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An approximate Kerr–Newman-like metric endowed with a magnetic dipole and mass quadrupole

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Abstract

Approximate all-terrain spacetimes for astrophysical applications are presented. The metrics possess five relativistic multipole moments, namely, mass, rotation, mass quadrupole, charge, and magnetic dipole moment. All these spacetimes approximately satisfy the Einstein–Maxwell field equations. The first metric is generated using the Hoenselaers–Perjés method from given relativistic multipoles. The second metric is a perturbation of the Kerr–Newman metric, which makes it a relevant approximation for astrophysical calculations. The last metric is an extension of the Hartle–Thorne metric that is important for obtaining internal models of compact objects perturbatively. The electromagnetic field is calculated using Cartan forms for locally non-rotating observers. These spacetimes are relevant for inferring properties of compact objects from astrophysical observations. Furthermore, the numerical implementations of these metrics are straightforward, making them versatile for simulating potential astrophysical applications.

Keywords: Gravitation, Einstein-Maxwell equations, Compact objects

1. Introduction

A spacetime for a real compact object is useful for many applications in astrophysics. Real compact objects possess mass, rotation, a mass quadrupole, and a magnetic dipole, so that one needs a spacetime with these features. Some exact metrics containing some or all of these parameters have been obtained [1–9]. Konno *et al* calculated the flattening of neutron stars caused by rotation and a magnetic field [10, 11]. Using the inverse scattering method (soliton) technique of Belinskii and Zakharov [12, 13], metrics with these characteristics have also been obtained by [14]. Another technique to obtain solutions of the Einstein–Maxwell equations (EME) is the Sibgatullin method [15], which uses the Ernst formalism [16, 17]; solutions with these features have been found by [18–23]. For slowly rotating pulsars and magnetars, there have been some attempts at relativistic solutions, for example [24–27]. Another technique to generate new solutions from old ones is the Hoenselaers–Kinnersley–Xanthopoulos transformations [28]. This method was used by Quevedo and Mashhoon to find a solution containing Kerr and Erez–Rosen metrics [29, 30]. The forms of most exact metrics are cumbersome to work with and are not easy to implement numerically. Approximations are usually sufficient to obtain relevant results; for example, in [31], a ray-tracing program was used with approximate solutions compared to exact solutions.

The detection of gravitational waves marked a historic milestone for humanity [32]. Phenomena such as the identification of supermassive black holes in the galactic center of the Milky Way, the testing of black hole spacetimes, the study of the innermost stable circular orbits (ISCO), the ring polarization, and the observation of shadows in the Milky Way and M87 bring us closer to a better understanding of compact objects [33–36]. These findings are the motivation for this work; the metrics generated in the following sections can be applied to study these astrophysical phenomena.

In 1989, Fodor *et al* developed a procedure to obtain the relativistic multipoles using the Ernst functions for solutions of the Einstein field equations [37]. Hoenselaers and Perjés extended the results to electrovac solutions in 1990 [38]. In addition, they proposed the reverse method, i.e. finding the metric from the relativistic multipoles. Pappas employed this method in 2017 to find the metric of a set of five multipoles, namely, mass, spin, quadrupole, spin octupole, and mass hexadecapole [39]. To date, the electrovac algorithm has not been used to explicitly determine the metric with a set of massive, spin, and electromagnetic multipoles. The Hoenselaers–Perjés method had two mistakes; the first one was found by Sotiriou and Apostolatos [40], and the second one by Perjés [41–43].

In this contribution, we present approximate solutions of the EME. The first solution is an approximation using the improved Hoenselaers–Perjés method that employs the relativistic multipoles to generate the metric in a power series [37, 38]. The second one is an approximation employing the Kerr–Newman metric as a seed metric; this approximation is valid up to the third order in mass quadrupoles and magnetic dipoles, making it attractive for computational implementations. The third one is a new version of the Hartle–Thorne metric that includes charge and a magnetic dipole. In this case, the seed metric is Reissner–Nordström instead of Schwarzschild. The results were found with the help of REDUCE programs. These programs are available upon request.

The paper is organized as follows: the second section is devoted to the Hoenselaers method to generate the metric from the relativistic multipoles. In the third section, a Kerr–Newman-like metric with all these parameters is developed in a perturbative way. A new version of Hartle–Thorne, including charge and magnetic dipole, is also generated, perturbing a series-expanded version. Some conclusions and applications are discussed in the last section. The appendices contain a summary of the relevant formulas and formalisms used in this article.

2. Generation of the approximate spacetime from relativistic multipoles à la Hoenselaers–Perjés

Fodor *et al* developed an algorithm to compute the gravitational multipole moment of a stationary axisymmetric spacetime [37]. Following this idea, Hoenselaers and Perjés showed in [38] that the metric could be generated if the multipoles of the object are known. Sotiriou and Apostolatos corrected some typos in Fodor's article [40]. Recently, Perjés found another error when computing relativistic axial-symmetric electrovacuum multipoles [41–43]. Moreover, Pappas found a spacetime with five relativistic multipoles, namely, mass, spin, mass quadrupole, spin octupole, and mass hexadecapole, using this technique [39]. In this section, we generate the metric for a massive (mass, M) rotating (spin, S) charged object (charge, q_e) endowed with a magnetic dipole (μ, magnetic dipole) and mass quadrupole (M_2, mass quadrupole) employing this formalism.

The Lewis–Weyl–Papapetrou (LWP) metric in canonical cylindrical coordinates (t, ρ, z, ϕ) is given by

$$\mathrm{d}s^2 = -f(\mathrm{d}t - \omega\mathrm{d}\phi)^2 + \frac{\mathrm{e}^{2\gamma}}{f}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \frac{\rho^2}{f}\mathrm{d}\phi^2, \tag{1}$$

where f, ω, and γ depend upon ρ and z. This metric has to fulfill the EME with an electromagnetic four-potential $A_\mu = (-A_t, 0, 0, A_\phi)$ (see appendix A).

Hoenselaers and Perjés devised the Ernst formalism [16, 17] to find approximate metric components given the values of a set of relativistic multipoles. The Ernst formalism is based in two functions or potentials for the metric, equation (1). The first ones are the complex potentials, $\mathcal{E}$ and Φ. The second ones, ξ and q, are defined through the latter. A brief overview of the Ernst formalism is presented in appendix B.

Following this technique, the secondary Ernst functions are expanded in Taylor series

$$\begin{aligned}\tilde{\xi} &= \frac{1}{\bar{r}}\xi = \sum_{i,j=0}^{\infty} a_{ij}\bar{\rho}^i\bar{z}^j,\\ \tilde{q} &= \frac{1}{\bar{r}}q = \sum_{i,j=0}^{\infty} b_{ij}\bar{\rho}^i\bar{z}^j,\end{aligned} \tag{2}$$

where

$$\begin{aligned}&\bar{\rho} = \frac{\rho}{\eta^2}, \quad \bar{z} = \frac{z}{\eta^2}, \eta = \sqrt{\rho^2 + z^2} \quad \text{and}\\ &\bar{r}^2 = \bar{\rho}^2 + \bar{z}^2 = r^{-2} = \eta^{-2}\end{aligned} \tag{3}$$

with the condition that a_{ij} and b_{ij} vanish when i is odd. Now, on the axis of symmetry ($\rho = \bar{\rho} = 0$), we have

$$\begin{aligned}\tilde{\xi}(\bar{\rho} = 0) &= \sum_{i=0}^{\infty} m_i\bar{z}^i,\\ \tilde{q}(\bar{\rho} = 0) &= \sum_{i=0}^{\infty} q_i\bar{z}^i.\end{aligned} \tag{4}$$

The values of m_i and q_i are related to the relativistic multipoles and to the values of the remaining non-zero values of a_{ij} and b_{ij} through a type of recurrence relationship (see appendix D). To obtain an approximate metric, we need to truncate these Taylor series [39]. In our case, the truncation occurs if $i + j \leqslant 6$. Employing the recurrence relationships, the functions f and A_t are found directly ($f = \mathrm{Re}[\mathcal{E}] + \Phi\Phi$ and $A_t = \mathrm{Re}[\Phi]$). To find ω, A_ϕ, and γ, we have to integrate equations (B3), (B4), (A2), or (A2). The results for f, ω, and γ are

$$\begin{aligned}f = {} & 1 - 2M\mathcal{U} + (2M^2 + q_e^2)\mathcal{U}^2\\ & -2M^3\mathcal{U}^3 + 2M^4\mathcal{U}^4\\ & +(-2M^5 + M^3\rho^2 + M_2\rho^2 - 2M_2z^2)\mathcal{U}^5\\ & +(-2M^4\rho^2 - 2MM_2\rho^2\\ & +4MM_2z^2 + \mu^2z^2 - 2S^2z^2)\mathcal{U}^6\\ & +3M^2(M^3\rho^2 + M_2\rho^2 - 2M_2z^2)\mathcal{U}^7\\ & +\frac{1}{28}M^2(-21M^3\rho^4 + 28M^3\rho^2z^2\\ & -52M_2\rho^4 + 220M_2\rho^2z^2 - 8M_2z^4)\mathcal{U}^9,\end{aligned} \tag{5}$$

$$\begin{aligned}
&\omega = 2S\rho^2u^3 + (2MS - \mu q_e)\rho^2\mathcal{U}^4\\
&+4M^2S\rho^2\mathcal{U}^5 + 2S(2M^3 - M_2)\rho^2\mathcal{U}^6\\
&-3M^2S\rho^4\mathcal{U}^7 + \frac{1}{2}S(-8M^3 + 3M_2)\rho^4\mathcal{U}^8,\\
&\gamma = \frac{1}{2}(-M^2 + q_e^2)\rho^2\mathcal{U}^4\\
&+(-M^4 - 3MM_2 + 2\mu^2 - 2S^2)\rho^2\mathcal{U}^6\\
&+\frac{1}{4}(5M^4 + 15MM_2 - 9\mu^2 + 9S^2)\rho^4\mathcal{U}^8,
\end{aligned} \tag{6}$$

where $\mathcal{U} = 1/\eta$.

