Communications in Theoretical Physics

Commun. Theor. Phys. 76 (2024) 065002 (13pp) https://doi.org/10.1088/1572-9494/ad361b

The Sasa–Satsuma equation with high-order discrete spectra in space-time solitonic regions: soliton resolution via the mixed $\bar{\partial}$-Riemann–Hilbert problem

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Received 5 February 2024, revised 18 March 2024
Accepted for publication 21 March 2024
Published 13 May 2024

Abstract

In this paper, we investigate the Cauchy problem of the Sasa–Satsuma (SS) equation with initial data belonging to the Schwartz space. The SS equation is one of the integrable higher-order extensions of the nonlinear Schrödinger equation and admits a 3×3 Lax representation. With the aid of the $\bar{\partial}$-nonlinear steepest descent method of the mixed $\bar{\partial}$-Riemann–Hilbert problem, we give the soliton resolution and long-time asymptotics for the Cauchy problem of the SS equation with the existence of second-order discrete spectra in the space-time solitonic regions.

Keywords: Sasa–Satsuma equation, inverse scattering, $\bar{\partial}$-Riemann–Hilbert problem, $\bar{\partial}$ steepest descent method, soliton resolution

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1993, based on the scheme of the Riemann–Hilbert (RH) problem, Deift and Zhou discussed the long-time asymptotic behaviors of the solutions of the mKdV equation by combining classical Fourier analysis and the steepest descent method [1]. Based on this method, the long-time asymptotic behaviors of many integrable equations were explored, such as the sine-Gordon equation [2], the Korteweg-de Vries equation [3], the Camassa–Holm equation [4], the short pulse equation [5, 6], the Fokas–Lenells equation [7], the extended mKdV equation [8, 9], and so on.

As a development of the Deift–Zhou nonlinear steepest descent method, a powerful tool called the $\bar{\partial}$-steepest descent method was first proposed by Mclaughlin and miller to analyze the asymptotic behaviors of orthogonal polynomials [10, 11]. Later, this method was successfully used to analyze the long-time behaviors of solutions to integrable nonlinear wave equations, such as the focusing NLS equation [12, 13], the defocusing NLS equation [14, 15], the derivative NLS equation [16], the mKdV equation [17, 18], the fifth-order mKdV equation [19], the complex short pulse equation [20], the modified Camassa–Holm equation [21], the Novikov equation [22], etc.

As a new-type integrable high-order equation of the nonlinear Schrödinger equation, the Sasa–Satsuma (SS) equation was presented [23]

$$q_t + q_{xxx} + 3\,|q|^2 q_x + 3(|q|^2 q)_x = 0, \qquad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \tag{1.1}$$

which admits a 3×3 Lax pair

$$\Phi_x + ik\sigma\Phi = U(x, t; k)\Phi, \qquad \Phi_t + 4ik^3\sigma = W(x, t; k)\Phi, \tag{1.2}$$

where $\Phi = \Phi(x, t; k)$ is a matrix function of x, t and iso-spectral parameter $k \in \mathbb{C}$,

$$U(x,t;k)=\begin{pmatrix}0_{2\times 2} & \boldsymbol{q}(x,t)\\ -\boldsymbol{q}^{\dagger}(x,t) & 0_{1\times 2}\end{pmatrix},$$
$$\boldsymbol{q}(x,t)=\begin{pmatrix}q(x,t)\\ q^{*}(x,t)\end{pmatrix},$$
$$\sigma=\begin{pmatrix}\mathbb{I}_{2\times 2} & 0_{2\times 1}\\ 0_{1\times 2} & -1\end{pmatrix},$$
$$W(x,t;k)=4k^2U+2ik\sigma(U_x-U^2)+2U^3-U_{xx}+[U_x,U], \tag{1.3}$$

with '$*$' and '$\dagger$' denoting the complex conjugation and Hermite transformation, respectively. In fact, the SS equation (1.1) can also be regarded as the special reduction ($r=q^*$) of the two-component integrable complex modified KdV equations [24, 25]

$$q_t+q_{xxx}+6|q|^2q_x+3(qr)_xr^*=0,$$
$$r_t+r_{xxx}+6|r|^2r_x+3(qr)_xq^*=0. \tag{1.4}$$

The SS equation admits many other integrable properties, such as N-soliton solutions, infinite conservation laws, nonlocal symmetries, Painlevé property, dark soliton solutions, and rogue wave solutions [26–30]. Recently, Liu *et al* studied the long-time asymptotic behaviors of the SS equation via the Deift–Zhou nonlinear steepest descent method [31]. Recently, Xun and Fan used this method to study the long-time and Painlevé-type asymptotics of the SS equation under the assumption of scattering data admitting only finitely simple zeros [32].

Based on the above-mentioned situations, in this paper, we focus on the long-time asymptotic behaviors of solutions for the Cauchy problem of the integrable SS equation (1.1) with the initial data:

$$q(x,0)=q_0(x)\in\mathcal{S}(\mathbb{R}), \tag{1.5}$$

under the assumption of the scattering data possessing finitely double zeros, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space. We then obtain the long-time behaviors of the potential $q(x, t)$.

The rest of this paper is organized as follows. In section 2, we review the direct and inverse scattering transforms about the 3×3 Lax pair of equation (1.1) and deduce the analytic region about the Jost functions. Furthermore, we set up the original RH problem. Based on the RH problem, in section 3, using the ideas from [13, 32], we give a series of the transformation of the RH problem to make it a model RH problem whose solution is a parabolic function. In section 4, through the transformations of the RH problem, the potential of RHP1 can be reconstructed by three parts. One is the double-pole soliton solutions by solving the RHP in the reflectionless case, and the other terms are provided by the error function $E(r)$ and the pure $\bar{\partial}$-problem.

2. The direct scattering problem

2.1. Jost solutions of the Lax pair and scattering data

Based on the boundary-value condition $\lim_{|x|\to\infty}q_0(x)=0$, the eigenfunction of the Lax pair (1.2) has the following asymptotic form

$$\Phi(k,x,t)\sim \mathrm{e}^{-\mathrm{i}(kx+4k^3t)\sigma},\quad |x|\to\infty. \tag{2.1}$$

To change the large-space asymptotics of the eigenfunction of the Lax pair (1.2) into a unit matrix, let

$$\Psi(k,x,t)=\Phi(k,x,t)\mathrm{e}^{\mathrm{i}(kx+4k^3t)\sigma}.$$

Then $\Psi(k, x, t)$ satisfies the following modified Lax pair

$$\Psi_x+\mathrm{i}k[\sigma,\Psi]=U\Psi,\qquad \Psi_t+4\mathrm{i}k^3[\sigma,\Psi]=W\Psi, \tag{2.2}$$

which can be written as a fully differential form

$$\mathrm{d}(\mathrm{e}^{\mathrm{i}(kx+4k^3t)\hat{\sigma}}\Psi)=\mathrm{e}^{\mathrm{i}(kx+4k^3t)\hat{\sigma}}(U\Psi\mathrm{d}x+W\Psi\mathrm{d}t), \tag{2.3}$$

from which the Jost solutions $\Psi_+(k, x, t)$ and $\Psi_-(k, x, t)$ can be rewritten as follows:

$$\Psi_{\pm}(k,x,t)=I-\int_x^{\pm\infty}\mathrm{e}^{\mathrm{i}k(\xi-x)\hat{\sigma}}U(\xi,t)\Psi_{\pm}(k,y,t)\mathrm{d}\xi. \tag{2.4}$$

Let $\Psi_{\pm}=(\Psi_{\pm1}(k,x,t),\Psi_{\pm2}(k,x,t))$, where $\Psi_{\pm1}(k, x, t)$ and $\Psi_{\pm2}(k, x, t)$ represent their first two columns and third column, respectively. It follows from equation (2.4) that Ψ_{-1}, Ψ_{+2} are analytic in $\mathbb{C}_+$, and Ψ_{+1}, Ψ_{-2} are analytic in $\mathbb{C}_-$. Moreover,

$$(\Psi_{\mp1}(k,x,t),\Psi_{\pm2}(k,x,t))=I+\mathcal{O}(k^{-1}),\quad k\in\mathbb{C}_{\pm}\to\infty. \tag{2.5}$$

By using Abel's lemma and $\mathrm{tr}(U)=\mathrm{tr}(W)=0$, one knows that $\det\Psi_{\pm}(k,x,t)$ are independent of variable x and $\det\Psi_{\pm}=1$. Furthermore, $\Psi_{\pm}\mathrm{e}^{-\mathrm{i}(kx+4k^3t)\sigma}$ are linearly dependent to lead to

$$\Psi_{-}(k)\mathrm{e}^{-\mathrm{i}(kx+4k^3t)\sigma}=\Psi_{+}(k)\mathrm{e}^{-\mathrm{i}(kx+4k^3t)\sigma}S(k), \tag{2.6}$$

where $S(k)$ is a 3×3 scattering matrix. Moreover, together with $\det(S)=1$, one can know that $\Psi_{\pm}$ and $S(k)$ admit the symmetries:

$$\Psi_{\pm}^{\dagger}(k^*;x,t)=\Psi^{-1}(k;x,t),$$
$$\Psi_{\pm}(k;x,t)=\varrho\Psi_{\pm}^{*}(-k^*,x,t)\varrho, \tag{2.7}$$

$$S^{\dagger}(k^*)=S^{-1}(k),$$
$$S(k)=\varrho S^*(-k^*)\varrho, \tag{2.8}$$

based on the two symmetries of U

$$U^{\dagger}=-U,\quad \varrho U\varrho=U^*,\quad \varrho=\varrho^{-1}=\begin{pmatrix}\sigma_1 & 0\\ 0 & 1\end{pmatrix},$$
$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2.9}$$

