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High-dimensional nonlinear variable separation solutions and novel wave excitations for the (4+1)-dimensional Boiti–Leon–Manna–Pempinelli equation

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Abstract

Considering the importance of higher-dimensional equations that are widely applied to real nonlinear problems, many $(4+1)$-dimensional integrable systems have been established by uplifting the dimensions of their corresponding lower-dimensional integrable equations. Recently, an integrable $(4+1)$-dimensional extension of the Boiti–Leon–Manna–Pempinelli (4DBLMP) equation has been proposed, which can also be considered as an extension of the famous Korteweg–de Vries equation that is applicable in fluids, plasma physics and so on. It is shown that new higher-dimensional variable separation solutions with several arbitrary lower-dimensional functions can also be obtained using the multilinear variable separation approach for the 4DBLMP equation. In addition, by taking advantage of the explicit expressions of the new solutions, versatile $(4+1)$-dimensional nonlinear wave excitations can be designed. As an illustration, periodic breathing lumps, multi-dromion-ring-type instantons, and hybrid waves on a doubly periodic wave background are discovered to reveal abundant nonlinear structures and dynamics in higher dimensions.

Keywords: $(4+1)$-dimensional Boiti-Leon-Manna-Pempinelli equation, variable separation solution, periodic breathing lumps, multi-dromion-ring-type instanton, hybrid waves on a doubly periodic wave background

(Some figures may appear in colour only in the online journal)

1. Introduction

It is well known that differential equations play a crucial role in the investigation of natural phenomena. Lower-dimensional differential equations, being relatively easier to solve, have been extensively utilized to explore various issues in different fields, such as physics, fluids, economics, engineering and management. In fact, a number of interesting problems need to be modelled by higher-dimensional systems. For instance, physically, in the AdS/CFT correspondence and the brane world scenario, attention has been paid to gravity theories in $D=5$ spacetime dimensions. The coupling to gravity in $D=5$ spacetime dimensions was introduced for the particle-like and vortex-type solutions by uplifting the $D=4$ Yang–Mills instantons and $D=3$ Yang–Mills–Higgs monopoles [1]. Spherically and axially symmetric monopoles of the SU(2) Einstein–Yang–Mills–Higgs-dilaton system were studied with a new coupling between the dilaton field

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and the covariant derivative of the Higgs field arising from a $(4+1)$-dimensional Einstein–Yang–Mills model [2]. Mathematically, global well-posedness and scattering of the energy-critical $(4+1)$-dimensional Maxwell–Klein–Gordon equation were proved in [3].

Considering the importance and wide applications of higher-dimensional equations to real nonlinear problems, many $(4+1)$-dimensional integrable nonlinear partial differential equations (NPDEs) have been proposed by uplifting dimensions of their corresponding (3+1)-or (2+1)-dimensional integrable equations. For instance, Fokas [4] proposed an integrable generalization of the (2+1)-dimensional Kadomtsev–Petviashvili (KP) equation by extending its Lax pair, formulated in four spatial and one temporal dimensions, which is known as the Fokas equation. Its various solutions, including solitons, periodic waves, breathers and rogue waves, have been obtained using different mathematical techniques [5–8]. A new $(4+1)$-dimensional variable-coefficient KP(4DVCKP) equation was introduced, and its lump, rogue-soliton and lump-kink solutions were obtained by using the Hirota bilinear method in [9]. More solutions and wave interactions of the 4DVCKP equation were obtained in [10]. An integrable $(4+1)$-dimensional extension of the Boiti–Leon–Manna–Pempinelli (BLMP) equation was established in [11], which can also be considered as an extension of the famous Korteweg–de Vries equation. With regard to the $(4+1)$-dimensional BLMP equation, the Painlevé property, bilinear representation, Lax pair, infinite conservation laws and a group of wave solutions, including stripe solitons, kinks, periodic waves, complexitons and mixed lump-solitons, and their interaction phenomena have been discussed in [11–16].