The components of the metric tensor are

$$\begin{aligned}
&g_{tt} = f,\\
&g_{t\phi} = f\omega\\
&=2\rho^2S\mathcal{U}^3 - \rho^2(2MS + \mu q_e)\mathcal{U}^4\\
&+4M^2\rho^2S\mathcal{U}^5 - 2\rho^2S(2M^3 + M_2)\mathcal{U}^6\\
&-3M^2\rho^4S\mathcal{U}^7\\
&+\frac{1}{2}S(8M^3\rho^2 + 7M_2\rho^2 - 8M_2z^2)\rho^2\mathcal{U}^8,\\
&g_{\rho\rho} = \frac{e^{2\gamma}}{f}\\
&=1 + 2M\mathcal{U} + (2M^2 - q_e^2)\mathcal{U}^2\\
&+2M^3\mathcal{U}^3 + (2M^4 - M^2\rho^2 + q_e^2\rho^2)\mathcal{U}^4\\
&+(2M^5 - 3M^3\rho^2 - M_2\rho^2 + 2M_2z^2)\mathcal{U}^5\\
&+(-6M^4\rho^2 - 8MM_2\rho^2 + 4MM_2z^2\\
&+4\mu^2\rho^2 - \mu^2z^2 - 4S^2\rho^2 + 2S^2z^2)\mathcal{U}^6\\
&+3M^2(-3M^3\rho^2 - 5M_2\rho^2 + 2M_2z^2)\mathcal{U}^7\\
&+\frac{3}{2}(2M^4 + 5MM_2 - 3\mu^2 + 3S^2)\rho^4\mathcal{U}^8\\
&+\frac{1}{28}M^2(217M^3\rho^4 - 28M^3\rho^2z^2\\
&+500M_2\rho^4 - 276M_2\rho^2z^2 + 8M_2z^4)\mathcal{U}^9,\\
&g_{zz} = g_{\rho\rho},\\
&g_{\phi\phi} = \frac{\rho^2}{f} - f\omega^2\\
&=\rho^2 + 2M\rho^2\mathcal{U} + (2M^2 - q_e^2)\rho^2\mathcal{U}^2\\
&+2M^3\rho^2\mathcal{U}^3 + 2M^4\rho^2\mathcal{U}^4\\
&+\rho^2(2M^5 - M^3\rho^2 - M_2\rho^2 + 2M_2z^2)\mathcal{U}^5\\
&+\rho^2(-2M^4\rho^2 - 2MM_2\rho^2 + 4MM_2z^2\\
&-\mu^2z^2 - 4\rho^2S^2 + 2S^2z^2)\mathcal{U}^6\\
&+3M^2\rho^2(-M^3\rho^2 - M_2\rho^2 + 2M_2z^2)\mathcal{U}^7\\
&+\frac{1}{28}M^2\rho^2(21M^3\rho^4 - 28M^3\rho^2z^2\\
&+52M_2\rho^4 - 220M_2\rho^2z^2 + 8M_2z^4)\mathcal{U}^9.
\end{aligned} \tag{7}$$

The components of the four-potential $A_\nu = (-A_t, 0, 0, A_\phi)$ are

$$\begin{aligned}
&A_t = q_e\mathcal{U} - Mq_e\mathcal{U}^2\\
&+\frac{1}{2}(M_2q_e\rho^2 - 2M_2q_ez^2 + 2\mu Sz^2)\mathcal{U}^6,\\
&A_\phi = -\mu\rho^2\mathcal{U}^3\\
&+\frac{1}{2}(-M\mu + 3q_eS)\rho^2\mathcal{U}^4\\
&+\frac{1}{2}M_2\mu\rho^2\mathcal{U}^6 - \frac{3}{8}M_2\mu\rho^4\mathcal{U}^8.
\end{aligned} \tag{8}$$

The electromagnetic field components for a locally non-rotating observer (LNRO) can be found by using the formulas in appendix C.

To see which form has this metric in spherical-like coordinates, we use the Kerr–Newman mapping (Schwarzschild mapping)

$$\begin{aligned}
&\rho = \sqrt{\Delta}\,\sin\theta,\\
&z = (r - M)\cos\theta,
\end{aligned} \tag{9}$$

with $a = 0$ [44, 45]. Then, the function η is

$$\begin{aligned}
\eta^2 &= \Delta + (M^2 - a^2 - q_e^2)\cos^2\theta\\
&= r(r - 2M) + (M^2 - q_e^2)\cos^2\theta.
\end{aligned} \tag{10}$$

Using equation (9), the components of the metric tensor in spherical-like coordinates (t, r, θ, ϕ) can be obtained. For comparison with the Kerr–Newman metric with perturbations, we change the signs of $A_t \to -A_t$, $S \to -S$, $\mu \to -\mu$, so the metric components take the form

$$\begin{aligned}
&g_{tt} = 1 - 2Mu + q_e^2u^2 - 2M_2P_2u^3\\
&+\frac{1}{3}(3\mu^2\cos^2\theta - 6MM_2P_2 - 4P_2S^2 - 2S^2)u^4\\
&+\frac{2}{63}M^2M_2(-35P_2^2 - 44P_2 + 7)u^5,\\
&g_{t\phi} = (-2Su + \mu q_eu^2 + M_2S(5P_2 + 1)u^4)\sin^2\theta,\\
&g_{rr} = ((r - M)^2\sin^2\theta + \Delta\cos^2\theta)\frac{g_{\rho\rho}}{\Delta}\\
&=1 + 2Mu + (4M^2 - q_e^2)u^2 + 2(4M^3 + M_2P_2)u^3\\
&+\frac{1}{3}(48M^4 + 10MM_2P_2^2 + 22MM_2P_2\\
&-2MM_2 - 6\mu^2P_2^2 + 2\mu^2P_2 + \mu^2\\
&+6S^2P_2^2)u^4 + \frac{2}{63}M^2(1008M^3 + 665M_2P_2^2\\
&+548M_2P_2 - 133M_2)u^5,\\
&g_{\theta\theta} = ((r - M)^2\sin^2\theta + \Delta\cos^2\theta)g_{zz}
\end{aligned}$$

$$
\begin{aligned}
&= r^2\Bigg(1 + 2M_2P_2u^3 + \frac{1}{3}(10MM_2P_2^2 + 10MM_2P_2 \\
&-2MM_2 - 6\mu^2P_2^2 + 2\mu^2P_2 + \mu^2 \\
&+6S^2P_2^2)u^4 + \frac{2}{63}M^2M_2(455P_2^2 + 86P_2 - 91)u^5\Bigg), \\
&g_{\phi\phi} = r^2(1 + 2M_2P_2u^3 \\
&+\frac{1}{3}(18MM_2P_2 - 2\mu^2P_2 - \mu^2 + 12P_2S^2 - 6S^2)u^4 \\
&+\frac{2}{63}M^2M_2(35P_2^2 + 422P_2 - 7)u^5\Big)\sin^2\theta,
\end{aligned}
\tag{11}
$$

where $u = 1/r$, and P_2 is the Legendre polynomial of the second degree.

In this case, the components of the four-potential in spherical-like coordinates are

$$
\begin{aligned}
A_t &= -q_e u + (-\mu S\cos^2\theta \\
&+M_2q_eP_2)u^4, \\
A_\phi &= \sin^2\theta\Bigg(\mu u + \frac{3}{2}(M\mu - q_eS)u^2 \\
&-\frac{1}{4}\mu M_2(1 + P_2)u^4\Bigg).
\end{aligned}
\tag{12}
$$

The electromagnetic field components for an LNRO are (see appendix C)

$$
\begin{aligned}
E_r &= q_e u^2 + 4P_2(-M_2q_e + \mu S)u^5, \\
E_\theta &= 3\cos\theta\sin\theta(-M_2q_e + 2\mu S)u^5, \\
H_r &= (2\mu u^3 + 3(M\mu - q_eS)u^4 \\
&-5M_2\mu P_2u^6)\cos\theta, \\
H_\theta &= (\mu u^3 + (2M\mu - 3q_eS)u^4 \\
&-M_2\mu(1 + 3P_2)u^6)\sin\theta.
\end{aligned}
\tag{13}
$$

The metric, equation (7), is the solution of the EME up to the second order in M_2, up to the third order in the following parameters (q_e, μ, S), and up to the sixth order in M. For the metric functions, equation (5), not all the terms in equations (D4) and (D5) were kept to comply with the EME. To keep all terms, the integration of ω, A_ϕ, and γ should be improved. The improvement of these integrations can be done by adding new suitable terms to the metric functions f, ω, γ, and the four-potential functions A_t and A_ϕ. The first terms in equation (13) for E_r, H_r, and H_θ are the usual formulas of the electric field of a charged object and the magnetic dipole field, respectively.

3. Generation of the approximate spacetime for the Kerr–Newman metric

In this section, we generate a perturbed metric using the Kerr–Newman metric as a seed metric. To find this approximate metric, we proceed as in [44, 45], i.e. the magnetic dipole is included perturbatively. The mass quadrupole was already added in [44], and in [45] other parameters like mass hexadecapole and spin octupole were taken into account. In this case, we have to solve the EME for the Lewis metric, see appendix A. This metric in spherical-like coordinates is

$$
\begin{aligned}
ds^2 &= -V dt^2 + 2W dt d\phi + X dr^2 \\
&+Y d\theta^2 + Z d\phi^2,
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
g_{tt} &= V = V_K + V_q + V_\mu, \\
g_{t\phi} &= W = W_K + W_q + W_\mu, \\
g_{rr} &= X = X_K + X_q + X_\mu, \\
g_{\theta\theta} &= Y = Y_K + r^2(Y_q + Y_\mu), \\
g_{\phi\phi} &= Z = Z_K + r^2(Z_q + Z_\mu)\sin^2\theta.
\end{aligned}
\tag{15}
$$

The Kerr–Newman metric potentials are given by

$$
\begin{aligned}
V_K &= \frac{1}{\rho^2}(\Delta - a^2\sin^2\theta), \\
W_K &= \frac{a}{\rho^2}(\Delta - (r^2 + a^2))\sin^2\theta \\
&= -2(J - q_e^2)\frac{r}{\rho^2}, \\
X_K &= \frac{\rho^2}{\Delta}, \\
Y_K &= \rho^2, \\
Z_K &= \frac{1}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta]\sin^2\theta
\end{aligned}
\tag{16}
$$

with $J = Ma$,

$$
\begin{aligned}
\rho^2 &= r^2 + a^2\cos^2\theta, \\
\Delta &= r^2 - 2Mr + a^2 + q_e^2.
\end{aligned}
\tag{17}
$$

The perturbations due to the mass quadrupole were obtained in [44, 45]

$$
\begin{aligned}
V_q &= -2qP_2u^3 - 2MqP_2u^4 + 2q^2P_2^2u^6, \\
W_q &= MaqP_3^1u^4\sin\theta, \\
X_q &= 2qP_2u^3 + \frac{2}{7}Mq(29P_2 + 6P_4)u^4 \\
&+\frac{2}{385}q^2(44 + 55P_2 + 36P_4 + 250P_6)u^6, \\
Y_q &= 2qP_2u^3 + \frac{6}{7}Mq(5P_2 + 2P_4)u^4 \\
&+\frac{2}{385}q^2(44 + 55P_2 + 36P_4 + 250P_6)u^6, \\
Z_q &= 2qP_2u^3 + 6MqP_2u^4 + 2q^2P_2^2u^6,
\end{aligned}
\tag{18}
$$

where $u = 1/r$, $P_n(\cos\theta)$, $n = 1,\ldots,6$, and $P_3^1(\cos\theta) = (5P_2 + 1)\sin\theta$ are Legendre polynomials.