One can further rewrite $S(k)$ as

$$S(k)=\begin{pmatrix}A(k) & -\mathrm{adj}[A^{\dagger}(k^*)]B^{\dagger}(k^*)\\ B(k) & \det[A^{\dagger}(k^*)]\end{pmatrix}=\lim_{x\to\infty}\mathrm{e}^{\mathrm{i}kx\hat{\sigma}}\Psi_{-}(k;x,0), \tag{2.10}$$

where $A(k)=\sigma_1A^*(-k^*)\sigma_1=(a_{ij}(k))_{2\times 2}$, $\mathrm{adj}[A^{\dagger}(k^*)]$ denotes the adjoint matrix of $A^{\dagger}(k^*)$, and $B(k)=B^*(-k^*)\sigma_1=(B_1(k), B_2(k))$, and

$$A(k) = I + \int_{\mathbb{R}} \boldsymbol{q}(x, 0)\Psi_{-12}(k; x, 0)\mathrm{d}x,$$
$$B(k) = -\int_{\mathbb{R}} \boldsymbol{q}^{\dagger}(x, 0)\Psi_{-11}(k; x, 0)\mathrm{e}^{-2ikx}\mathrm{d}x, \tag{2.11}$$

which imply that $A(k)$ is analytic in $\mathbb{C}_+$ by virtue of the analyticity of $\Psi_{-12}(k; x, 0)$.

For the convenience of the following analysis, an assumption of scattering data is that the functions $A(k)$ and $\det A(k)$ have no zeros on $\mathbb{R}$ and $A(k)$ has finite double zeros in $\mathbb{C}\backslash\mathbb{R}$, $\gamma(k) := B(k)A^{-1}(k) \in H^{1,1}(\mathbb{R})$.

2.2. The Riemann–Hilbert problem with higher-order poles

Let $A(k)$ have $2N$ double zeros $k_1, k_2,\ldots,k_{2N}$ in $\mathbb{C}_+$ with $k_{N+j} = -k_j^*, j = 1, 2,\ldots, N$ since there is the symmetry $A(k) = \sigma_1 A^*(-k^*)\sigma_1$, that is, $A(k_j) = A'(k_j) = 0$, $A''(k_j) \neq 0$ $(j = 1, 2,\ldots, 2N)$. To establish a RH problem, we define the following sectionally meromorphic matrix $M(k; x, t)$ with the aid of the analyticity of Jost functions

$$M(k; x, t) = \begin{cases} (\Psi_{-1}(k)A^{-1}(k), \Psi_{+2}(k)), & k \in \mathbb{C}^+, \\ \left(\Psi_{+1}(k), \dfrac{\Psi_{-2}(k)}{\det A^{\dagger}(k^*)}\right), & k \in \mathbb{C}^-, \end{cases} \tag{2.12}$$

such that $M(k; x, t)$ has $2N$ double poles $K = \{k_j, j = 1,\ldots,2N\}$ in $\mathbb{C}^+$ and $2N$ double poles $\overline{K} = \{k_j^*, j = 1,\ldots,2N\}$ in $\mathbb{C}^-$. According to equations (2.6) and (2.12), one can find that the matrix-valued function $M(k; x, t)$ satisfies the following RH problem:

RHP-1. Find a matrix-valued function solution $M(k; x, t)$ satisfying the following conditions:

- Analyticity : $M(k; x, t)$ is a meromorphic function in $\mathbb{C}\backslash\mathbb{R}$ and has double poles at $k_j \in K$ and $k_j^* \in \overline{K}$;
- Jump relation: $M(k; x, t)$ has continuous boundary values $M_{\pm}(k; x, t)$ on $\mathbb{R}$, and

$$M_+(k) = M_-(k)V(k; x, t), \qquad k \in \mathbb{R}, \tag{2.13}$$

where the jump matrix is

$$V(k; x, t) = \begin{pmatrix} \mathbb{I}_{2\times 2} + \gamma^{\dagger}(k^*)\gamma(k) & \gamma^{\dagger}(k^*)\mathrm{e}^{-2it\theta(k;x,t)} \\ \gamma(k)\mathrm{e}^{2it\theta(k;x,t)} & 1 \end{pmatrix},$$
$$\theta(k; x, t) = k\left(\frac{x}{t} + 4k^2\right). \tag{2.14}$$

- Asymptotics:

$$M(k; x, t) = I + O\left(\frac{1}{k}\right), \quad k \to \infty. \tag{2.15}$$

Therefore, $M(k; x, t)$ has double poles at each point in $K \cup \overline{K}$ with:

$$\underset{k=k_j}{\mathrm{P}_{-2}} M(k; x, t) = \lim_{k\to k_j} M(k; x, t)\begin{pmatrix} 0 & 0 \\ \mathcal{A}_j \mathrm{e}^{2it\theta(k)} & 0 \end{pmatrix}, \tag{2.16}$$

$$\mathrm{Res}_{k=k_j} M(k; x, t) = \lim_{k\to k_j} M(k; x, t)\begin{pmatrix} 0 & 0 \\ \mathcal{B}_j \mathrm{e}^{2it\theta(k)} & 0 \end{pmatrix} + M'(k; x, t)\begin{pmatrix} 0 & 0 \\ \mathcal{A}_j \mathrm{e}^{2it\theta(k)} & 0 \end{pmatrix}, \tag{2.17}$$

$$\underset{k=k_j^*}{\mathrm{P}_{-2}} M(k; x, t) = \lim_{k\to k_j^*} M(k; x, t)\begin{pmatrix} 0 & -\mathcal{A}_j^{\dagger}\mathrm{e}^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix}, \tag{2.18}$$

$$\mathrm{Res}_{k=k_j^*} M(k; x, t) = \lim_{k\to k_j^*} M(k; x, t)\begin{pmatrix} 0 & -\mathcal{B}_j^{\dagger}\mathrm{e}^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix} + M'(k; x, t)\begin{pmatrix} 0 & -\mathcal{A}_j^{\dagger}\mathrm{e}^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix}, \tag{2.19}$$

where

$$\mathcal{A}_j = \frac{2B(k_j)\mathrm{adj}[A(k_j)]}{\ddot{\det}[A(k_j)]},$$
$$\mathcal{B}_j = \left(2it\theta'(k_j) + \frac{2}{3}\frac{\dddot{\det}[A(k_j)]}{\ddot{\det}[A(k_j)]}\right)\mathcal{A}_j + \mathcal{A}_j'. \tag{2.20}$$

We now give the reconstruction formula for the solution of (1.1). Let Ψ have the following Laurent expansion

$$\Psi = \Psi^{(0)} + \frac{\Psi^{(1)}}{k} + \frac{\Psi^{(2)}}{k^2} + \cdots \tag{2.21}$$

Then, by substituting (2.21) into (2.2) and comparing the k^{-2} in t-part and k^0 in the x part, one can obtain

$$4i[\sigma, \Psi^{(1)}] = 4U\Psi^{(0)}, \quad \Psi_x^{(0)} + i[\sigma, \Psi^{(1)}] = U\Psi^{(0)}. \tag{2.22}$$

From the above two equations, we have

$$U = i[\sigma, \Psi^{(1)}], \tag{2.23}$$

i.e.

$$\boldsymbol{q}(x, t) = (q(x, t), q^*(x, t))^{\mathrm{T}} = 2i\Psi_{12}^{(1)} = 2i\lim_{k\to\infty}(kM(k; x, t))_{12}, \tag{2.24}$$

where $q(x, t)$ solves the SS equation (1.1).

In the following, we mainly consider the solution of $M(k; x, t)$ of RHP-1.

3. The mixed $\bar{\partial}$-RH problem and its decomposition

3.1. Two factorizations of jump matrix $V(k)$

Since the jump matrix $V(k; x, t)$ given by equation (2.14) admits the two different oscillatory terms for $t > 0$

$$O_{\pm} = \mathrm{e}^{\pm 2it\theta(k)} = \mathrm{e}^{\pm 2it(4k^3 + \frac{x}{t}k)},$$
$$\theta(k) = \tfrac{kx}{t} + 4k^3 = 4(k^3 - 3k_0^2 k), \tag{3.1}$$

where $\theta(k)$ admits two phase points $k = \pm k_0$, $k_0 = \sqrt{-\frac{x}{12t}}$ with $xt < 0$. To analyze their properties, one needs to consider the properties of $\mathrm{Re}[i\theta(k)]$ of $O_{\pm}$

$$\mathrm{Re}[i\theta(k)] = 4\,\mathrm{Im}\,k(\mathrm{Im}^2 k - 3\,\mathrm{Re}^2 k + 3k_0^2), \tag{3.2}$$

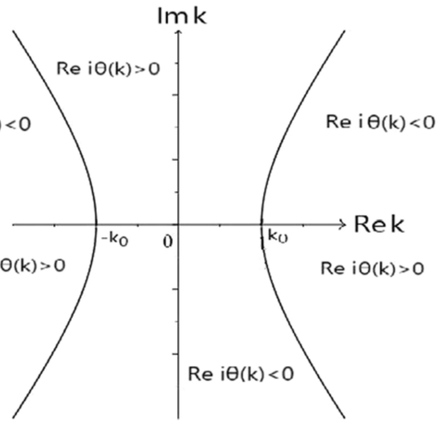

Figure 1. The signature table of $\mathrm{Re}\, i\theta(k) = 4\,\mathrm{Im}\,k\,(\mathrm{Im}^2 k - 3\,\mathrm{Re}^2 k + 3k_0^2)$ with $\pm k_0$ being phase points.

whose signature table is given in figure 1.

