)
[2] Benney D J and Newell A C 1967 The propagation of nonlinear wave envelopes *J. Math. Phys.* **46** 133–9
[3] Zakharov V E 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid *J. Appl. Mech. Tech. Phys.* **9** 190–4
[4] Zakharov V E 1972 Collapse of Langmuir waves *Sov. Phys. JETP* **35** 908–14
[5] Hasegawa A and Tappert F 1973 Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers: I. Anomalous dispersion *Appl. Phys. Lett.* **23** 142–4
[6] Hasegawa A and Tappert F 1973 Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers: II. Normal dispersion *Appl. Phys. Lett.* **23** 171–2
[7] Gross E P 1961 Structure of a quantized vortex in boson systems *Il Nuovo Cimento* **20** 454–77
[8] Pitaevskii L P 1961 Vortex lines in an imperfect Bose gas *Sov. Phys. JETP* **13** 451–4
[9] Gross E P 1963 Hydrodynamics of a superfluid condensate *J. Math. Phys.* **4** 195–207
[10] Stamper-Kurn D M, Andrews M R, Chikkatur A P, Inouye S, Miesner H J, Stenger J and Ketterle W 1998 Optical confinement of a Bose−Einstein condensate *Phys. Rev. Lett.* **80** 2027–30
[11] Liu W M and Kengne E 2019 *Schrödinger Equations in Nonlinear Systems* (Springer) (https://link.springer.com/book/10.1007/978-981-13-6581-2)
[12] Chen H H and Liu C S 1976 Solitons in nonuniform media *Phys. Rev. Lett.* **37** 693–7
[13] Calogero F and Degasperis A 1978 Conservation laws for classes of nonlinear evolution equations solvable by the spectral transform *Commun. Math. Phys.* **63** 155–76
[14] Gupta M R 1979 Exact inverse sacttering solution of a nonlinear evolution equation in a nonuniform medium *Phys. Lett.*A **72** 420–2
[15] Burtsev S P, Zakharov V E and Mikhailov A V 1987 Inverse scattering method with variable spectral parameter *Theor.. Math. Phys.* **70** 227–40
[16] Silem A, Zhang C and Zhang D J 2019 Dynamics of three nonisospectral nonlinear Schrödinger equations *Chin. Phys.* B **28** 020202
[17] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Theory of Bose–Einstein condensation in trapped gases *Rev. Mod. Phys.* **71** 463–512
[18] Liu S M, Wu H and Zhang D J 2020 New results on the classical and nonlocal Gross-Pitaevskii equation with a parabolic potential *Rep. Math. Phys.* **86** 271–92
[19] Silem A, Wu H and Zhang D J 2021 Nonisospectral effects on generating localized waves *Commun. Theor. Phys.* **73** 115002
[20] Nijhoff F W, Atkinson J and Hietarinta J 2009 Soliton solutions for ABS lattice equations: I. Cauchy matrix approach *J Phys A: Math Theor.* **42** 404005
[21] Zhang D J and Zhao S L 2013 Solutions to ABS lattice equations via generalized Cauchy matrix approach *Stud. Appl. Math.* **131** 72–103
[22] Xu D D, Zhang D J and Zhao S L 2014 The Sylvester equation and integrable equations: I. The Korteweg-de Vries system and sine-Gordon equation *J. Nonl. Math. Phys.* **21** 382–406
[23] Zhao S L 2018 The Sylvester equation and integrable equations: the Ablowitz–Kaup–Newell–Segur system *Rep. Math. Phys.* **82** 241–63
[24] Tian H J and Zhang D J 2020 Cauchy matrix structure of the Mel'nikov model of long-short wave interaction *Commun. Theor. Phys.* **72** 125006

[25] Li S S, Qu C Z, Yi X X and Zhang D J 2022 Cauchy matrix approach to the SU(2) self-dual Yang–Mills equation *Stud. Appl. Math.* **148** 1703–21
[26] Li S S, Qu C Z and Zhang D J 2023 Solutions to the SU(N) self-dual Yang–Mills equation *Physica*D **453** 133828
[27] Zhao S L, Shen S F and Feng W 2014 Kadomtsev–Petviashvili system and reduction: generalized Cauchy matrix approach arXiv: 1404.3043
[28] Sylvester J 1884 Sur l'equation en matrices $px = xq$ *C.R. Acad. Sci. Paris* **99** 67–76
[29] Zhang D J, Zhao S L, Sun Y Y and Zhou J 2014 Solutions to the modified Korteweg-de Vries equation *Rev. Math. Phys.* **26** 1430006
[30] Zhang D J, Bi J B and Hao H H 2006 Modified KdV equation with self-consistent sources in non-uniform media and soliton dynamics *J. Phys. A: Math. Gen.* **39** 14627–48