The ansatz to include the magnetic dipole in the metric is

$$
\begin{aligned}
V_\mu &= \mu^2(\beta_1P_2 + \beta_2)u^4, \\
W_\mu &= \mu q_e(\beta_3P_2 + \beta_4)u^2\sin^2\theta, \\
X_\mu &= \mu^2(\beta_5P_2 + \beta_6)u^4, \\
Y_\mu &= \mu^2(\beta_7P_2 + \beta_8)u^4, \\
Z_\mu &= \mu^2(\beta_9P_2 + \beta_{10})u^4.
\end{aligned}
\tag{19}
$$

Furthermore, the ansatz for the electromagnetic four-potential $A_\nu = (A_t, 0, 0, A_\phi)$ is

$$\begin{aligned}
A_t &= K_t + A_{pt}\\
&= K_t - qq_e(\alpha_1 + \alpha_2 P_1 + \alpha_3 P_2 + \alpha_4 P_3)u^4\\
&- Ma\mu(\alpha_5 P_2 + \alpha_6)u^4,\\
A_\phi &= K_\phi + A_{p\phi}\\
&= K_\phi + (\sigma_1 \mu u + \mu M(\sigma_2 P_2 + \sigma_3)u^2\\
&+ \mu q(\sigma_4 P_2 + \sigma_5)u^4)\sin^2\theta,
\end{aligned} \tag{20}$$

where $K_\nu = (K_t, 0, 0, K_\phi)$ is the four-potential for the Kerr–Newman metric [46]

$$\begin{aligned}
K_t &= -\frac{q_e r}{\rho^2},\\
K_\phi &= -aK_t\sin^2\theta.
\end{aligned} \tag{21}$$

Solving the EME (see equations (A3) to (A10) in appendix A), the unknowns $\alpha_i\,(i = 1,...,6)$, $\beta_j\,(j = 1,...,10)$, and $\sigma_k\,(k = 1,...,5)$ are found by solving algebraic equations:

$$\begin{aligned}
&\beta_1 = \frac{2}{3},\quad \beta_2 = \frac{1}{3},\quad \beta_3 = 0,\\
&\beta_4 = 1,\quad \beta_5 = 1,\\
&\beta_6 = -\frac{15}{4},\quad \beta_7 = -\frac{1}{3},\quad \beta_8 = 1,\\
&\beta_9 = -\frac{7}{18},\quad \beta_{10} = \frac{19}{18},\\
&\alpha_1 = 0,\quad \alpha_2 = 0,\quad \alpha_3 = -1,\quad \alpha_4 = \frac{2}{5},\\
&\alpha_5 = \frac{2}{3},\quad \alpha_6 = \frac{1}{3},\\
&\sigma_1 = 1,\quad \sigma_2 = 0,\quad \sigma_3 = \frac{2}{3},\\
&\sigma_4 = -\frac{1}{4},\quad \sigma_5 = -\frac{1}{4}.
\end{aligned} \tag{22}$$

The metric potentials can be written in a exponential form (except for W). It is better to implement numerical codes:

$$\begin{aligned}
V &= -\frac{1}{\rho^2}[(a\sin\theta)^2 - \Delta]\mathrm{e}^{2(-\psi_q + \psi_{1\mu})},\\
W &= \frac{a}{\rho^2}[(\Delta - (r^2 + a^2)]\sin^2\theta + W_q + W_\mu,\\
X &= \frac{\rho^2}{\Delta}\mathrm{e}^{2(\chi_q + \chi_{1\mu})},\\
Y &= \rho^2\mathrm{e}^{2(\chi_q + \chi_{2\mu})},\\
Z &= \frac{1}{\rho^2}[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta]\mathrm{e}^{2(\psi_q + \psi_{2\mu})}\sin^2\theta,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
\psi_q &= qP_2u^3 + 3MqP_2u^4,\\
\psi_{1\mu} &= \frac{1}{6}\mu^2(2P_2 + 1)u^4 = \frac{1}{2}\mu^2u^4\cos^2\theta,\\
\psi_{2\mu} &= \frac{1}{36}\mu^2(19 - 7P_2)u^4,\\
\chi_q &= qP_2u^3 + \frac{1}{9}Mq(15P_2^2 + 15P_2 - 3)u^4\\
&+ \frac{1}{9}q^2(25P_2^3 - 21P_2^2 - 6P_2 + 2)u^6,\\
\chi_{1\mu} &= \frac{1}{8}\mu^2(4P_2 - 15)u^4,\\
\chi_{2\mu} &= \frac{1}{6}\mu^2(3 - P_2)u^4.
\end{aligned} \tag{24}$$

The perturbed four-potential components are

$$\begin{aligned}
A_{pt} &= -\frac{1}{5}qq_e(-5P_2 + 2P_3)u^4 - Ma\mu u^4\cos^2\theta,\\
A_{p\phi} &= \left(\mu u + \frac{3}{2}M\mu u^2 - \frac{1}{4}\mu q(P_2 + 1)u^4\right)\sin^2\theta.
\end{aligned} \tag{25}$$

The electromagnetic field components for an LNRO are (see appendix C)

$$\begin{aligned}
E_r &= E_{Kr} + E_{pr},\\
E_\theta &= E_{K\theta} + E_{p\theta}\\
H_r &= H_{Kr} + H_{pr},\\
H_\theta &= H_{K\theta} + H_{p\theta}.
\end{aligned} \tag{26}$$

The expressions for the Kerr–Newman electromagnetic fields are determined in appendix C. The perturbation terms are found to be

$$\begin{aligned}
E_{pr} &= \frac{Z_K\partial_r A_{pt} - W_K\partial_r A_{p\phi}}{\sqrt{X_K Z_K(V_K Z_K + W_K^2)}}\\
&= \partial_r A_{pt} + 2Ju^3\partial_r A_{p\phi}\\
&= \frac{4}{15}qq_e u^5(10P_2\cos\theta - 15P_2 - 4\cos\theta)\\
&+ 4aM\mu u^5 P_2,\\
E_{p\theta} &= \frac{Z_K\partial_\theta A_{pt} - W_K\partial_\theta A_{p\phi}}{\sqrt{Y_K Z_K(V_K Z_K + W_K^2)}}\\
&= u(\partial_\theta A_{pt} + 2Ju^3\partial_\theta A_{p\phi})\\
&= \frac{1}{5}qq_e u^5\sin\theta(10P_2 - 15\cos\theta + 2)\\
&+ 6aM\mu u^5\cos\theta\sin\theta,\\
H_{pr} &= \frac{\partial_\theta A_{p\phi}}{\sqrt{Y_K Z_K}} = \frac{u^2}{\sin\theta}\partial_\theta A_{p\phi}\\
&= 2\mu u^3\cos\theta + 3M\mu u^4\cos\theta - \mu qu^6 P_2\cos\theta,\\
H_{p\theta} &= -\frac{\partial_r A_{p\phi}}{\sqrt{X_K Z_K}} = -\frac{u}{\sin\theta}(1 - Mu)\partial_\theta A_{p\phi}\\
&= \mu u^3\sin\theta + 2M\mu u^4\sin\theta - \mu qu^6(1 + P_2)\sin\theta.
\end{aligned} \tag{27}$$

The spacetime components, equation (23), are the solution of the EME up to the third order in the following parameters (μ, q).

4. Hartle–Thorne spacetime with charge and magnetic dipole

The Hartle–Thorne metric is an approximate solution of the Einstein equations proposed in 1968 to study the spacetime outside neutron stars, white dwarfs, and supermassive stars with small rotation and small mass quadrupoles [47]. Hartle and Thorne matched the interior and the exterior solutions, so that it is considered as a standard to know if a solution has physical meaning. In this section, we perturbatively include charge and magnetic dipole moments in this spacetime. Therefore, to include these additional features we need the Reissner–Nordström metric as a seed metric. Other parameters, such as mass hexadecapole and spin octupole, were taken into account in [45]. Again, we have to solve the EME for the Lewis metric, equation (14), see appendix A. The ansatz for the metric potentials in spherical-like coordinates is

$$\begin{aligned} V &= \left(1 - 2Mu + q_e^2 u^2 - \frac{2}{3}J^2 u^4\right) \mathrm{e}^{2\psi_1},\\ W &= -2Ju \sin^2\theta - JqP_3^1 u^4 \sin\theta + W_\mu,\\ X &= (1 - 2Mu + q_e^2 u^2 + 2J^2 u^4)^{-1} \mathrm{e}^{-2\psi_2},\\ Y &= r^2 \mathrm{e}^{-2\psi_3},\\ Z &= Y \sin^2\theta, \end{aligned} \tag{28}$$

where $u = 1/r$, and the functions ψ_i ($i = 1, 2, 3$) and W_μ depend upon (r, θ) and the parameters q and μ. The ansatz for the functions ψ_i ($i = 1, 2, 3$) is

$$\begin{aligned} &\psi_1 = qP_2 u^3 + 3MqP_2 u^4\\ &-(2/3)J^2 P_2 u^4 + V_\mu,\\ &\psi_2 = qP_2 u^3 + 3MqP_2 u^4 - 8J^2 P_2 u^4\\ &+\frac{1}{24} q^2 (16P_2^2 + 16P_2 - 77) u^6 - X_\mu,\\ &\psi_3 = qP_2 u^3 + \frac{5}{2} MqP_2 u^4 - \frac{1}{2} J^2 P_2 u^4\\ &+\frac{1}{72} q^2 (28P_2^2 - 8P_2 + 43) u^6 - Y_\mu, \end{aligned} \tag{29}$$

where the quadrupole perturbations were obtained in [45]. The functions V_μ, W_μ, X_μ, and Y_μ have a similar form as in equation (19). After solving the EME (see equations (A3) to (A10) in appendix A), the functions V_μ, W_μ, X_μ, and Y_μ are

$$\begin{aligned} V_\mu &= \frac{1}{2}\mu^2 u^4 \cos^2\theta,\\ W_\mu &= \mu q_e u^2 \sin^2\theta,\\ X_\mu &= \frac{1}{3}\mu^2 (P_2 - 1) u^4,\\ Y_\mu &= -\frac{1}{6}\mu^2 P_2 u^4. \end{aligned} \tag{30}$$

The components of the four-potential A_ν are

$$\begin{aligned} A_t &= -q_e u - J\mu u^4 \cos^2\theta - qq_e P_2 u^4,\\ A_\phi &= \left(\mu u - \frac{3}{2} Jq_e u^2 + \frac{3}{2} M\mu u^2\right.\\ &\left.+\frac{1}{4}\mu q (P_2 + 1) u^4\right) \sin^2\theta. \end{aligned} \tag{31}$$

The electromagnetic field components for an LNRO are (see appendix C)

$$\begin{aligned} &E_r = q_e u^2 + 4(J\mu + qq_e) P_2 u^5,\\ &E_\theta = 3(2J\mu + qq_e) u^5 \cos\theta \sin\theta,\\ &H_r = (2\mu u^3 + 3(\mu M - Jq_e) u^4\\ &+5\mu q P_2 u^6) \cos\theta,\\ &H_\theta = (\mu u^3 + (2\mu M - 3Jq_e) u^4\\ &+\mu q (3P_2 + 1) u^6) \sin\theta. \end{aligned} \tag{32}$$

The metric, equation (28), is the solution of the EME up to the third order in the following parameters (J, μ, q). The first terms in equation (32) for E_r, and for H_r and H_θ are the electric field, and magnetic dipole common formulas, respectively.

5. Comparison among the spacetimes

In this section, we compare the spacetimes obtained in the previous sections. The principal characteristics of these metrics are:

- These spacetimes possess five parameters, namely, mass, angular momentum, mass quadrupole, electric charge, and magnetic dipole.
- These metrics are asymptotically flat.
- The Lense–Thirring metric, which can be deduced from the Kerr metric, in a series expansion to the second order of M and to the first order of J is contained in all spacetimes.
- The Reissner–Nordström metric in a series expansion to the second order of M and q_e is contained in all spacetimes.
- The four potentials of these metrics contain the first order of the charge and magnetic dipole of the objects.
- The electric field corresponds to the first order expected for a charged object, in the same way that the magnetic field corresponds to the first order expected for an object with a magnetic dipole.