To analyze the long-time asymptotics of RHP-1, we first divide all the poles into two parts:

$$\Delta^- = \left\{k \,\middle|\, \mathrm{Re}^2(k) - \tfrac{1}{3}\mathrm{Im}^2(k) < k_0^2\right\},$$
$$\Delta^+ = \left\{k \,\middle|\, \mathrm{Re}^2(k) - \tfrac{1}{3}\mathrm{Im}^2(k) > k_0^2\right\}. \quad (3.3)$$

We assume that there are no poles corresponding to the region Δ^- for simplicity.

The jump matrix $V(k, x, t)$ has two different decompositions of upper and lower triangular matrices:

$$V = \begin{cases} \begin{pmatrix} I & \gamma^\dagger(k^*)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} I & 0 \\ \gamma(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}\setminus[-k_0, k_0], \\ \begin{pmatrix} I & 0 \\ \dfrac{\gamma(k)}{1+\gamma(k)\gamma^\dagger(k^*)}\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}\begin{pmatrix} I + \gamma^\dagger(k^*)\gamma(k) & 0 \\ 0 & \dfrac{1}{1+\gamma(k)\gamma^\dagger(k^*)} \end{pmatrix}\begin{pmatrix} I & \dfrac{\gamma^\dagger(k^*)}{1+\gamma(k)\gamma^\dagger(k^*)}\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in (-k_0, k_0). \end{cases} \quad (3.4)$$

To offset the influence of the diagonal matrix of the second decomposition, one needs to introduce the 2×2 matrix function $\delta(k)$ satisfying the following property:

Proposition 1. *The matrix function $\delta(k)$ and scalar function $\det\delta(k)$ satisfy the following properties:*

- $\delta(k)$ and $\det(\delta(k))$ are analytic, and $\delta(k)\delta^\dagger(k^*) = I$, $\det(\delta(k))\det(\delta^*(k^*)) = 1$ in $\mathbb{C}\setminus[-k_0, k_0]$.
- For $k \in (-k_0, k_0)$,

$$\delta_+(k) = \delta_-(k)(1 + \gamma^\dagger(k)\gamma(k)),$$
$$\det(\delta_+(k)) = \det(\delta_-(k))(1 + |\gamma(k)|^2); \quad (3.5)$$

-
$$|\delta_+(k)|^2 = \begin{cases} |\gamma(k)|^2 + 2, & k \in (-k_0, k_0), \\ 2, & \text{otherwise}. \end{cases} \quad (3.6)$$

$$|\delta_-(k)|^2 = \begin{cases} 2 - \dfrac{|\gamma(k)|^2}{1+|\gamma(k)|^2}, & k \in (-k_0, k_0), \\ 2, & \text{otherwise}. \end{cases} \quad (3.7)$$

- As $|k| \to \infty$ with $|\arg(k)| \leqslant c < \pi$,

$$\delta(k) = I + \mathcal{O}(k^{-1}), \quad \det(\delta(k)) = 1 + \frac{\mathrm{i}}{k}\left[\frac{1}{2\pi}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(k_0)|^2}\right)\mathrm{d}\xi - 2\nu k_0\right] + \mathcal{O}(k^{-2}); \quad (3.8)$$

-
$$\det(\delta(k)) = \left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu(k_0)}\mathrm{e}^{\mathcal{X}(k)}, \quad (3.9)$$

where

$$\nu(k_0) = -\frac{1}{2\pi}\log(1 + |\gamma(k_0)|^2),$$
$$\mathcal{X}(k) = \frac{1}{2\pi\mathrm{i}}\int_{-k_0}^{k_0}\log\left(\frac{1+|\gamma(\xi)|^2}{1+|\gamma(k_0)|^2}\right)\frac{\mathrm{d}\xi}{\xi-k}. \quad (3.10)$$

- Along the ray $k = \pm k_0 + \mathbb{R}^+\mathrm{e}^{\mathrm{i}\phi}$ with $|\phi| \leqslant c < \pi$, as $k \to \pm k_0$,

$$\left|\det(\delta(k)) - \left(\frac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu(k_0)}\mathrm{e}^{\mathcal{X}(\pm k_0)}\right| \lesssim |k \mp k_0|^{1/2}. \quad (3.11)$$

Proof. The proof of the above properties is similar to the proof of proposition 3.1 in [12]. □

In what follows, our aim is to find a transform of $M(k; x, t) \to M^{(1)}(k; x, t)$ such that the jump matrix of $M^{(1)}(k; x, t)$ can be well decomposed. Let

$$M^{(1)}(k; x, t) = M(k; x, t)T(k), \quad (3.12)$$

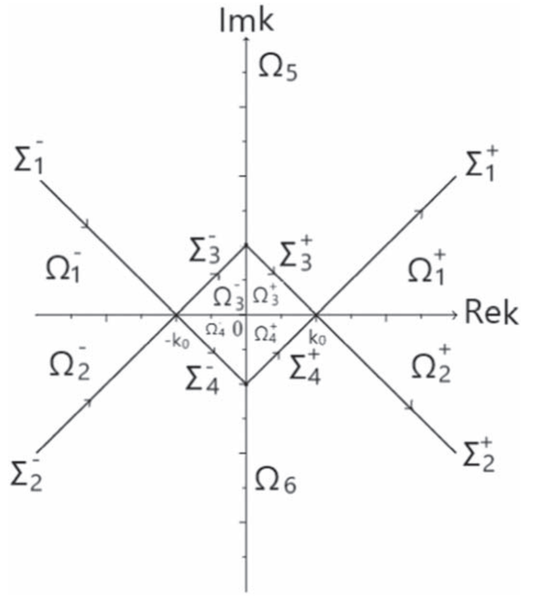

Figure 2. Deformation of the jump countor from $\mathbb{R}$ to $\Sigma^{(2)}$.

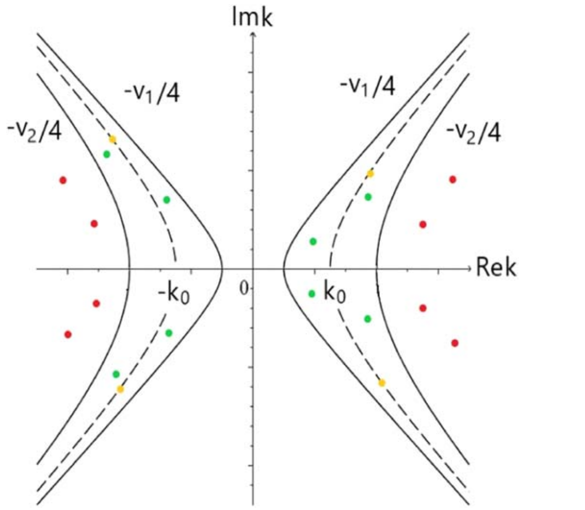

Figure 3. Pole distribution. The red, green and yellow points generate the breather solutions. Moreover, the red points lie in the region $K^{+}(\mathcal{I})$, the green points lie in the region $K(\mathcal{I})$, and the yellow points on the line $\mathrm{Re}\, i\theta(k) = 0$.

where

$$T(k) = \begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det[\delta(k)] \end{pmatrix} = \begin{pmatrix} T_1^{-1}(k) & 0 \\ 0 & T_2(k) \end{pmatrix} \tag{3.13}$$

with $\delta(k)$, $\det[\delta(k)]$ are given by proposition 1, then the modified $M^{(1)}(k; x, t)$ satisfies the following RHP-2:

RHP-2. Find a matrix-valued function $M^{(1)}(k; x,\ t)$ satisfying

- Analyticity: $M^{(1)}(k)$ is analytic in $\mathbb{C}\backslash(\mathbb{R} \cup K \cup \bar{K})$.
- Jump condition: $M_{+}^{(1)}(k) = M_{-}^{(1)}(k) V^{(1)}(k), \quad k \in \mathbb{R}$, where the jump matrix is

$$V^{(1)}(k) = \begin{cases} \begin{pmatrix} I & T_1 T_2 \gamma^{\dagger}(k) \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ (T_1 T_2)^{-1} \gamma(k) \mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & k \in \mathbb{R}\backslash[-k_0, k_0], \\ \begin{pmatrix} I & 0 \\ \dfrac{(T_{1-} T_{2-})^{-1} \gamma(k)}{1 + \gamma(k) \gamma^{\dagger}(k)} \mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix} \begin{pmatrix} I & \dfrac{T_{1+} T_{2+} \gamma^{\dagger}(k)}{1 + \gamma(k)\gamma^{\dagger}(k)} \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & k \in (-k_0, k_0). \end{cases} \tag{3.14}$$

- Asymptotics: $M^{(1)}(k) = I + \mathcal{O}(k^{-1}), \quad \text{as} \quad k \to \infty$.