Let us see the Taylor series expressions of the metrics up to the fourth order in M, the second order in q_e and μ, and the first order in J, S, M_2 and q.

From equation (11), the Hoenselaers–Perjés metric with charge and magnetic dipole is

$$
\begin{aligned}
V &= 1 - 2Mu + q_e^2 u^2 - 2M_2 p2u^3 + \mu^2 u^4 \cos^2\theta,\\
W &= -2Su\sin^2\theta,\\
X &= 1 + 2Mu + (4M^2 - q_e^2)u^2 + 2(4M^3 + M_2P_2)u^3\\
&\quad + \frac{1}{3}(48M^4 + \mu^2(-6P_2^2 + 2P_2 + 1))u^4,\\
Y &= r^2\left(1 + 2M_2P_2u^3 + \frac{1}{3}\mu^2(-6P_2^2 + 2P_2 + 1)u^4\right),\\
Z &= r^2\left(1 + 2M_2P_2u^3 - \frac{1}{3}\mu^2(2P_2 + 1)u^4\right)\sin^2\theta. \qquad (33)
\end{aligned}
$$

From equation (23), the Kerr–Newman with a magnetic dipole and mass quadrupole is

$$
\begin{aligned}
V &= 1 - 2Mu + q_e^2 u^2 - 2qP_2u^3\\
&\quad + \mu^2 u^4 \cos^2\theta,\\
W &= -2Ju\sin^2\theta,\\
X &= 1 + 2Mu + (4M^2 - q_e^2)u^2\\
&\quad + 2(4M^3 + qP_2)u^3\\
&\quad + \frac{1}{4}(64M^4 + \mu^2(4P_2 - 15))u^4,\\
Y &= r^2\left(1 + 2qP_2u^3 + \frac{1}{3}\mu^2(3 - P_2)u^4\right),\\
Z &= r^2\left(1 + 2qP_2u^3 + \frac{1}{18}\mu^2(19 - 7P_2)u^4\right)\sin\theta^2. \qquad (34)
\end{aligned}
$$

From equation (28), the Hartle–Thorne metric with charge and a magnetic dipole is

$$
\begin{aligned}
V &= 1 - 2Mu + q_e^2 u^2 + 2qP_2u^3\\
&\quad + \mu^2 u^4 \cos\theta,\\
W &= -2Ju\sin^2\theta,\\
X &= 1 + 2Mu + (4M^2 - q_e^2)u^2\\
&\quad + 2(4M^3 - 2qP_2)u^3\\
&\quad + \frac{1}{3}(48M^4 + 2\mu^2(P_2 - 1))u^4,\\
Y &= r^2\left(1 - 2qP_2u^3 - \frac{1}{3}\mu^2P_2u^4\right),\\
Z &= r^2\left(1 - 2qP_2u - \frac{1}{3}\mu^2P_2u^2\right)\sin^2\theta. \qquad (35)
\end{aligned}
$$

The first order of the four-potential components for the metrics are

$$
\begin{aligned}
A_t &= -q_e u,\\
A_\phi &= \mu u \sin^2\theta. \qquad (36)
\end{aligned}
$$

The components of the electric and magnetic fields at the first order are

$$
\begin{aligned}
E_r &= q_e u^2,\\
E_\theta &= 0,\\
H_r &= 2\mu u^3 \cos\theta,\\
H_\theta &= \mu u^3 \cos\theta. \qquad (37)
\end{aligned}
$$

From (33, 34, 35, 36) it can be seen that the structure of the metrics is similar. The Hoenselaers–Perjés metric was deduced by assuming the multipolar structure from the beginning. In [45], the multipole structure of a metric with five parameters, namely, the mass, spin, mass quadrupole, spin octupole, and mass hexadecapole was obtained using the formalism of Fodor *et al* [37]. Moreover, the metrics derived in [45] are contained in these new metrics without an electric charge and magnetic dipole. Since the new metrics contain Reissner–Nordström, the remaining parameter μ is the only one that remains to be interpreted; however, from the structure of the electromagnetic four-potential and the electromagnetic field, it can be inferred that it corresponds to a magnetic dipole.

6. Conclusions

In this contribution, we found three spacetimes that can be employed in astrophysical calculations. The metrics include mass, rotation, quadrupole moment, charge, and magnetic dipole moment. The first spacetime was found using the Hoenselaers–Perjés formalism. This metric is in canonical cylindrical coordinates and was transformed to spherical-like coordinates using Kerr–Newman mapping. The second metric is a generalization of the Kerr–Newman metric. This is an excellent approximation, since the Kerr–Newman metric is exact, and the parameters q and μ are small compared to the mass and spin for an ample range of real values. The last metric is an improvement of the Hartle–Thorne spacetime, including charge and magnetic dipole moment. It is important to explore the matching of interior with exterior solutions with magnetic dipole moments. The metrics were found using REDUCE programs that are available on request.

These metrics are relevant in astrophysics, since they include the approximation of the magnetic dipole moment, which makes them suitable for simulating real compact objects. Our approximations can easily be determined with a higher order of precision by taking into account the higher-order terms in the expansions. It can be done without significantly changing the form of the analytic spacetimes. Furthermore, they can easily be implemented numerically.

These metrics have many potential applications. For instance, they could be used to infer features of the structure of a compact object from astrophysical observations. The ISCO and the shadow of the compact object are other characteristics that can be calculated from these spacetimes. In addition, a ray-tracing program including these spacetimes can be useful for studying the chaotic behavior of geodesics, light scattering, determination of the shape of the neutron star, the thermal spectrum, and pulse profiles.

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Appendix A. Einstein–Maxwell equations

The EMEs are given by

$$\begin{aligned} R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} &= \kappa T_{\mu\nu}, \\ \nabla_\nu[\sqrt{-g}\, F^{\mu\nu}] &= 0, \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} &= 0, \end{aligned} \tag{A1}$$

where $\kappa = 8\pi G/c^4$, $g_{\mu\nu}$ are the components of the metric tensor, $R_{\mu\nu}$ are the components of the Ricci tensor, R is the curvature scalar, and $T_{\mu\nu}$ is the stress-energy tensor given by

$$T_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $A_\mu = (-A_t, 0, 0, A_\phi)$.

The EMEs for the LWP metric. equation (1), are explicitly

$$\begin{aligned} &\rho^2 f \nabla^2 f = \rho^2 (\nabla f)^2 - f^4 (\nabla\omega)^2 \\ &+2f[\rho^2(\nabla A_t)^2 + f^2(\nabla A_\phi - \omega\nabla A_t)^2], \\ &0 = \nabla \cdot \left[\frac{f^2}{\rho^2}\nabla\omega - 4\frac{f}{\rho^2}A_t(\nabla A_\phi - \omega\nabla A_t)\right], \\ &\partial_\rho\gamma = \frac{\rho}{4f^2}[(\partial_\rho f)^2 - (\partial_z f)^2] \\ &-\frac{f^2}{4\rho}[(\partial_\rho\omega)^2 - (\partial_z\omega)^2] \\ &+\frac{f}{\rho}[(\partial_\rho A_\phi)^2 - (\partial_z A_\phi)^2] \\ &+\left(\frac{f\omega^2}{\rho} - \frac{\rho}{f}\right)[(\partial_\rho A_t)^2 - (\partial_z A_t)^2] \\ &-\frac{2f\omega}{\rho}(\partial_\rho A_t \partial_\rho A_\phi - \partial_z A_t \partial_z A_\phi), \\ &\partial_z\gamma = \frac{\rho}{2f^2}\partial_\rho f \partial_z f - \frac{f^2}{2\rho}\partial_\rho\omega\partial_z\omega \\ &+\frac{2f}{\rho}\partial_\rho A_\phi \partial_z A_\phi \\ &+2\left(\frac{f\omega^2}{\rho} - \frac{\rho}{f}\right)\partial_\rho A_t \partial_z A_t \\ &-\frac{2f\omega}{\rho}(\partial_\rho A_t \partial_z A_\phi + \partial_\rho A_\phi \partial_z A_t) \\ &0 = \nabla \cdot \left[\frac{f}{\rho^2}(\nabla A_\phi - \omega\nabla A_t)\right], \\ &0 = \nabla \cdot \left[\frac{1}{f}\nabla A_t - \frac{f\omega}{\rho^2}(\nabla A_\phi - \omega\nabla A_t)\right]. \end{aligned} \tag{A2}$$

These expressions were verified using REDUCE programs that are available upon request.

The EMEs for the Lewis metric, equation (14), are solved by introducing the ansatz and solving for algebraic equations of the unknown variables. These EMEs are explicitly

$$\begin{aligned} 0 = {} & Y((X(\mathcal{R}\partial_r Y + VY\partial_r Z) \\ & -\mathcal{R}Y\partial_r X - 2WXY\partial_r W)\partial_r V \\ & +X(2V(X(\partial_\theta W)^2 + Y(\partial_r W)^2) \\ & -YZ(\partial_r V)^2) + 2\mathcal{R}XY\partial_r\partial_r V) \\ & -X(X(\mathcal{R}\partial_\theta Y - VY\partial_\theta Z) - \mathcal{R}Y\partial_\theta X \\ & +2WXY\partial_\theta W + XYZ\partial_\theta V)\partial_\theta V \\ & +2\mathcal{R}X^2Y\partial_\theta\partial_\theta V - 4(\mathcal{R} + W^2)XY^2(\partial_r A_t)^2 \\ & -4(\mathcal{R} + W^2)X^2Y(\partial_\theta A_t)^2 \\ & -4VXY^2(V\partial_r A_\phi - 2W\partial_r A_t)\partial_r A_\phi \\ & -4VX^2Y(V\partial_\theta A_\phi - 2W\partial_\theta A_t)\partial_\theta A_\phi, \end{aligned} \tag{A3}$$

$$\begin{aligned} 0 = {} & Y((X(\mathcal{R}\partial_r Y - VY\partial_r Z) - \mathcal{R}Y\partial_r X) \\ & \times\partial_r W + 2\mathcal{R}XY\partial_r\partial_r W) \\ & -X(X(\mathcal{R}\partial_\theta Y + VY\partial_\theta Z) - \mathcal{R}Y\partial_\theta X) \\ & \times\partial_\theta W + 2\mathcal{R}X^2Y\partial_\theta\partial_\theta W \\ & -XY^2(Z\partial_r W - 2W\partial_r Z)\partial_r V \\ & -X^2Y(Z\partial_\theta W - 2W\partial_\theta Z)\partial_\theta V \\ & +4WXY^2Z(\partial_r A_t)^2 + 4WX^2YZ(\partial_\theta A_t)^2 \\ & -4XY^2(VW\partial_r A_\phi + 2\mathcal{R}\partial_r A_t - 2W^2\partial_r A_t)\partial_r A_\phi \\ & -4X^2Y(VW\partial_\theta A_\phi + 2\mathcal{R}\partial_\theta A_t - 2W^2\partial_\theta A_t)\partial_\theta A_\phi, \end{aligned} \tag{A4}$$

$$\begin{aligned} 0 = {} & X(VY^2(2\mathcal{R}\partial_r\partial_r Z - V\partial_r Z^2V) \\ & -\mathcal{R}^2(\partial_r Y)^2 + 2\mathcal{R}^2Y\partial_r\partial_r Y) \\ & -\mathcal{R}Y(\mathcal{R}\partial_r Y + VY\partial_r Z)\partial_r X \\ & -\mathcal{R}(X(\mathcal{R}\partial_\theta Y - VY\partial_\theta Z) + \mathcal{R}Y\partial_\theta X)\partial_\theta X \\ & +2\mathcal{R}^2XY\partial_\theta\partial_\theta X - 2Y^2(W(\mathcal{R}\partial_r X + 2VX\partial_r Z) \\ & -(\mathcal{R} - 2W^2)X\partial_r W)\partial_r W + 4\mathcal{R}WXY^2\partial_r\partial_r W \\ & +2\mathcal{R}WXY\partial_\theta W\partial_\theta X \\ & -Y^2(\mathcal{R}Z\partial_r X - 2W^2X\partial_r Z \\ & +4WXZ\partial_r W + XZ^2\partial_r V)\partial_r V \\ & +2\mathcal{R}XY^2Z\partial_r\partial_r V + \mathcal{R}XYZ\partial_\theta V\partial_\theta X \\ & -4\mathcal{R}XY^2Z(\partial_r A_t)^2 + 4\mathcal{R}X^2YZ(\partial_\theta A_t)^2 \\ & +4\mathcal{R}XY^2(V\partial_r A_\phi - 2W\partial_r A_t)\partial_r A_\phi \\ & -4\mathcal{R}X^2Y(V\partial_\theta A_\phi - 2W\partial_\theta A_t)\partial_\theta A_\phi, \end{aligned} \tag{A5}$$

$$\begin{aligned} 0 = {} & V(X(Y(2\mathcal{R}\partial_r\partial_\theta Z - V\partial_\theta Z\partial_r Z) \\ & -\mathcal{R}\partial_r Y\partial_\theta Z) - \mathcal{R}Y\partial_\theta X\partial_r Z) \\ & -2WY(\mathcal{R}\partial_\theta X + VX\partial_\theta Z)\partial_r W \\ & -2X(W(\mathcal{R}\partial_r Y + VY\partial_r Z) \\ & -(\mathcal{R} - 2W^2)Y\partial_r W)\partial_\theta W + 4\mathcal{R}WXY\partial_r\partial_\theta W \\ & -Y(\mathcal{R}Z\partial_\theta X - W^2X\partial_\theta Z + 2WXZ\partial_\theta W)\partial_r V \\ & -X(\mathcal{R}Z\partial_r Y - W^2Y\partial_r Z \\ & +2WYZ\partial_r W + YZ^2\partial_r V)\partial_\theta V \\ & +2\mathcal{R}XYZ\partial_r\partial_\theta V \\ & -8\mathcal{R}XYZ\partial_\theta A_t\partial_r A_t - 8\mathcal{R}WXY\partial_r A_\phi\partial_\theta A_t \\ & +8\mathcal{R}XY(V\partial_r A_\phi - W\partial_r A_t)\partial_\theta A_\phi, \end{aligned} \tag{A6}$$

$$
\begin{aligned}
0 &= X(\mathcal{R}(\mathcal{R}\partial_r Y - VY\partial_r Z)\partial_r Y\\
&-VXY(2\mathcal{R}\partial_\theta\partial_\theta Z - V(\partial_\theta Z)^2)\\
&-2\mathcal{R}^2 Y\partial_r\partial_r Y + \mathcal{R}VX\partial_\theta Y\partial_\theta Z) + \mathcal{R}^2 Y\partial_r X\partial_r Y\\
&+\mathcal{R}^2(Y\partial_\theta X + X\partial_\theta Y)\partial_\theta X - 2\mathcal{R}^2 XY\partial_\theta\partial_\theta X - 2\mathcal{R}WXY\partial_r W\partial_r Y\\
&+2X^2(W(\mathcal{R}\partial_\theta Y + 2VY\partial_\theta Z) - (\mathcal{R} - 2W^2)Y\partial_\theta W)\partial_\theta W\\
&-4\mathcal{R}WX^2 Y\partial_\theta\partial_\theta W - \mathcal{R}XYZ\partial_r V\partial_r Y\\
&+X^2(\mathcal{R}Z\partial_\theta Y - 2W^2 Y\partial_\theta Z + 4WYZ\partial_\theta W + YZ^2\partial_\theta V)\partial_\theta V\\
&-2\mathcal{R}X^2 YZ\partial_\theta\partial_\theta V - 4\mathcal{R}XY^2 Z(\partial_r A_t)^2 + 4\mathcal{R}X^2 YZ(\partial_\theta A_t)^2\\
&+4\mathcal{R}XY^2(V\partial_r A_\phi - 2W\partial_r A_t)\partial_r A_\phi - 4\mathcal{R}X^2 Y(V\partial_\theta A_\phi - 2W\partial_\theta A_t)\partial_\theta A_\phi,
\end{aligned}
\tag{A7}
$$

$$
\begin{aligned}
0 &= X(Y(Y(2\mathcal{R}\partial_r\partial_r Z - V\partial_r Z^2) - VX\partial_\theta Z^2 + 2\mathcal{R}X\partial_\theta\partial_\theta Z + \mathcal{R}\partial_r Y\partial_r Z)\\
&-\mathcal{R}X\partial_\theta Y\partial_\theta Z) - \mathcal{R}Y^2\partial_r X\partial_r Z + \mathcal{R}XY\partial_\theta X\partial_\theta Z\\
&+2XY^2(Z\partial_r W - W\partial_r Z)\partial_r W + 2X^2 Y(Z\partial_\theta W - W\partial_\theta Z)\partial_\theta W\\
&+XY^2 Z\partial_r V\partial_r Z + X^2 YZ\partial_\theta V\partial_\theta Z\\
&+4XY^2 Z^2(\partial_r A_t)^2 + 4X^2 YZ^2(\partial_\theta A_t)^2\\
&+4XY^2((\mathcal{R} + W^2)\partial_r A_\phi + 2WZ\partial_r A_t)\partial_r A_\phi\\
&+4X^2 Y((\mathcal{R} + W^2)\partial_\theta A_\phi + 2WZ\partial_\theta A_t)\partial_\theta A_\phi,
\end{aligned}
\tag{A8}
$$

$$
\begin{aligned}
0 &= Y((X((\mathcal{R} + W^2)Y\partial_r Z + \mathcal{R}Z)\partial_r Y - \mathcal{R}YZ\partial_r X - 2WXYZ\partial_r W\\
&-XYZ^2\partial_r V)\partial_r A_t + 2\mathcal{R}XYZ\partial_r\partial_r A_t)\\
&+X(X((\mathcal{R} + W^2)Y\partial_\theta Z - \mathcal{R}Z\partial_\theta Y) + \mathcal{R}YZ\partial_\theta X - 2WXYZ\partial_\theta W\\
&-XYZ^2\partial_\theta V)\partial_\theta A_t + 2\mathcal{R}X^2 YZ\partial_\theta\partial_\theta A_t\\
&+Y(W(X(\mathcal{R}\partial_r Y - VY\partial_r Z) - \partial_r X\mathcal{R}Y) + 2(\mathcal{R} - W^2)XY\partial_r W\\
&-WXYZ\partial_r V)\partial_r A_\phi + 2\mathcal{R}WXY^2\partial_r\partial_r A_\phi\\
&-X(W(X(\mathcal{R}\partial_\theta Y + VY\partial_\theta Z) - \mathcal{R}Y\partial_\theta X) - 2(\mathcal{R} - W^2)XY\partial_\theta W\\
&+WXYZ\partial_\theta V)\partial_\theta A_\phi + 2\mathcal{R}WX^2 Y\partial_\theta\partial_\theta A_\phi,
\end{aligned}
\tag{A9}
$$

$$
\begin{aligned}
0 &= Y((2\mathcal{R}XY\partial_r W - 2W^2 XY\partial_r W - \mathcal{R}WY\partial_r X + \mathcal{R}WX\partial_r Y\\
&-VWXY\partial_r Z - WXYZ\partial_r V)\partial_r A_t + 2\mathcal{R}WXY\partial_r\partial_r A_t)\\
&+X(2\mathcal{R}XY\partial_\theta W - 2W^2 XY\partial_\theta W + \mathcal{R}WY\partial_\theta X - \mathcal{R}WX\partial_\theta Y\\
&-VWXY\partial_\theta Z - WXYZ\partial_\theta V)\partial_\theta A_t + 2\mathcal{R}WX^2 Y\partial_\theta\partial_\theta A_t\\
&-Y(V(X(\mathcal{R}\partial_r Y - VY\partial_r Z) - \mathcal{R}Y\partial_r X - 2WXY\partial_r W)\\
&+(\mathcal{R} + W^2)XY\partial_r V)\partial_r A_\phi - 2\mathcal{R}VXY^2\partial_r\partial_r A_\phi\\
&+X(V(X(\mathcal{R}\partial_\theta Y + VY\partial_\theta Z) - \mathcal{R}Y\partial_\theta X + 2WXY\partial_\theta W)\\
&-(\mathcal{R} + W^2)XY\partial_\theta V)\partial_\theta A_\phi - 2\mathcal{R}VX^2 Y\partial_\theta\partial_\theta A_\phi,
\end{aligned}
\tag{A10}
$$

where $\mathcal{R} = VZ + W^2$. If one chooses $A_\mu = (A_t, 0, 0, A_\phi)$, one has to change the signs of $A_t \rightarrow -A_t$ in expressions (A2), and subsequent ones, as well as in equation (A3) and subsequent ones. These expressions were found using REDUCE programs that are available upon request.