Moreover, $M^{(1)}(k; x,\ t)$ satisfies the following residue conditions at double poles $k_j \in K$ and $k_j^{*} \in \overline{K}$:

$$\underset{k=k_j}{\mathrm{P}_{-2}} M^{(1)}(k;\ x,\ t) = \lim_{k \to k_j} M^{(1)}(k) \begin{pmatrix} 0 & 0 \\ \mathcal{A}_j T_1^{-1} T_2^{-1} \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \tag{3.15}$$

$$\begin{aligned} \mathrm{Res}_{k=k_j} M^{(1)}(k;\ x,\ t) &= \lim_{k \to k_j} M^{(1)}(k) \\ &\times \begin{pmatrix} 0 & 0 \\ (\mathcal{B}_j T_1^{-1} T_2^{-1} + \mathcal{A}_j T_1^{-1} T_2^{-1\prime}) \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix} \\ &+ M^{(1)\prime}(k) \begin{pmatrix} 0 & 0 \\ \mathcal{A}_j T_1^{-1} T_2^{-1} \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \end{aligned} \tag{3.16}$$

$$\underset{k=k_j^{*}}{\mathrm{P}_{-2}} M^{(1)}(k;\ x,\ t) = \lim_{k \to k_j^{*}} M^{(1)} \begin{pmatrix} 0 & -\mathcal{A}_j^{\dagger} T_1 T_2 \mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 0 \end{pmatrix}, \tag{3.17}$$

$$\begin{aligned} \mathrm{Res}_{k=k_j^{*}} M^{(1)}(k;\ x,\ t) &= \lim_{k \to k_j^{*}} M^{(1)}(k) \\ &\times \begin{pmatrix} 0 & (-\mathcal{B}_j^{\dagger} T_1 T_2 - \mathcal{A}_j^{\dagger} T_2 T_1') \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix} \\ &+ M^{(1)\prime}(k) \begin{pmatrix} 0 & -\mathcal{A}_j^{\dagger} T_1 T_2 \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{3.18}$$

To open the original jump curve $\mathbb{R}$ along the steepest descent lines arising from the phase points $\pm k_0$, let these contours be

$$\begin{aligned}
&\Sigma_1^{\pm}=\{k \mid k\mp k_0=\mathbb{R}^{+}\mathrm{e}^{\mathrm{i}\frac{(2\pm1)\pi}{4}}\},\\
&\Sigma_3^{\pm}=\{k \mid k\mp k_0=d\mathrm{e}^{\mathrm{i}\frac{(2\pm1)\pi}{4}},\quad d\in(0,\sqrt{2}k_0)\},\\
&\Sigma_2^{\pm}=\{k \mid k\mp k_0=\mathbb{R}^{+}\mathrm{e}^{\mathrm{i}\frac{(-2\pm1)\pi}{4}}\},\\
&\Sigma_4^{\pm}=\{k \mid k\mp k_0=d\mathrm{e}^{\mathrm{i}\frac{(-2\mp1)\pi}{4}},\quad d\in(0,\sqrt{2}k_0)\}.
\end{aligned}\tag{3.19}$$

Then the complex plane $\mathbb{C}$ is divided into ten open domains, denoted by $\Omega_j^{\pm}$, $j=1,2,3,4$ and Ω_5, Ω_6 (see figure 2). In what follows, we will introduce the continuous functions related to the jump matrix $V^{(1)}$ in these regions.

Proposition 2. *Let $D=(-k_0,k_0)$, $D_-=(-\infty,-k_0)$, $D_+=(k_0,+\infty)$. Then there exists the continuous functions $R_j^{\pm}$: $\overline{\Omega}_j^{\pm}\to C$, $j=1,2,3,4$ such that*

$$R_1^{\pm}(k)=\begin{cases}\cos(2\arctan(k\mp k_0))g_1^{\pm}+[1-\cos(2\arctan(k\mp k_0))]f_1^{\pm}, & k\in\overline{\Omega}_1^{\pm},\\ g_1^{\pm}=-\gamma(k)T_1^{-1}(k)T_2^{-1}(k), & k\in D_{\pm},\\ f_1^{\pm}=-\gamma(\pm k_0)T_1^{-1}(k)\mathrm{e}^{-\mathcal{X}(\pm k_0)}\left(\dfrac{k-k_0}{k+k_0}\right)^{-\mathrm{i}\nu}(1-\mathcal{X}_K(k)), & k\in\Sigma_1^{\pm},\end{cases}\tag{3.20}$$

$$R_2^{\pm}(k)=\begin{cases}\cos(2\arctan(k\mp k_0))g_2^{\pm}+[1-\cos(2\arctan(k\mp k_0))]f_2^{\pm}, & k\in\overline{\Omega}_2^{\pm},\\ g_2^{\pm}=T_1(k)T_2(k)\gamma^{\dagger}(k), & k\in D_{\pm},\\ f_2^{\pm}=T_1(k)\mathrm{e}^{\mathcal{X}(\pm k_0)}\left(\dfrac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu}\gamma^{\dagger}(\pm k_0)(1-\mathcal{X}_K(k)), & k\in\Sigma_2^{\pm},\end{cases}\tag{3.21}$$

$$R_3^{\pm}(k)=\begin{cases}\cos(2\arctan(k\mp k_0))g_3+[1-\cos(2\arctan(k\mp k_0))]f_3^{\pm}, & k\in\overline{\Omega}_3^{\pm},\\ g_3=-T_{1+}(k)T_{2+}(k)\dfrac{\gamma^{\dagger}(k)}{1+\gamma(k)\gamma^{\dagger}(k)}, & k\in D,\\ f_3^{\pm}=-T_1(k)\mathrm{e}^{\mathcal{X}(\pm k_0)}\left(\dfrac{k-k_0}{k+k_0}\right)^{\mathrm{i}\nu}\dfrac{\gamma^{\dagger}(\pm k_0)}{1+\gamma(\pm k_0)\gamma^{\dagger}(\pm k_0)}(1-\mathcal{X}_K(k)), & k\in\Sigma_3^{\pm},\end{cases}\tag{3.22}$$

$$R_4^{\pm}(k)=\begin{cases}\cos(2\arctan(k\mp k_0))g_4+[1-\cos(2\arctan(k\mp k_0))]f_4^{\pm}, & k\in\overline{\Omega}_4^{\pm},\\ g_4=\dfrac{\gamma(k)}{1+\gamma(k)\gamma^{\dagger}(k)}T_{1-}^{-1}(k)T_{2-}^{-1}(k), & k\in D,\\ f_4^{\pm}=\dfrac{\gamma(\pm k_0)}{1+\gamma(\pm k_0)\gamma^{\dagger}(\pm k_0)}T_1^{-1}(k)\mathrm{e}^{-\mathcal{X}(\pm k_0)}\left(\dfrac{k-k_0}{k+k_0}\right)^{-\mathrm{i}\nu}(1-\mathcal{X}_K(k)), & k\in\Sigma_4^{\pm},\end{cases}\tag{3.23}$$

and $R_j^{\pm}(k)(j=1,2,3,4)$ have these estimates

$$\begin{aligned}
&|R_j^{\pm}(k)|\lesssim 1+\langle\mathrm{Re}(k)\rangle^{-1/2},\\
&|\bar\partial R_j^{\pm}(k)|\lesssim|\bar\partial\chi_K(k)|+|h_j{}'(k)(\mathrm{Re}(k))|+|k\mp k_0|^{-1/2},\\
&\bar\partial R_j\pm(k)=0,\ \text{for}\ k\in\Omega_5\cup\Omega_6\qquad\text{or}\qquad \mathrm{dist}(k,K\cup\bar K)<\rho/3,
\end{aligned}\tag{3.24}$$

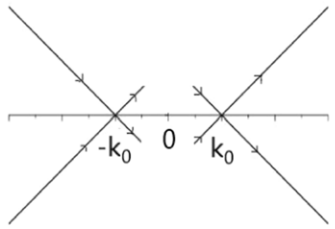

Figure 4. The jump contour for the jump matrix $V^{(\mathrm{in})}$.

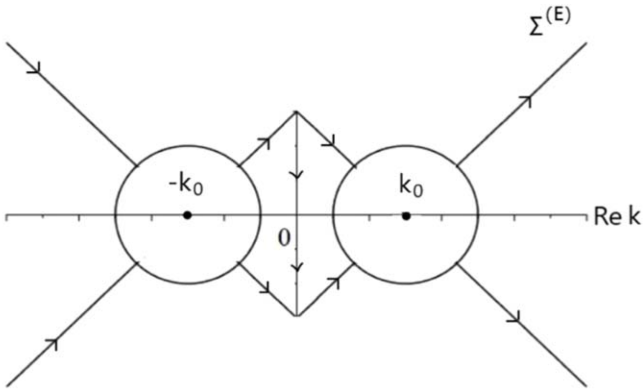

Figure 5. The jump contour $\Sigma^{(E)}$ for the error function $E(k)$.

where

$$h_1(k) = \gamma(k), \quad h_2(k) = \gamma^{\dagger}(k^*),$$
$$h_3(k) = \frac{h_2(k)}{1 + h_1(k)h_2(k)},$$
$$h_4(k) = \frac{h_1(k)}{1 + h_1(k)h_2(k)}.$$

and

$$\rho = \frac{1}{2}\min_{\lambda\neq\mu\in K\cup\bar{K}}|\lambda-\mu|, \quad \mathcal{X}_K(k) = \begin{cases} 1, & \mathrm{dist}(k, K\cup\bar{K}) < \frac{\rho}{3}, \\ 0, & \mathrm{dist}(k, K\cup\bar{K}) > \frac{2\rho}{3}. \end{cases}$$

Proof. The proof is similar to [12, 32]. □

Let

$$M^{(2)}(k;x,t) = M^{(1)}(k;x,t)R^{(2)}(k;x,t), \tag{3.25}$$

where $R^{(2)}$ is defined as

$$R^{(2)}(k) = \begin{cases} \begin{pmatrix} I & 0 \\ R_j^{\pm}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 1 \end{pmatrix}, & j = 1, 3, \\ \begin{pmatrix} I & R_j^{\pm}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 1 \end{pmatrix}, & j = 2, 4, \\ I_{3\times3}, & \text{otherwise} \end{cases} \tag{3.26}$$

with $R_j^{\pm}(k)$ being defined by proposition 2 (notice that the transform causes the previous contour $\mathbb{R}$ to change into contour $\Sigma^{(2)}$). Then $M^{(2)}(k;x,t)$ solves a mixed $\bar{\partial}$-RH problem:

Mixed $\bar{\partial}$-RHP. Find a matrix-valued function $M^{(2)}(k) = M^{(2)}(k;x,t)$ solving

- Continuity: $M^{(2)}(k)$ is continuous in $\mathbb{C}\setminus(\Sigma^{(2)}\cup K\cup\bar{K})$.
- Jump condition: $M_+^{(2)}(k) = M_-^{(2)}(k)V^{(2)}(k)$, $k\in\Sigma^{(2)}$, where

$$V^{(2)}(k) = \begin{cases} \begin{pmatrix} I & 0 \\ (-1)^j R_j^{\pm}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k\in\Sigma_j^{\pm},\ j = 1, 4, \\ \begin{pmatrix} I & (-1)^j R_j^{\pm}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k\in\Sigma_j^{\pm},\ j = 2, 3, \\ \begin{pmatrix} I & (R_3^+ - R_3^-)\mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & k\in(\mathrm{i}k_0\tan(\frac{\pi}{12}),\ \mathrm{i}k_0), \\ \begin{pmatrix} I & 0 \\ (R_4^+ - R_4^-)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & k\in(-\mathrm{i}k_0\tan(\frac{\pi}{12}),\ -\mathrm{i}k_0), \\ I_{3\times3}, & k\in(-\mathrm{i}k_0\tan(\frac{\pi}{12}),\ \mathrm{i}k_0\tan(\frac{\pi}{12})). \end{cases} \tag{3.27}$$

- Asymptotics: $M^{(2)}(k)\to I$, $k\to\infty$;

Moreover, for any $k\in\mathbb{C}\setminus(\Sigma^{(2)}\bigcup K\bigcup\bar{K})$, one finds that

$$\bar{\partial}M^{(2)}(k) = M^{(1)}(k)\bar{\partial}R^{(2)}(k), \tag{3.28}$$

where

$$\bar{\partial}R^{(2)}(k) = \begin{cases} \begin{pmatrix} 0 & 0 \\ \bar{\partial}R_j^{\pm}(k)\mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, & j = 1, 3, \\ \begin{pmatrix} 0 & \bar{\partial}R_j^{\pm}(k)\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix}, & j = 2, 4, \\ 0_{3\times3}, & \text{otherwise}, \end{cases} \tag{3.29}$$

and, $M^{(2)}(k;x,t)$ has double poles at k_j and k_j^* with

$$\underset{k=k_j}{\mathrm{P}_{-2}}M^{(2)}(k;x,t) = \lim_{k\to k_j}M^{(2)}(k)\begin{pmatrix} 0 & 0 \\ \mathcal{A}_jT_1^{-1}T_2^{-1}\mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \tag{3.30}$$

$$\begin{aligned} \mathrm{Res}_{k=k_j}M^{(2)}(k;x,t) &= \lim_{k\to k_j}M^{(2)}(k) \\ &\times\begin{pmatrix} 0 & 0 \\ (\mathcal{B}_jT_1^{-1}T_2^{-1} + \mathcal{A}_jT_1^{-1}(T_2^{-1})')\mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix} \\ &+ (M^{(2)})'(k)\begin{pmatrix} 0 & 0 \\ \mathcal{A}_jT_1^{-1}T_2^{-1}\mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \end{aligned} \tag{3.31}$$

$$\underset{k=k_j^*}{\mathrm{P}_{-2}}M^{(2)}(k;x,t) = \lim_{k\to k_j^*}M^{(2)}\begin{pmatrix} 0 & \hat{A}_jT_1T_2\mathrm{e}^{2\mathrm{i}t\theta} \\ 0 & 0 \end{pmatrix}, \tag{3.32}$$

$$\mathrm{Res}_{k=k_j^*} M^{(2)}(k;x,t) = \lim_{k\to k_j^*} M^{(2)}(k) \times \begin{pmatrix} 0 & (\hat{B}_j T_1 T_2 + \hat{A}_j T_2 (T_1)')\mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix} + (M^{(2)})'(k) \begin{pmatrix} 0 & \hat{A}_j T_1 T_2 \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix}. \tag{3.33}$$

3.2. Analysis on a pure RH problem

Throughout this section, our aim is to decompose the above-mentioned mixed $\bar{\partial}$-RHP into a pure RHP with $\bar{\partial}R^{(2)} = 0$ and a pure $\bar{\partial}$-problem with $\bar{\partial}R^{(2)} \neq 0$. The decomposition of $M^{(2)}(k;x,t)$ can be given as follows:

$$M^{(2)}(k;x,t) = \begin{cases} M^{(2)}_{\mathrm{RHP}}(k;x,t), & \text{as } \bar{\partial}R^{(2)} = 0, \\ M^{(3)}(k;x,t) M^{(2)}_{\mathrm{RHP}}(k;x,t), & \text{as } \bar{\partial}R^{(2)} \neq 0, \end{cases} \tag{3.34}$$

where $M^{(2)}_{\mathrm{RHP}}(k;x,t)$ and $M^{(3)}(k;x,t)$ correspond to the pure RHP part and the pure $\bar{\partial}$ part without jumps and poles of $M^{(2)}(k)$, respectively.

RHP-3. Find a matrix-valued function $M^{(2)}_{\mathrm{RHP}}(k)$ solving the following RHP

- Analyticity: $M^{(2)}_{\mathrm{RHP}}(k)$ is analytic in $\mathbb{C}\backslash(\Sigma^{(2)} \cup K \cup \bar{K})$.
- Jump condition: $M^{(2)}_{\mathrm{RHP}+}(k) = M^{(2)}_{\mathrm{RHP}-}(k) V^{(2)}(k)$, $k \in \Sigma^{(2)}$, where $V^{(2)}(k)$ is given by equation (3.27).
- Asymptotics: $M^{(2)}_{\mathrm{RHP}}(k) \to I,\ k \to \infty$.

Proposition 3. *The jump matrix $V^{(2)}$ has the following estimate:*

$$\|V^{(2)}(k;x,t) - I\|_{L^\infty(\Sigma^{(2)})} = \begin{cases} \mathcal{O}(\mathrm{e}^{-6k_0\rho^2 t}), & k \in \Sigma_j^{\pm}\backslash\partial\mathcal{U}_{\pm k_0}, \quad j=1,2, \\ \mathcal{O}(\mathrm{e}^{-8k_0^2\rho t}), & k \in \Sigma_j^{\pm}\backslash\partial\mathcal{U}_{\pm k_0}, \quad j=3,4, \\ \mathcal{O}((k_0|k \mp k_0|)^{-1} t^{-1/2}), & k \in \Sigma^{(2)} \bigcap \mathcal{U}_{\pm k_0}, \\ \mathcal{O}(\mathrm{e}^{-14k_0^3 \tan^3(\frac{\pi}{12})t}), & k \in [\pm \mathrm{i}k_0, \pm \mathrm{i}k_0 \tan(\frac{\pi}{12})], \\ 0, & k \in [-\mathrm{i}k_0 \tan(\frac{\pi}{12}), \mathrm{i}k_0 \tan(\frac{\pi}{12})], \end{cases} \tag{3.35}$$

where $\mathcal{U}_{\pm k_0} = \{k \mid |k \pm k_0| < \rho/2\}$.

3.2.1. Soliton solutions corresponding to discrete spectra.

In order to analyze the leading term of the solution, we firstly consider RHP-1. RHP-1 reduces to the following RH problem:

RHP-4. A matrix-valued function $M(k;x,t|\sigma_d)$ with the scattering data $\sigma_d = \{(k_j, \mathcal{A}_j, \mathcal{B}_j)\}_{k=1}^{2N}$ and $K = \{k_j\}_{j=1}^{2N}$ and satisfies the following condition:

- Analyticity: $M(k;x,t|\sigma_d)$ is analytical in $\mathbb{C}\backslash(\Sigma^{(2)} \cup K \cup \bar{K})$.
- Jump condition:

$$M_+(k;x,t|\sigma_d) = M_-(k;x,t|\sigma_d) V(k), \tag{3.36}$$

- Asymptotics: $M(k;x,t|\sigma_d) = I + \mathcal{O}(k^{-1}),\ k \to \infty$.