Appendix B. Ernst equations

In 1968, Ernst reformulated the EME for the LWP metric, equation (1), in a complex form using the potential [16, 17]

$$
\mathcal{E} = (f - |\Phi|^2) + \mathrm{i}\varphi,
\tag{B1}
$$

where the function $\mathcal{E}$ is the Ernst potential, and

$$
\Phi = A_t + \mathrm{i}\tilde{A}_\phi,
\tag{B2}
$$

with A_t and A_ϕ as the components of the electromagnetic four-potential. The function φ is the twist scalar, and $\tilde{A}_\phi$ is an auxiliary potential. Both functions are defined via

$$
\hat{\boldsymbol{n}} \times \nabla A_\phi = -\frac{\rho}{f}\nabla\tilde{A}_\phi + \omega\hat{\boldsymbol{n}} \times \nabla A_t,
\tag{B3}
$$

$$
\hat{\boldsymbol{n}} \times \nabla\omega = -\frac{\rho}{f^2}[\nabla\varphi + 2\,\mathrm{Im}(\Phi\nabla\Phi)],
\tag{B4}
$$

where $\hat{\boldsymbol{n}}$ is a unit vector in the azimuthal direction. From these equations, the EME can be rewritten as

$$
\begin{aligned}
&(\mathrm{Re}\,\mathcal{E} + |\Phi|^2)\nabla^2\mathcal{E} = (\nabla\mathcal{E} + 2\Phi\nabla\Phi)\cdot\nabla\mathcal{E},\\
&(\mathrm{Re}\,\mathcal{E} + |\Phi|^2)\nabla^2\Phi = (\nabla\mathcal{E} + 2\Phi\nabla\Phi)\cdot\nabla\Phi,\\
&\partial_\rho\gamma = \frac{1}{4}\frac{\rho}{f^2}[(\partial_\rho\mathcal{E} + 2\Phi\partial_\rho\Phi)(\partial_\rho\mathcal{E} + 2\Phi\partial_\rho\Phi)\\
&-(\partial_z\mathcal{E} + 2\Phi\partial_z\Phi)(\partial_z\mathcal{E} + 2\Phi\partial_z\Phi)]\\
&-\frac{\rho}{f}(\partial_\rho\Phi\partial_\rho\Phi - \partial_z\Phi\partial_z\Phi),\\
&\partial_z\gamma = \frac{1}{4}\frac{\rho}{f^2}[(\partial_\rho\mathcal{E} + 2\Phi\partial_\rho\Phi)(\partial_z\mathcal{E} + 2\Phi\partial_z\Phi)\\
&-(\partial_z\mathcal{E} + 2\Phi\partial_z\Phi)(\partial_\rho\mathcal{E} + 2\Phi\partial_\rho\Phi)]\\
&-\frac{\rho}{f}(\partial_\rho\Phi\partial_z\Phi - \partial_z\Phi\partial_\rho\Phi).
\end{aligned}
\tag{B5}
$$

These equations are the Ernst equations. Ernst introduced new functions q, ξ such that

$$
\begin{aligned}
\mathcal{E} &= \frac{\xi - 1}{\xi + 1},\\
\Phi &= \frac{q}{\xi + 1}.
\end{aligned}
\tag{B6}
$$

Under the transformation, equation (B6), the EMEs become

$$
\begin{aligned}
&(\xi\xi + qq - 1)\nabla^2\xi = 2[\xi\nabla\xi + q\nabla q]\cdot\nabla\xi\\
&(\xi\xi + qq - 1)\nabla^2 q = 2[\xi\nabla\xi + q\nabla q]\cdot\nabla q,\\
&\partial_\rho\gamma = \frac{\rho}{|\xi + 1|^6 f^2}((q((\xi + 1)\partial_\rho q - q\partial_\rho\xi)\\
&+(\xi + 1)\partial_\rho\xi)\\
&\times(q((\xi + 1)\partial_\rho q - q\partial_\rho\xi) + (\xi + 1)\partial_\rho\xi)\\
&-(q((\xi + 1)\partial_z q - q\partial_z\xi) + (\xi + 1)\partial_z\xi)\\
&\times(q((\xi + 1)\partial_z q - q\partial_z\xi) + (\xi + 1)\partial_z\xi)\\
&-f|\xi + 1|^2(((\xi + 1)\partial_\rho q - q\partial_\rho\xi)\\
&\times((\xi + 1)\partial_\rho q - q\partial_\rho\xi)\\
&+((\xi + 1)\partial_z q - q\partial_z\xi)((\xi + 1)\partial_z q - q\partial_z\xi))),\\
&\partial_z\gamma = \frac{\rho}{|\xi + 1|^6 f^2}((q((\xi + 1)\partial_\rho q - q\partial_\rho\xi)\\
&+(\xi + 1)\partial_\rho\xi)\\
&\times(q((\xi + 1)\partial_z q - q\partial_z\xi) + (\xi + 1)\partial_z\xi)\\
&-(q((\xi + 1)\partial_z q - q\partial_z\xi) + (\xi + 1)\partial_z\xi)\\
&\times(q((\xi + 1)\partial_\rho q - q\partial_\rho\xi) + (\xi + 1)\partial_\rho\xi)\\
&-f|\xi + 1|^2(((\xi + 1)\partial_\rho q - q\partial_\rho\xi)\\
&\times((\xi + 1)\partial_z q - q\partial_z\xi)\\
&+((\xi + 1)\partial_z q - q\partial_z\xi)((\xi + 1)\partial_\rho q - q\partial_\rho\xi))).
\end{aligned}
\tag{B7}
$$

The complex transformation, equation (B1), leads to powerful tools for solving the EME for the LWP spacetime under initial conditions [12, 13, 15, 28].

Appendix C. Determination of the electromagnetic field

To determine the electromagnetic field we use the Cartan formalism [48–51]. For this purpose, local non-rotating observers are chosen. The metric LWP can be written as follows

$$\begin{aligned}
&\mathrm{d}s^2 = -F_2\mathrm{d}t^2 + \frac{\mathrm{e}^{2\gamma}}{f}(\mathrm{d}\rho^2 + \mathrm{d}z^2)\\
&+F_1(\mathrm{d}\phi - \tilde{\omega}\mathrm{d}t)^2\\
&=-(\omega^t)^2 + (\omega^\rho)^2 + (\omega^z)^2 + (\omega^\phi)^2,
\end{aligned}\tag{C1}$$

where

$$\begin{aligned}
F_1 &= \frac{\rho^2}{f} - f\omega^2,\\
\tilde{\omega} &= -\frac{f\omega}{F_1} = -\frac{f^2\omega}{\rho^2 - f^2\omega^2},\\
F_2 &= \frac{\rho^2}{F_1} = \frac{\rho^2 f}{\rho^2 - f^2\omega^2}.
\end{aligned}\tag{C2}$$

The 1-forms for equation (C1) are

$$\begin{aligned}
\omega^t &= \sqrt{F_2}\,\mathrm{d}t,\\
\omega^\rho &= \frac{\mathrm{e}^\gamma}{\sqrt{f}}\mathrm{d}\rho,\\
\omega^z &= \frac{\mathrm{e}^\gamma}{\sqrt{f}}\mathrm{d}z,\\
\omega^\phi &= \sqrt{F_1}\,(\mathrm{d}\phi - \tilde{\omega}\mathrm{d}t).
\end{aligned}\tag{C3}$$

The potential 1-form is

$$A = A_t\mathrm{d}t + A_\phi\mathrm{d}\phi,\tag{C4}$$

where $A_t = A_t(\rho,\, z)$ and $A_\phi = A_\phi(\rho,\, z)$.

The Faraday 2-form is

$$\begin{aligned}
&F = \mathrm{d}A\\
&=\partial_\rho A_t\mathrm{d}\rho \wedge \mathrm{d}t + \partial_z A_\phi\mathrm{d}z \wedge \mathrm{d}t\\
&+\partial_\rho A_\phi\mathrm{d}\rho \wedge \mathrm{d}\phi + \partial_z A_\phi\mathrm{d}z \wedge \mathrm{d}\phi\\
&=(\partial_\rho A_t + \tilde{\omega}\partial_\rho A_\phi)\mathrm{d}\rho \wedge \mathrm{d}t\\
&+(\partial_z A_t + \tilde{\omega}\partial_z A_\phi)\mathrm{d}z \wedge \mathrm{d}t\\
&+\partial_\rho A_\phi\mathrm{d}\rho \wedge (\mathrm{d}\phi - \tilde{\omega}\mathrm{d}t)\\
&+\partial_z A_\phi\mathrm{d}z \wedge (\mathrm{d}\phi - \tilde{\omega}\mathrm{d}t)\\
&=E_\rho\omega^\rho \wedge \omega^t + E_z\omega^z \wedge \omega^t\\
&-H_z\omega^\rho \wedge \omega^\phi + H_\rho\omega^z \wedge \omega^\phi,
\end{aligned}\tag{C5}$$

where the orthonormal components of the electromagnetic field for an LNRO are

$$\begin{aligned}
&E_\rho = \sqrt{\frac{f}{F_2}}\mathrm{e}^{-\gamma}(\partial_\rho A_t + \tilde{\omega}\partial_\rho A_\phi)\\
&=\frac{\sqrt{\rho^2 - f^2\omega^2}}{\rho}\mathrm{e}^{-\gamma}\left(\partial_\rho A_t - \frac{f^2\omega}{\rho^2 - f^2\omega^2}\partial_\rho A_\phi\right),\\
&E_z = \sqrt{\frac{f}{F_2}}\mathrm{e}^{-\gamma}(\partial_z A_t + \tilde{\omega}\partial_z A_\phi)\\
&=\frac{\sqrt{\rho^2 - f^2\omega^2}}{\rho}\mathrm{e}^{-\gamma}\left(\partial_z A_t - \frac{f^2\omega}{\rho^2 - f^2\omega^2}\partial_z A_\phi\right),\\
&H_\rho = \sqrt{\frac{f}{F_1}}\mathrm{e}^{-\gamma}\partial_z A_\phi = \frac{f\mathrm{e}^{-\gamma}}{\sqrt{\rho^2 - f^2\omega^2}}\partial_z A_\phi,\\
&H_z = -\sqrt{\frac{f}{F_1}}\mathrm{e}^{-\gamma}\partial_\rho A_\phi = -\frac{f\mathrm{e}^{-\gamma}}{\sqrt{\rho^2 - f^2\omega^2}}\partial_\rho A_\phi.
\end{aligned}\tag{C6}$$

In the case of the Lewis metric we have

$$\begin{aligned}
&\mathrm{d}s^2 = -\left(\sqrt{V + \frac{W^2}{Z}}\mathrm{d}t\right)^2\\
&+[\sqrt{X}\mathrm{d}r]^2 + [\sqrt{Y}\mathrm{d}\theta]^2\\
&+\left(\sqrt{Z}\mathrm{d}\phi + \frac{W}{\sqrt{Z}}\mathrm{d}t\right)^2.
\end{aligned}\tag{C7}$$

The 1-forms are

$$\begin{aligned}
\omega^t &= \sqrt{V + \frac{W^2}{Z}}\mathrm{d}t,\\
\omega^r &= \sqrt{X}\mathrm{d}r,\\
\omega^\theta &= \sqrt{Y}\mathrm{d}\theta,\\
\omega^\phi &= \sqrt{Z}\left(\mathrm{d}\phi + \frac{W}{Z}\mathrm{d}t\right).
\end{aligned}\tag{C8}$$

The potential 1-form is written as in equation (C4), but where $A_t = A_t(r,\, \theta)$ and $A_\phi = A_\phi(r,\, \theta)$.