Moreover, $M(k;x,t|\sigma_d)$ has double poles at k_j and k_j^* with

$$\underset{k=k_j}{\mathrm{P}_{-2}} M(k;x,t|\sigma_d) = \lim_{k\to k_j} M(k;x,t|\sigma_d) \begin{pmatrix} 0 & 0 \\ A_j \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \tag{3.37}$$

$$\mathrm{Res}_{k=k_j} M(k;x,t|\sigma_d) = \lim_{k\to k_j} M(k;x,t) \begin{pmatrix} 0 & 0 \\ B_j \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix} + M'(k;x,t|\sigma_d) \begin{pmatrix} 0 & 0 \\ A_j \mathrm{e}^{2\mathrm{i}t\theta(k)} & 0 \end{pmatrix}, \tag{3.38}$$

$$\underset{k=k_j^*}{\mathrm{P}_{-2}} M(k;x,t|\sigma_d) = \lim_{k\to k_j^*} M(k;x,t|\sigma_d) \times \begin{pmatrix} 0 & -A_j^\dagger \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix}, \tag{3.39}$$

$$\mathrm{Res}_{k=k_j^*} M(k;x,t|\sigma_d) = \lim_{k\to k_j^*} M(k;x,t|\sigma_d) \times \begin{pmatrix} 0 & -B_j^\dagger \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix} + M'(k;x,t|\sigma_d) \begin{pmatrix} 0 & -A_j^\dagger \mathrm{e}^{-2\mathrm{i}t\theta(k)} \\ 0 & 0 \end{pmatrix}. \tag{3.40}$$

Proposition 4. *Given scattering data $\sigma_d = (k_j, \mathcal{A}_j, \mathcal{B}_j))_{k=1}^{2N}$ and discrete spectra $K = \{k_j\}_{j=1}^{2N}$, the RH problem has a unique solution*

$$q_{\mathrm{sol}}(x,t|\sigma_d) = (q(x,t), q^*(x,t))^{\mathrm{T}} = 2\mathrm{i} \lim_{k\to\infty} (kM(k|\sigma_d))_{12}. \tag{3.41}$$

Proof. The uniqueness of the solution can be guaranteed by Liouville's theorem. For the reflectionless case $V(k) = I$, it follows from equation (3.36) and the Plemelj formula that one has

$$M(k|\sigma_d) = I + \sum_{j=1}^{2N} \frac{\mathrm{Res}_{k=k_j} M(k \mid \sigma_d)}{k - k_j} + \sum_{j=1}^{2N} \frac{\mathrm{Res}_{k=k_j^*} M(k \mid \sigma_d)}{k - k_j^*} + \sum_{j=1}^{2N} \frac{\underset{k=k_j}{\mathrm{P}_{-2}} M(k \mid \sigma_d)}{(k-k_j)^2} + \frac{\underset{k=k_j^*}{\mathrm{P}_{-2}} M(k \mid \sigma_d)}{(k-k_j^*)^2}. \tag{3.42}$$

One can find that $M(k|\sigma_d)$ has the following formulation of the sum of sparse matrices:

$$M(k) = I + \sum_{l=1}^{2N}\left(\frac{(\beta_l \quad 0_{3\times 1})}{k - k_l} + \frac{(0_{3\times 2} \quad \widetilde{\beta_l})}{k - k_l^*} + \frac{(\alpha_l \quad 0_{3\times 1})}{(k - k_l)^2} + \frac{(0_{3\times 2} \quad \widetilde{\alpha_l})}{(k - k_l^*)^2}\right), \tag{3.43}$$

which, together with the residue condition, can further lead to the following equations:

$$(\alpha_j \quad 0) = \begin{pmatrix} 0 & 0 \\ \eta_j & 0 \end{pmatrix} + \sum_{l=1}^{2N}\left(\frac{(\widetilde{\beta_l}\eta_j \quad 0)}{k_j - k_l^*} + \frac{(\alpha_l^*\eta_j \quad 0)}{(k_j - k_l^*)^2}\right), \tag{3.44}$$

$$(\beta_j \quad 0) = \begin{pmatrix} 0 & 0 \\ \eta_j & 0 \end{pmatrix} + \sum_{l=1}^{2N}\left(\frac{(\beta_l^*\zeta_j \quad 0)}{k_j - k_l^*} + \frac{(\alpha_l^*\zeta_j \quad 0)}{(k_j - k_l^*)^2} - \frac{(\beta_l^*\eta_j \quad 0)}{(k_j - k_l^*)^2} - 2\frac{(\alpha_l^*\eta_j \quad 0)}{(k_j - k_l^*)^3}\right), \tag{3.45}$$

$$(0 \quad \alpha_j^*) = \begin{pmatrix} 0 & -\eta_j^\dagger \\ 0 & 0 \end{pmatrix} + \sum_{l=1}^{2N}\left(\frac{(0 \quad -\beta_l\eta_j^\dagger)}{k_j^* - k_l} + \frac{(0 \quad -\alpha_l\eta_j^\dagger)}{(k_j^* - k_l)^2}\right), \tag{3.46}$$

$$(0 \quad \beta_j^*) = \begin{pmatrix} 0 & -\eta_j^\dagger \\ 0 & 0 \end{pmatrix} + \sum_{l=1}^{2N}\left(\frac{(0 \quad -\beta_l\zeta_j^\dagger)}{k_j^* - k_l} + \frac{(0 \quad -\alpha\zeta_j^\dagger)}{k_j^* - k_l} + \frac{(0 \quad \beta_l\eta_j^\dagger)}{(k_j - k_l^*)^2} + 2\frac{(0 \quad \alpha_l\eta_j^\dagger)}{(k_j - k_l^*)^3}\right), \tag{3.47}$$

where $\zeta_j = B_j e^{2it\theta(k_j)}$, $\eta_j = A_j e^{2it\theta(k_j)}$. Then α_l, β_l, $\widetilde{\alpha_l}$, $\widetilde{\beta_l}$ can be solved from the above equations. □

In what follows, we separate $M_{\mathrm{RHP}}^{(2)}(k)$ into two parts:

$$M_{\mathrm{RHP}}^{(2)}(k) = \begin{cases} E(k)M^{(\mathrm{out})}(k), & k \in \mathbb{C}\backslash\mathcal{U}_{\pm k_0}, \\ E(k)M^{(\mathrm{LC})}(k),\ M^{(\mathrm{LC})}(k) = M^{(\mathrm{out})}(k)M^{(\mathrm{in})}(k), & k \in \mathcal{U}_{\pm k_0}, \end{cases} \tag{3.48}$$

where $M^{(\mathrm{out})}$ is used to find the pure solitions outside $\mathcal{U}_{\pm k_0}$, which is defined in $\mathbb{C}$ and only admits discrete spectra without a jump. $M^{(\mathrm{in})}$ is defined in $\mathcal{U}_{\pm k_0}$ without discrete spectra, and the model RHP considered by Liu [31]. Moreover, $E(k)$ denotes the error between $M_{\mathrm{RHP}}^{(2)}(k)$ and $M^{(\mathrm{out})}(k)$ outside $\mathcal{U}_{\pm k_0}$.

Let

$$M^{(\mathrm{out})}(k;\, x,\, t|\sigma_d^{(\mathrm{out})}) = M(k;\, x,\, t|\sigma_d)\begin{pmatrix} \delta^{-1}(k) & 0 \\ 0 & \det\delta(k) \end{pmatrix}, \tag{3.49}$$

with the scattering data

$$\sigma_d^{(\mathrm{out})} = \{(k_j,\, \widetilde{\mathcal{A}}_j,\, \widetilde{\mathcal{B}}_j,\, k_j \in K\}_{j=1}^{2N},$$
$$\{\widetilde{\mathcal{A}}_j,\, \widetilde{\mathcal{B}}_j\} = \{\mathcal{A}_j,\, \mathcal{B}_j\}\,\delta^{-1}(k_j)(\det\delta(k_j))^{-1}. \tag{3.50}$$

Then $M^{(\mathrm{out})}(k|\sigma_d^{(\mathrm{out})})$ satisfies the following RH problem:

RHP-5. Find a matrix-valued function $M^{(\mathrm{out})}(k;\, x,\, t|\sigma_d^{\mathrm{out}})$ without the jump condition solving the following problem:

- Analyticity: $M^{(\mathrm{out})}(k|\sigma_d^{\mathrm{out}})$ is analytic in $\mathbb{C}\backslash(\Sigma^{(2)} \cup K \cup \overline{K})$.
- Asymptotics: $M^{(\mathrm{out})}(k|\sigma_d^{\mathrm{out}}) \to I,\ k \to \infty$,

$M^{(\mathrm{out})}(k|\sigma_d^{\mathrm{out}})$ has double poles at k_j and k_j^* with

$$\underset{k=k_j}{\mathrm{P}_{-2}}M^{(1)}(k;\, x,\, t) = \lim_{k\to k_j} M^{(1)}(k)\begin{pmatrix} 0 & 0 \\ A_jT_1^{-1}T_2^{-1}e^{2it\theta(k)} & 0 \end{pmatrix}, \tag{3.51}$$

$$\mathrm{Res}_{k=k_j}M^{(1)}(k;\, x,\, t) = \lim_{k\to k_j} M^{(1)}(k) \times \begin{pmatrix} 0 & 0 \\ (B_jT_1^{-1}T_2^{-1} + A_jT_1^{-1}(T_2^{-1})')e^{2it\theta(k)} & 0 \end{pmatrix} + (M^{(1)})'(k)\begin{pmatrix} 0 & 0 \\ A_jT_1^{-1}T_2^{-1}e^{2it\theta(k)} & 0 \end{pmatrix}, \tag{3.52}$$

$$\underset{k=k_j^*}{\mathrm{P}_{-2}}M^{(1)}(k;\, x,\, t) = \lim_{k\to k_j^*} M^{(1)}\begin{pmatrix} 0 & \hat{A}_jT_1T_2e^{2it\theta} \\ 0 & 0 \end{pmatrix}, \tag{3.53}$$

$$\mathrm{Res}_{k=k_j^*}M^{(1)}(k;\, x,\, t) = \lim_{k\to k_j^*} M^{(1)}(k) \times \begin{pmatrix} 0 & (\hat{B}_jT_1T_2 + \hat{A}_jT_2(T_1)')e^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix} + (M^{(1)})'(k)\begin{pmatrix} 0 & \hat{A}_jT_1T_2e^{-2it\theta(k)} \\ 0 & 0 \end{pmatrix}. \tag{3.54}$$

Proposition 5. *RHP-5 has the uniqueness solution, and its potential is equivalent to one of the reflectionless cases of RHP-4, that is*

$$q_{\mathrm{sol}}(x,\, t|\sigma_d^{(\mathrm{out})}) = q_{\mathrm{sol}}(x,\, t|\sigma_d) = (q(x,\, t),\, q^*(x,\, t))^T = 2i\lim_{k\to\infty}(kM(k|\sigma_d))_{12}. \tag{3.55}$$