The Faraday 2-form is

$$\begin{aligned}
&F = \mathrm{d}A\\
&= \partial_r A_t\mathrm{d}r \wedge \mathrm{d}t + \partial_\theta A_t\mathrm{d}\theta \wedge \mathrm{d}t\\
&+ \partial_r A_\phi\mathrm{d}r \wedge \mathrm{d}\phi + \partial_\theta A_\phi\mathrm{d}\theta \wedge \mathrm{d}\phi\\
&= \left(\partial_r A_t - \frac{W}{Z}\partial_r A_\phi\right)\mathrm{d}r \wedge \mathrm{d}t\\
&+ \left(\partial_\theta A_t - \frac{W}{Z}\partial_\theta A_\phi\right)\mathrm{d}\theta \wedge \mathrm{d}t\\
&+ \partial_r A_\phi\mathrm{d}r \wedge \left(\mathrm{d}\phi + \frac{W}{Z}\mathrm{d}t\right)\\
&+ \partial_\theta A_\phi\mathrm{d}\theta \wedge \left(\mathrm{d}\phi + \frac{W}{Z}\mathrm{d}t\right)\\
&= E_r\omega^r \wedge \omega^t + E_\theta\omega^\theta \wedge \omega^t\\
&- H_\theta\omega^r \wedge \omega^\phi + H_r\omega^\theta \wedge \omega^\phi,
\end{aligned}\tag{C9}$$

where the orthonormal components of the electromagnetic field for an LNRO are

$$\begin{aligned}
E_r &= \frac{1}{\sqrt{X(VZ+W^2)}}\left(\sqrt{Z}\,\partial_r A_t - \frac{W}{\sqrt{Z}}\partial_r A_\phi\right),\\
E_\theta &= \frac{1}{\sqrt{Y(VZ+W^2)}}\left(\sqrt{Z}\,\partial_\theta A_t - \frac{W}{\sqrt{Z}}\partial_\theta A_\phi\right),\\
H_r &= \frac{1}{\sqrt{YZ}}\partial_\theta A_\phi,\\
H_\theta &= -\frac{1}{\sqrt{XZ}}\partial_r A_\phi.
\end{aligned} \tag{C10}$$

For the Kerr–Newman metric, the electromagnetic fields for an LNRO are [51, 52]

$$\begin{aligned}
E_{Kr} &= \frac{q_e}{\rho^4\sqrt{\Sigma}}(r^2+a^2)(r^2-a^2\cos^2\theta),\\
E_{K\theta} &= -2\frac{a^2 q_e r}{\rho^4}\sqrt{\frac{\Delta}{\Sigma}}\cos\theta\sin\theta,\\
H_{Kr} &= 2\frac{a q_e r}{\rho^4\sqrt{\Sigma}}(r^2+a^2)\cos\theta,\\
H_{K\theta} &= \frac{a q_e}{\rho^4}\sqrt{\frac{\Delta}{\Sigma}}(r^2-a^2\cos^2\theta)\sin\theta,
\end{aligned} \tag{C11}$$

where $\Sigma = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$.

Appendix D. Hoenselaers–Perjés relationships for the relativistic multipoles

The parameters P_i represent the mass or spin relativistic multipoles; if i is even, it is a massive multipole, and if i is odd, it is a spin multipole (complex value). The parameters Q_i represent the electromagnetic relativistic multipoles; if i is even, the value is real, and if i is odd, it is a complex value. They are given by [42, 43]

$$\begin{aligned}
&P_0 = m_0,\\
&P_1 = m_1,\\
&P_2 = m_2,\\
&P_3 = m_3 + \frac{1}{5}q_0 S_{10},\\
&P_4 = m_4 - \frac{1}{7}m_0 M_{20}\\
&+\frac{3}{35}q_1 S_{10} + \frac{1}{7}q_0(3S_{20} - 2H_{20}),\\
&P_5 = m_5 - \frac{1}{3}m_0 M_{30} - \frac{1}{21}m_1 M_{20}\\
&+\frac{1}{21}q_2 S_{10} + \frac{1}{21}q_1(4S_{20} - 3H_{20})\\
&+\frac{1}{21}q_0((q_0 q_0 - m_0 m_0)S_{10}\\
&+14S_{30} + 13S_{21} - 7H_{30}),\\
&P_6 = m_6 - \frac{5}{231}m_2 M_{20}\\
&-\frac{4}{33}m_1 M_{30} + \frac{1}{33}m_0^2 m_0 M_{20}\\
&-\frac{1}{33}m_0(18M_{40} + 8M_{31}) + \frac{1}{33}q_3 S_{10}\\
&+\frac{1}{231}q_2(25S_{20} - 20H_{20})\\
&+\frac{2}{231}q_1(35S_{30} + 37S_{21} - 21H_{30})\\
&-\frac{1}{1155}(37q_1 m_0 + 13q_0 m_1)m_0 S_{10}\\
&+\frac{1}{33}q_0^2(5q_0 S_{20} - 4m_0 Q_{20} + 3q_1 S_{10})\\
&+\frac{10}{231}q_1 q_0 q_0 S_{10} + \frac{2}{33}q_0 m_0\\
&\times(2m_0 H_{20} - 3q_0 M_{20} - 2m_1 S_{10})\\
&+\frac{1}{33}q_0(30S_{40} + 32S_{31} - 24H_{31} - 12H_{40})\\
&Q_0 = q_0,\\
&Q_1 = q_1,\\
&Q_2 = q_2,\\
&Q_3 = q_3 - \frac{1}{5}m_0 H_{10},\\
&Q_4 = q_4 + \frac{1}{7}q_0 Q_{20} - \frac{3}{35}m_1 H_{10}\\
&-\frac{1}{7}m_0(3H_{20} - 2S_{20}),\\
&Q_5 = q_5 + \frac{1}{3}q_0 Q_{30} + \frac{1}{21}q_1 Q_{20}\\
&-\frac{1}{21}m_2 H_{10} - \frac{1}{21}m_1(4H_{20} - 3S_{20})\\
&+\frac{1}{21}m_0((m_0 m_0 - q_0 q_0)H_{10}\\
&-14H_{30} - 13H_{21} + 7S_{30}),\\
&Q_6 = q_6 + \frac{5}{231}q_2 Q_{20} + \frac{4}{33}q_1 Q_{30}\\
&+\frac{1}{33}q_0^2 q_0 Q_{20} + \frac{1}{33}q_0(18Q_{40} + 8Q_{31})\\
&-\frac{1}{33}m_3 H_{10} - \frac{1}{231}m_2(25H_{20} - 20S_{20})\\
&-\frac{2}{231}m_1(35H_{30} + 37H_{21} - 21S_{30})\\
&-\frac{1}{1155}(37m_1 q_0 + 13m_0 q_1)q_0 H_{10}\\
&+\frac{1}{33}m_0^2(5m_0 H_{20} - 4q_0 M_{20} + 3m_1 H_{10})\\
&+\frac{10}{231}m_1 m_0 m_0 H_{10} + \frac{2}{33}m_0 q_0\\
&\times(2q_0 S_{20} - 3m_0 Q_{20} - 2q_1 H_{10})\\
&-\frac{1}{33}m_0(30H_{40} + 32H_{31} - 24S_{31} - 12S_{40}),
\end{aligned} \tag{D1}$$

where $m_i = a_{0i}$, $q_i = b_{0i}$, and

$$
\begin{aligned}
M_{ij} &= m_i m_j - m_{i-1} m_{j+1},\\
Q_{ij} &= q_i q_j - q_{i-1} q_{j+1},\\
S_{ij} &= m_i q_j - m_{i-1} q_{j+1},\\
H_{ij} &= q_i m_j - q_{i-1} m_{j+1}.
\end{aligned}
\tag{D2}
$$

Now, we choose the relativistic multipoles as follows

$$
\begin{aligned}
P_0 &= m, & Q_0 &= q_e,\\
P_1 &= \mathrm{i}S, & Q_1 &= \mathrm{i}\mu,\\
P_2 &= M_2, & Q_2 &= 0,\\
P_n &= 0, & Q_n &= 0, \ \forall\ n \geqslant 3.
\end{aligned}
\tag{D3}
$$

From equation (D1), one can invert the relationships to get the values m_i and q_i

$$
\begin{aligned}
m_0 &= M,\\
m_1 &= \mathrm{i}S,\\
m_2 &= M_2,\\
m_3 &= \frac{\mathrm{i}}{5}(M\mu q_e - q_e^2 S),\\
m_4 &= \frac{1}{35}(5M^2M_2 + 3M\mu^2\\
&+5MS^2 - 15M_2 q_e^2 - 8\mu q_e S),\\
m_5 &= \frac{\mathrm{i}}{21}(-8MM_2 S\\
&+3M_2\mu q_e + \mu^2 S - S^3),\\
m_6 &= \frac{1}{1155}(55M^4M_2 + 66M^3\mu^2\\
&+55M^3S^2 - 330M^2M_2q_e^2\\
&-66M^2\mu q_e S - 255MM_2^2\\
&-121M\mu^2q_e^2 - 110Mq_e^2S^2 - 20M_2\mu^2\\
&+275M_2q_e^4 - 115M_2S^2 + 176\mu q_e^3 S),\\
q_0 &= q_e,\\
q_1 &= \mathrm{i}\mu,\\
q_2 &= 0,\\
q_3 &= \frac{\mathrm{i}}{5}M(M\mu - q_e S),\\
q_4 &= \frac{1}{35}(-10MM_2q_e + 8M\mu S\\
&-5\mu^2 q_e - 3q_e S^2),\\
q_5 &= \frac{\mathrm{i}}{105}(9M^4\mu\\
&-9M^3q_eS - 9M^2\mu q_e^2 - 25MM_2\mu + 9Mq_e^3S\\
&+10M_2q_eS + 5\mu^3 - 5\mu S^2),\\
q_6 &= \frac{1}{1155}(-220M^3M_2q_e + 176M^3\mu S\\
&-110M^2\mu^2q_e - 121M^2q_eS^2\\
&+220MM_2q_e^3 - 66M\mu q_e^2S\\
&-100M_2^2q_e - 135M_2\mu S + 55\mu^2q_e^3\\
&+66q_e^3S^2).
\end{aligned}
\tag{D4}
$$

Using the following recurrence relationships, we obtain the non-zero values of a_{ij} and b_{ij} for $i+j \leqslant 6$ using equation (D4)

$$
\begin{aligned}
a_{20} &= -\frac{1}{2}(a_{02} + a_{00}^2 a_{00})\\
&=-\frac{1}{2}(M^3 + M_2),\\
a_{21} &= \frac{1}{2}(-3a_{03} - 4a_{01}a_{00}a_{00} - a_{00}^2 a_{01})\\
&=\frac{3\mathrm{i}}{10}(-5M^2S - M\mu q_e + q_e^2 S),\\
a_{22} &= \frac{1}{2}(2a_{20}a_{00}a_{00} - 6a_{04}\\
&-5a_{02}a_{00}a_{00} - 4a_{01}^2 a_{00}\\
&-4a_{01}a_{00}a_{01} - a_{00}^2 a_{02})\\
&=\frac{1}{70}(-35M^5 - 275M^2M_2\\
&-18M\mu^2 - 30MS^2 + 90M_2q_e^2 + 48\mu q_e S),\\
a_{23} &= \frac{1}{2}(2a_{21}a_{00}a_{00} + 2a_{20}a_{01}a_{00}\\
&+2a_{20}a_{00}a_{01} - 10a_{05} - 5a_{03}a_{00}a_{00}\\
&-11a_{02}a_{01}a_{00} - 5a_{02}a_{00}a_{01}\\
&-4a_{01}^2 a_{01} - 4a_{01}a_{00}a_{02} - a_{00}^2 a_{03})\\
&=\frac{\mathrm{i}}{210}(-315M^4S - 237M^3\mu q_e\\
&+237M^2q_e^2S - 650MM_2S\\
&+90M\mu q_e^3 - 250M_2\mu q_e\\
&-50\mu^2 S - 90q_e^4 S - 370S^3),\\
a_{24} &= \frac{1}{2}(2a_{22}a_{00}a_{00} + 2a_{21}a_{01}a_{00}\\
&+2a_{21}a_{00}a_{01} + 2a_{20}a_{02}a_{00} + 2a_{20}a_{01}a_{01}\\
&+2a_{20}a_{00}a_{02} - 15a_{06} - 4a_{04}a_{00}a_{00}\\
&-13a_{03}a_{01}a_{00} - 5a_{03}a_{00}a_{01}\\
&-8a_{02}^2 a_{00} - 11a_{02}a_{01}a_{01} - 5a_{02}a_{00}a_{02}\\
&-4a_{01}^2 a_{02} - 4a_{01}a_{00}a_{03} - a_{00}^2 a_{04})\\
&=\frac{1}{770}(-385M^7 - 4345M^4M_2\\
&-693M^3\mu^2 - 1265M^3S^2\\
&+3465M^2M_2q_e^2 + 1606M^2\mu q_e S\\
&-4500MM_2^2 + 605M\mu^2q_e^2\\
&+242Mq_e^2S^2 + 100M_2\mu^2\\
&-1375M_2q_e^4 - 2505M_2S^2 - 880\mu q_e^3 S),\\
a_{40} &= \frac{1}{8}(-a_{22} - 4a_{20}a_{00}a_{00}\\
&+a_{02}a_{00}a_{00} - a_{01}^2 a_{00} - a_{00}^2 a_{20})\\
&=\frac{1}{280}(105M^5 + 260M^2M_2 + 9M\mu^2\\
&+50MS^2 - 45M_2q_e^2 - 24\mu q_e S),
\end{aligned}
$$