Proof. According to the reconstruction formula (2.24), the proof is similar to [32]. □

We now consider the long-time asymptotic behavior of soliton solutions. Firstly, we define a space-time region

$$\mathcal{D}(v_1,\, v_2) = \{(x,\, t) \in \mathbb{R}^2|x = vt,\, v \in [v_1,\, v_2]\}, \tag{3.56}$$

where $\nu_2 \leqslant \nu_1 < 0$. Let

$$\begin{aligned}
&\mathcal{I} = [-\frac{\nu_1}{4}, -\frac{\nu_2}{4}],\\
&K(\mathcal{I}) = \left\{k_j \in K | -\frac{\nu_1}{4} \leqslant K_j \leqslant -\frac{\nu_2}{4}\right\},\\
&N(\mathcal{I}) = |K(\mathcal{I})|,\\
&K^-(\mathcal{I}) = \left\{k_j \in K | K_j < -\frac{\nu_1}{4}\right\},\\
&K^+(\mathcal{I}) = \left\{k_j \in K | K_j > -\frac{\nu_2}{4}\right\},\\
&K_j = 3\,\mathrm{Re}^2 k_j - \mathrm{Im}^2 k_j\\
&c_j(\mathcal{I}) = c_j \delta^{-1}(k_j) \mathrm{e}^{\frac{\mathrm{i}}{2\pi}\int_{-k_0}^{k_0} \frac{\log(1+|\gamma(\zeta)|^2)}{\zeta - k_j}\mathrm{d}\zeta}.
\end{aligned} \tag{3.57}$$

See figure 3. Then we have the following proposition:

Proposition 6. *Given scattering data $\sigma_d = \{(k_j, \mathcal{A}_j, \mathcal{B}_j)\}_{j=1}^{2N}$ and $\sigma_d(\mathcal{I}) = \{(k_j, c_j(\mathcal{I})) | k_j \in K(\mathcal{I})\}$. At $t \to +\infty$ with $(x, t) \in \mathcal{D}(\nu_1, \nu_2)$, we have*

$$M(k; x, t|\sigma_d) = [I + \mathcal{O}(\mathrm{e}^{-8\mu t})] M^{\Delta_{\mathcal{I}}}(k; x, t|\sigma_d(\mathcal{I})), \tag{3.58}$$

where $\mu(\mathcal{I}) = \min_{k_j \in K/K(\mathcal{I})}\{\mathrm{Im}\,k_j \cdot \mathrm{dist}(3\,\mathrm{Re}^2 k_j - \mathrm{Im}^2 k_j, \mathcal{I})\}$.

Proof. The proof is similar to [13, 32]. □

Corollary 1. *Suppose that q_{sol} is the soliton solution of the SS equation corresponding to its scattering data $\sigma_d = \{(k_j, \mathcal{A}_j, \mathcal{B}_j)\}_{j=1}^{2N}$, then one has*

$$q_{\mathrm{sol}}(x, t|\sigma_d^{(\mathrm{out})}) = q_{\mathrm{sol}}(x, t|\sigma_d(\mathcal{I})) + \mathcal{O}(\mathrm{e}^{-8\mu t}), \quad t \to +\infty, \tag{3.59}$$

where $q_{\mathrm{sol}}(x, t|\sigma_d(\mathcal{I}))$ is the soliton solution corresponding to the scattering data $\sigma_d(\mathcal{I})$ of the SS equation.

3.2.2. The solvable local RH problem. **RHP-6**. Find a matrix-valued function $M^{(\mathrm{in})}(k; x, t)$ which satisfies

- Analyticity: $M^{(\mathrm{in})}(k; x, t)$ is analytical in $\mathbb{C}\backslash\Sigma^{(2)}$ with symmetry: $M^{(\mathrm{in})}(k) = \varrho M^{(\mathrm{in})*}(-k^*)\varrho$.
- Jump condition: $M^{(\mathrm{in})}(k; x, t)$ has the jump condition

$$M_+^{(\mathrm{in})}(k) = M_-^{(\mathrm{in})}(k) V^{(\mathrm{in})}(k), \; k \in \Sigma^{(2)}. \tag{3.60}$$

where the jump matrix $V^{(\mathrm{in})}(k) = V^{(2)}(k)$ is given by equation (3.27). See figure 4.

- Asymptotics: $M^{(\mathrm{in})}(k) \to I, \; k \to \infty$.

RHP-6 is a solvable model for the SS equation. Here, we mainly adopt the final results for solving the model RHP (see [31] for more details), whose solution has the asymptotics:

$$\begin{aligned}
M^{(\mathrm{in})}(k) = I &- \frac{1}{\sqrt{48tk_0}(k + k_0)} M_1^{(\mathrm{in})}\\
&+ \frac{1}{\sqrt{48tk_0}(k - k_0)} \varrho (M_1^{(\mathrm{in})})^* \varrho + \mathcal{O}(\frac{\log t}{t}),
\end{aligned} \tag{3.61}$$

with $\|M^{(\mathrm{in})}\|_\infty \lesssim 1$, where

$$M_1^{(\mathrm{in})} = \begin{pmatrix} 0 & \mathrm{i}\varpi^{-2}\beta_{12} \\ -\mathrm{i}\varpi^2\beta_{21} & 0 \end{pmatrix}, \tag{3.62}$$

with

$$\begin{aligned}
&\varpi = (192\tau)^{\frac{\mathrm{i}\nu}{2}} \mathrm{e}^{\mathcal{X}(-k_0) - 8\mathrm{i}\tau},\\
&\beta_{12} = -\beta_{21}^\dagger = \frac{\nu\Gamma(-\mathrm{i}\nu)\mathrm{e}^{\frac{\pi(2\nu - \mathrm{i})}{4}}}{\sqrt{2\pi}} \sigma_2 \gamma^T(k_0).
\end{aligned} \tag{3.63}$$

According to RHP-5 and RHP-6, one has the solvable local model RHP with $M^{(LC)}(k) = M^{(\mathrm{out})}(k) M^{(\mathrm{in})}(k)$ inside $\mathcal{U}_{\pm k_0}$ which is a bounded function in $\mathcal{U}_{\pm k_0}$ and has the same jump condition as $M_{\mathrm{RHP}}^{(2)}(k)$.

3.2.3. A small norm RH problem. In this section, we mainly consider the small norm RHP corresponding to the error matrix function $E(k)$ given by equation (3.48). Firstly, according to the definition of $M_{\mathrm{RHP}}^{(2)}(k)$ and $M^{(LC)}(k)$, we can obtain that $E(k)$ satisfies the following RHP:

RHP-7. Find a matrix-valued function $E(k)$ solving

- Analyticity: $E(k)$ is continuous in $\mathbb{C}\backslash\Sigma^{(E)}$, where $\Sigma^{(E)} = \partial\mathcal{U}_{\pm k_0} \cup (\Sigma^{(E)}\backslash\mathcal{U}_{\pm k_0})$.
- Jump condition: $E(k)$ has the following jump condition (see figure 5)

$$E_+(k) = E_-(k) V^{(E)}(k), \qquad k \in \Sigma^{(E)}, \tag{3.64}$$

where matrix $V^{(E)}(k)$ is defined by

$$V^{(E)}(k) = \begin{cases} M^{(\mathrm{out})}(k) V^{(2)}(k) M^{(\mathrm{out})}(k)^{-1}, & k \in \Sigma^{(2)}\backslash\mathcal{U}_{\pm k_0},\\ M^{(\mathrm{out})}(k) M^{(\mathrm{in})}(k) M^{(\mathrm{out})}(k)^{-1}, & k \in \partial\mathcal{U}_{\pm k_0}. \end{cases} \tag{3.65}$$

- Asymptotic behaviors: $E(k) \to I, \; k \to \infty$.

Proposition 7. *The jump matrix $V^{(E)}(k)$ has the following uniform estimate*

$$|V^{(E)}(k) - I| = \begin{cases} \mathcal{O}(\mathrm{e}^{-6k_0\rho^2 t}), & k \in \Sigma_j^{\pm}\backslash\mathcal{U}_{\pm k_0}, \quad j = 1, 2,\\ \mathcal{O}(\mathrm{e}^{-8k_0^2\rho t}), & k \in \Sigma_j^{\pm}\backslash\mathcal{U}_{\pm k_0}, \quad j = 3, 4,\\ \mathcal{O}(\mathrm{e}^{-14k_0^3\tan^3(\pi/12)t}), & k \in [\pm\mathrm{i}k_0, \pm\mathrm{i}k_0\tan(\frac{\pi}{12})],\\ \mathcal{O}(t^{-1/2}), & k \in \partial\mathcal{U}_{\pm k_0},\\ 0, & k \in [-\mathrm{i}k_0\tan(\frac{\pi}{12}), \mathrm{i}k_0\tan(\frac{\pi}{12}). \end{cases} \tag{3.66}$$

Proof. The proof can be seen in [32]. □

Based on the Beals–Coifman theorem, we can construct the solution of RHP-7 in the form

$$E(k) = I + \frac{1}{2\pi\mathrm{i}} \int_{\Sigma^{(E)}} \frac{\kappa_E(\xi)[V^{(E)}(\xi) - I]}{\xi - k} \mathrm{d}\xi, \tag{3.67}$$

where $\kappa_E \in L^2(\Sigma^{(E)})$ satisfies $(I - C_{\omega_E})\kappa_E = I$,

$$\omega_E = (\omega_E)_+ + (\omega_E)_- = V^{(E)} - I, \quad (\omega_E)_- = 0,$$
$$(\omega_E)_+ = V^{(E)} - I,$$
$$C_{\omega_E} g = C_-(g(\omega_E)_+) + C_+(g(\omega_E)_-) = C_-(g(V^{(E)} - I)) \tag{3.68}$$

with C_- denoting the Cauchy projection operator

$$C_- g(k) = \lim_{\zeta \to k \in \Sigma^{(E)}} \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{(E)}} \frac{g(\xi)}{\xi - \zeta} \mathrm{d}\xi, \tag{3.69}$$

and $\|C_-\|_{L^2}$ is a finite value.