$$\begin{aligned}
a_{41} &= \frac{1}{8}(-3a_{23} - 6a_{21}a_{00}a_{00} - 6a_{20}a_{01}a_{00} \\
&-4a_{20}a_{00}a_{01} + 3a_{03}a_{00}a_{00} \\
&-3a_{02}a_{01}a_{00} + a_{02}a_{00}a_{01} \\
&-a_{01}^2a_{01} - 4a_{01}a_{00}a_{20} - a_{00}^2a_{21}) \\
&= \frac{\mathrm{i}}{280}(525M^4S + 192M^3\mu q_e \\
&-192M^2q_e^2S + 290MM_2S - 45M\mu q_e^3 \\
&+125M_2\mu q_e + 25\mu^2S + 45q_e^4S + 150S^3),
\end{aligned}$$

$$\begin{aligned}
a_{42} &= \frac{1}{8}(8a_{40}a_{00}a_{00} - 6a_{24} \\
&-7a_{22}a_{00}a_{00} - 10a_{21}a_{01}a_{00} - 6a_{21}a_{00}a_{01} \\
&-2a_{20}^2a_{00} - 7a_{20}a_{02}a_{00} \\
&-6a_{20}a_{01}a_{01} + 2a_{20}a_{00}a_{20} - 4a_{20}a_{00}a_{02} \\
&+6a_{04}a_{00}a_{00} - 3a_{03}a_{01}a_{00} \\
&+3a_{03}a_{00}a_{01} - 3a_{02}^2a_{00} - 3a_{02}a_{01}a_{01} \\
&-5a_{02}a_{00}a_{20} + a_{02}a_{00}a_{02} - 4a_{01}^2a_{20} \\
&-a_{01}^2a_{02} - 4a_{01}a_{00}a_{21} - a_{00}^2a_{22}) \\
&=\frac{1}{3080}(3850M^7 + 31\,405M^4M_2 \\
&+3168M^3\mu^2 + 6380M^3S^2 \\
&-15\,840M^2M_2q_e^2 - 7260M^2\mu q_eS \\
&+15\,810MM_2^2 - 1815M\mu^2q_e^2 \\
&-1188Mq_e^2S^2 - 300M_2\mu^2 + 4125M_2q_e^4 \\
&+7130M_2S^2 + 2640\mu q_e^3S),
\end{aligned}$$

$$\begin{aligned}
a_{60} &= \frac{1}{18}(-a_{42} - 2a_{40}a_{00}a_{00} \\
&+a_{22}a_{00}a_{00} - 2a_{21}a_{01}a_{00} - 7a_{20}^2a_{00} \\
&+a_{20}a_{02}a_{00} - 4a_{20}a_{00}a_{20}
\end{aligned}$$

$$\begin{aligned}
&+a_{02}a_{00}a_{20} - a_{01}^2a_{20} - a_{00}^2a_{40}) \\
&=\frac{1}{18480}(-5775M^7 - 24035M^4M_2 \\
&-1419M^3\mu^2 - 6710M^3S^2 \\
&+7095M^2M_2q_e^2 + 2772M^2\mu q_eS \\
&-9120MM_2^2 + 605M\mu^2q_e^2 \\
&+1012Mq_e^2S^2 + 100M_2\mu^2 \\
&-1375M_2q_e^4 - 2890M_2S^2 - 880\mu q_e^3S),
\end{aligned}$$

$$b_{20} = -\frac{1}{2}(b_{02} + b_{00}^2b_{00}) = -\frac{1}{2}q_e^3,$$

$$\begin{aligned}
b_{21} &= \frac{1}{2}(-3b_{03} - 4b_{01}b_{00}b_{00} - b_{00}^2b_{01}) \\
&=\frac{3\mathrm{i}}{10}(-M^2\mu + Mq_eS - 5\mu q_e^2),
\end{aligned}$$

$$\begin{aligned}
b_{22} &= \frac{1}{2}(2b_{20}b_{00}b_{00} - 6b_{04} - 5b_{02}b_{00}b_{00} - 4b_{01}^2b_{00} \\
&-4b_{01}b_{00}b_{01} - b_{00}^2b_{02}) \\
&=\frac{1}{70}(60MM_2q_e - 48M\mu S \\
&+30\mu^2q_e - 35q_e^5 + 18q_eS^2),
\end{aligned}$$

$$\begin{aligned}
b_{23} &= \frac{1}{2}(2b_{21}b_{00}b_{00} + 2b_{20}b_{01}b_{00} \\
&+2b_{20}b_{00}b_{01} - 10b_{05} - 5b_{03}b_{00}b_{00} \\
&-11b_{02}b_{01}b_{00} - 5b_{02}b_{00}b_{01} \\
&-4b_{01}^2b_{01} - 4b_{01}b_{00}b_{02} - b_{00}^2b_{03}) \\
&= \frac{\mathrm{i}}{210}(-90M^4\mu + 90M^3q_eS
\end{aligned}$$

$$\begin{aligned}
&-57M^2\mu q_e^2 + 250MM_2\mu + 57Mq_e^3S \\
&-100M_2q_eS - 470\mu^3 - 315\mu q_e^4 + 50\mu S^2),
\end{aligned}$$

$$\begin{aligned}
b_{24} &= \frac{1}{2}(2b_{22}b_{00}b_{00} + 2b_{21}b_{01}b_{00} \\
&+2b_{21}b_{00}b_{01} + 2b_{20}b_{02}b_{00} + 2b_{20}b_{01}b_{01} \\
&+2b_{20}b_{00}b_{02} - 15b_{06} - 4b_{04}b_{00}b_{00} \\
&-13b_{03}b_{01}b_{00} - 5b_{03}b_{00}b_{01} \\
&-8b_{02}^2b_{00} - 11b_{02}b_{01}b_{01} - 5b_{02}b_{00}b_{02} \\
&-4b_{01}^2b_{02} - 4b_{01}b_{00}b_{03} - b_{00}^2b_{04}) \\
&=\frac{1}{770}(1100M^3M_2q_e - 880M^3\mu S \\
&+858M^2\mu^2q_e + 605M^2q_eS^2 \\
&+110MM_2q_e^3 - 946M\mu q_e^2S \\
&+500M_2^2q_e + 675M_2\mu S - 55\mu^2q_e^3 \\
&-385q_e^7 + 33q_e^3S^2),
\end{aligned}$$

$$\begin{aligned}
b_{40} &= \frac{1}{8}(-b_{22} - 4b_{20}b_{00}b_{00} \\
&+b_{02}b_{00}b_{00} - b_{01}^2b_{00} - b_{00}^2b_{20}) \\
&=\frac{1}{280}(-30MM_2q_e + 24M\mu S \\
&+20\mu^2q_e + 105q_e^5 - 9q_eS^2),
\end{aligned}$$

$$\begin{aligned}
b_{41} &= \frac{1}{8}(-3b_{23} - 6b_{21}b_{00}b_{00} \\
&-6b_{20}b_{01}b_{00} - 4b_{20}b_{00}b_{01} + 3b_{03}b_{00}b_{00} \\
&-3b_{02}b_{01}b_{00} + b_{02}b_{00}b_{01} \\
&-b_{01}^2b_{01} - 4b_{01}b_{00}b_{20} - b_{00}^2b_{21}) \\
&=\frac{\mathrm{i}}{280}(45M^4\mu - 45M^3q_eS \\
&+102M^2\mu q_e^2 - 125MM_2\mu - 102Mq_e^3S \\
&+50M_2q_eS + 200\mu^3 + 525\mu q_e^4 - 25\mu S^2),
\end{aligned}$$

$$\begin{aligned}
b_{42} &= \frac{1}{8}(8b_{40}b_{00}b_{00} - 6b_{24} - 7b_{22}b_{00}b_{00} \\
&- 10b_{21}b_{01}b_{00} - 6b_{21}b_{00}b_{01} \\
&- 2b_{20}^2b_{00} - 7b_{20}b_{02}b_{00} - 6b_{20}b_{01}b_{01} \\
&+ 2b_{20}b_{00}b_{20} - 4b_{20}b_{00}b_{02} \\
&+ 6b_{04}b_{00}b_{00} - 3b_{03}b_{01}b_{00} \\
&+ 3b_{03}b_{00}b_{01} - 3b_{02}^2b_{00} - 3b_{02}b_{01}b_{01} \\
&- 5b_{02}b_{00}b_{20} + b_{02}b_{00}b_{02} \\
&- 4b_{01}^2b_{20} - b_{01}^2b_{02} - 4b_{01}b_{00}b_{21} - b_{00}^2b_{22})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3080}(-3300M^3M_2q_e + 2640M^3\mu S\\
&-2112M^2\mu^2q_e - 1815M^2q_eS^2\\
&-3960MM_2q_e^3 + 5280M\mu q_e^2S\\
&-1500M_2^2q_e - 2025M_2\mu S - 880\mu^2q_e^3\\
&+3850q_e^7 - 1188q_e^3S^2),\\
&b_{60} = \frac{1}{18}(-b_{42} - 2b_{40}b_{00}b_{00}\\
&+b_{22}b_{00}b_{00} - 2b_{21}b_{01}b_{00} - 7b_{20}^2b_{00}\\
&+b_{20}b_{02}b_{00} - 4b_{20}b_{00}b_{20}\\
&+b_{02}b_{00}b_{20} - b_{01}^2b_{20} - b_{00}^2b_{40})\\
&=\frac{1}{18480}(1100M^3M_2q_e - 880M^3\mu S\\
&+88M^2\mu^2q_e + 605M^2q_eS^2\\
&+2530MM_2q_e^3 - 2112M\mu q_e^2S\\
&+500M_2^2q_e + 675M_2\mu S - 3080\mu^2q_e^3\\
&-5775q_e^7 + 759q_e^3S^2).
\end{aligned}
\tag{D5}
$$

As already mentioned, not all the terms in equations (D4) and (D5) held to satisfy the EME.

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