Since

$$\|C_{\omega_E}\|_{L^2(\Sigma^{(E)})} \lesssim \|C\|_{L^2(\Sigma^{(E)})} \|V^{(E)} - I\|_{L^\infty(\Sigma^E)} \lesssim \mathcal{O}(t^{-1/2}), \tag{3.70}$$

the matrix function κ_E exists and is unique, and the solution $E(k)$ of RHP-7 exists and is unique.

Proposition 8. *$V^{(E)}$ and κ_E admit the following important estimates*

$$\|\kappa_E - I\|_{L^2(\Sigma^{(E)})} = \mathcal{O}(t^{-1/2}),$$
$$\|V^{(E)} - I\|_{L^p} = \mathcal{O}(t^{-1/2}), \quad p \in [1, +\infty), \quad k \geqslant 0. \tag{3.71}$$

Proposition 9. *The matrix function* E(k) *has the following asymptotics*

$$E(k; x, t) = I + \frac{E_1(x, t)}{k} + \mathcal{O}(k^{-2}), \quad k \to \infty, \tag{3.72}$$

where

$$E_1(x, t) = \frac{\mathrm{i}}{2\pi} \int_{\Sigma^{(E)}} \kappa_E(\xi)(V^E - I)\mathrm{d}\xi. \tag{3.73}$$

Moreover, $E_1(x, t)$ *is given by*

$$E_1(x, t) = \frac{1}{\sqrt{48tk_0}} M^{(\mathrm{out})}(k_0) M_1^{(\mathrm{in})}(k_0) M^{(\mathrm{out})-1}(k_0)$$
$$+ \frac{1}{\sqrt{48tk_0}} M^{(\mathrm{out})}(-k_0) M_1^{B_0}(-k_0) M^{(\mathrm{out})-1}(-k_0)$$
$$+ \mathcal{O}(t^{-1} \log t). \tag{3.74}$$

Proof. The proof is similar as [32]. □

3.3. Analysis on a pure $\bar{\partial}$-problem

Here we consider the pure $\bar{\partial}$-problem which is obtained by removing the pure RHP part with $\bar{\partial} R^{(2)} = 0$. Let

$$M^{(3)}(k) = M^{(2)}(k) M_{\mathrm{RHP}}^{(2)}(k)^{-1}. \tag{3.75}$$

Then we know that $M^{(3)}$ is continuous and has no jumps in the complex plane, and solves a pure $\bar{\partial}$-problem. **Pure $\bar{\partial}$-problem**. Find a matrix-valued function $M^{(3)}(k, x, t)$ solving

- Continuity: $M^{(3)}(k)$ is continuous in $\mathbb{C} \backslash \Sigma^{(2)}$.
- Jump condition: $\bar{\partial} M^{(3)}(k) = M^{(3)}(k) W^{(3)}(k), \quad k \in \mathbb{C}$, where $W^{(3)} = M_{\mathrm{RHP}}^{(2)}(k) \bar{\partial} R^{(2)}(k) M_{\mathrm{RHP}}^{(2)}(k)^{-1}$.
- Asymptotic behaviors: $M^{(3)}(k) \to I, \ k \to \infty$.

The solution of the above pure $\bar{\partial}$-problem can be given by the following integral equation

$$(I - F) M^{(3)}(k) = I, \tag{3.76}$$

where F is the Cauchy operator defined as

$$F[f](k) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{f(\xi) W^{(3)}(\xi)}{\xi - k} \mathrm{d}A(\xi) \tag{3.77}$$

with $\mathrm{d}A(\xi)$ being the Lebesgue measure.

Proposition 10. *For large time* t, *there exists the estimate for* F*:*

$$\|F\|_{L^\infty \to L^\infty} \lesssim (k_0 t)^{-1/4}, \tag{3.78}$$

which implies that the operator $(I - F)^{-1}$ is invertible and the solution of the pure $\bar{\partial}$-problem exists and is unique.

$M^{(3)}(k)$ has the following asymptotic expansion

$$M^{(3)} = I + \frac{M_1^{(3)}}{k} + O(\tfrac{1}{k^2}),$$
$$M_1^{(3)} = \tfrac{1}{\pi} \iint_{\mathbb{C}} M^{(3)}(\xi) W^{(3)}(\xi) \mathrm{d}A(\xi),$$
$$k \to \infty. \tag{3.79}$$

According to proposition 10 and the asympotics of $M_1^{(3)}$, one has

Proposition 11. *For large time* t, *there exists the estimate for* $M_1^{(3)}$

$$|M_1^{(3)}| \lesssim (k_0 t)^{-3/4}. \tag{3.80}$$

4. Long-time asymptotics in the region $x < 0, |x/t| = \mathcal{O}(1)$ and $x > 0, |x/t| = \mathcal{O}(1)$

Based on the above discussions, our main result is summarized as follows:

Theorem 1. *Let $\sigma_d = \{(k_j, \mathcal{A}_j, \mathcal{B}_j), k_j \in K\}_{j=1}^{2N}$ denote the scattering data generated by initial data $q_0(x) \in \mathcal{S}(\mathbb{R})$ with the second-order discrete spectra. For fixed $v_2 \leqslant v_1 \in \mathbb{R}^-$, define $\mathcal{I} = [-v_1/4, -v_2/4]$ and a space-time cone $\mathcal{D}(v_1, v_2)$ for variables* x *and* t*. Let $q_{\mathrm{sol}}(x, t, \sigma_d(\mathcal{I}))$ be the $N(\mathcal{I})$ solution corresponding to the modified scattering data $\sigma_d(\mathcal{I}) = \{(k_j, c_j(\mathcal{I})), k_j \in K(\mathcal{I})\}$. When $x < 0$, as $t \to +\infty$ with $(x, t) \in \mathcal{D}(v_1, v_2)$, we have the long-time asymptotics of the SS equation*

$$q(x, t) = q_{\mathrm{sol}}(x, t | \sigma_d(\mathcal{I})) + p t^{-1/2} + \mathcal{O}(t^{-3/4}), \tag{4.1}$$

where

$$p = \frac{1}{\sqrt{48k_0}}(M^{(\text{out})}(k_0)M_1^{(\text{in})}(k_0)M^{(\text{out})-1}(k_0) - M^{(\text{out})}(-k_0)\varrho(M_1^{(\text{in})}(-k_0))^*\varrho M^{(\text{out})-1}(-k_0))_{12}. \tag{4.2}$$

Similarly, when $x > 0$, as $t \to +\infty$ with $(x, t) \in \mathcal{D}(v_1, v_2)$, we have

$$q(x, t) = q_{\text{sol}}(x, t|\sigma_d(\mathcal{I})) + \mathcal{O}(t^{-1}). \tag{4.3}$$

Proof. Based on a series of transformations (3.12), (3.25), (3.34) and (3.48), we find

$$M(k) = M^{(3)}(k)E(k)M^{(\text{out})}(k)R^{(2)-1}(k)T^{-1}(k).$$

In particular, by considering $k \to \infty$ along the imaginary axis (i.e. in Ω_5, Ω_6), we have

$$M = \left(I + \frac{M_1^{(3)}}{k} + \ldots\right)\left(I + \frac{E_1}{k} + \ldots\right)\left(I + \frac{M_1^{(\text{out})}}{k} + \ldots\right) \times \left(I + \frac{T_1}{k} + \ldots\right) = I + \frac{M_1}{k} + \ldots,$$

which generates

$$M_1 = M_1^{(\text{out})} + E_1 + M_1^{(3)} + T_1. \tag{4.4}$$

According to the reconstruction formula (2.24) and proposition 11, the following estimate holds

$$q(x, t) = 2\text{i}(M_1^{(\text{out})})_{12} + 2\text{i}(E_1)_{12} + \mathcal{O}(t^{-3/4}). \tag{4.5}$$

Notice that

$$2\text{i}(M_1^{(\text{out})})_{12} = q_{\text{sol}}(x, t|\sigma_d^{(\text{out})}), \tag{4.6}$$

which, together with proposition 9, yields

$$(E_1)_{12} = pt^{-1/2} + \mathcal{O}(t^{-1}\log t), \tag{4.7}$$

where p is given by equation (4.2). Substituting equations (4.6) and (4.7) into (4.5) yields

$$q(x, t) = q_{\text{sol}}(x, t|\sigma_d^{(\text{out})}) + pt^{-1/2} + \mathcal{O}(t^{-3/4}). \tag{4.8}$$

Based on equation (3.59), we find the final asymptotic expression (4.1) with $(x, t) \in \mathcal{D}(v_1, v_2)$. □

Remark 1. Though the large-time asymptotics of the potential given by equation (4.1) has the same form as in [32], they have different meanings. In (4.1), $q_{\text{sol}}(x, t|\sigma_d(\mathcal{I}))$ denotes the soliton solutions generated by the double poles of the scattering data of the spectral problem, while it denotes the simple poles case in [32].

Remark 2. Theorem 1 did not consider the Painlevé asymptotics in the Painlevé region, in which the main term of the potential has no soliton solution. Thus, one need not consider the order of the discrete spectra such that the formula of the corresponding asymptotic behavior is the same as the one in [32].

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