To better understand nonlinear waves in various disciplines, many effective methods, such as the inverse scattering method [17], Darboux transformation [18] and Hirota bilinear method [19], have been developed to search for explicit solutions of NPDEs, especially for those with integrable properties, such as possession of infinitely many conservation laws, Lax pairs and $N-$soliton solutions. However, as far as we are aware, the nonlinear variable separation approach, first established for (2+1)-dimensional integrable systems, has not yet been applied to higher-dimensional equations, except for (3+1)-dimensions. Therefore, in this paper, we concentrate on extending the multilinear variable separation approach to explore new variable separated solutions of integrable systems in $4+1$ dimensions. Accordingly, the $(4+1)$-dimensional BLMP equation is taken as an example, which is in the form of [11]

$$\begin{aligned} &u_{yt}+u_{zt}+u_{st}+a(u_y+u_z+u_s)_{xxx}\\ &\quad +b(u_x(u_y+u_z+u_s))_x=0, \end{aligned} \tag{1}$$

where u is a function of four spatial variables x, y, z, s and one temporal variable t, while a and b are nonzero parameters. Equation (1) contains several important (2+1)- and (3+1)-dimensional nonlinear models with physical interests for particular choices of parameters. The details of important BLMP equations of different dimensions reduced from equation (1) have been presented in [12]. Many results related to this equation have been mentioned above, and more details can be found in [11–16].

This paper is organized as follows. In section 2, the multilinear variable separation approach is applied to equation (1) to search for new variable separation solutions with arbitrary lower-dimensional functions. In section 3, the periodic breathing lumps, multi-dromion-ring-type instantons and hybrid waves on a doubly periodic wave background are illustrated to reveal abundant nonlinear structures and dynamics in higher dimensions. A discussion and summary are presented in the last section.

2. High-dimensional nonlinear variable separation solutions

It is straightforward to verify that under the transformation

$$u=\frac{6a}{b}(\ln f)_x+u_0, \tag{2}$$

where $u_0\equiv u_0(y, z, s)$ is an arbitrary function of the indicated arguments, the $(4+1)$-dimensional BLMP equation (1) can be transformed to the following bilinear form

$$\begin{aligned} &(D_yD_t+D_zD_t+D_sD_t+a(D_yD_x^3+D_zD_x^3+D_sD_x^3)\\ &+b(u_{0y}+u_{0z}+u_{0s})D_x^2)f\cdot f=0, \end{aligned} \tag{3}$$

where D_i^m $(i=x, y, z, s, t)$ are the Hirota's bilinear operators [19] defined by

$$\begin{aligned} D_i^m f\cdot f=&(\partial_i-\partial_{i'})^m f(x, y, z, s, t)\\ &\cdot f(x', y', z', s', t')|_{x'=x,y'=y,z'=z,s'=s,t'=t}, \end{aligned}$$

with m being a non-negative integer.

It is noted that the multilinear variable separation approach was first proposed to search for exact solutions with as many arbitrary (1+1)-dimensional functions as possible for (2+1)-dimensional nonlinear equations. In this case, there is only one way to divide the spatial arguments of the expansion function $f\equiv f(x, y, t)$ in the transformation, equation (2), namely, assume f to be a combination of two different functions with variables (x, t) and (y, t), respectively. However, it would be greatly varied for higher-dimensional systems. For instance, for the $(4+1)$-dimensional BLMP equation (1), variable separated functions can have different combinations of the spatial variables x, y, z, s, except for the time variable t. Here, we consider that four spatial variables are separated into two functions by requiring one of them to include as many variables as possible. In this case, f is assumed to be

$$f=a_0+a_1p+a_2q+a_3pq, \tag{4}$$

where a_0, a_1, a_2 and a_3 are constants, and $p\equiv p(y, z, s, t)$ and $q\equiv q(x, t)$ are variable separated functions of the indicated variables. Substituting equation (4) into equation (3) leads to

$$\begin{aligned}&[(a_0a_3 - a_1a_2)(q_t + aq_{xxx}) - (a_1 + a_3q)^2p_t\\&- (a_1 + a_3q)(a_0 + a_1p + a_2q\\&+ a_3pq)\partial_t](p_y + p_z + p_s)\\&- b(a_2 + a_3p)(u_{0y} + u_{0z} + u_{0s})[(a_2 + a_3p)q_x^2\\&- q_{xx}(a_0 + a_1p + a_2q + a_3pq)] = 0.\end{aligned} \tag{5}$$

It is not difficult to find that equation (5) can be identically satisfied in the following two cases.

Case 1: $p_t \neq 0$. In this case, equation (5) can be separated into two variable separated equations

$$u_{0y} + u_{0z} + u_{0s} = 0, \tag{6}$$

and

$$p_y + p_z + p_s = 0, \tag{7}$$

while q is an arbitrary function of (x, t). From the linear equations (6) and (7), it is easy to find that $u_0 \equiv u_0(-y+z, -y+s)$ and $p \equiv p(-y+z, -y+s, t)$, respectively, which are arbitrary functions of the indicated variables.

Case 2: $p_t = \mathbf{0}$. In this situation, p becomes an arbitrary function of (y, z, s), while q is determined by

$$q_t + aq_{xxx} = 0, \tag{8}$$

and u_0 still satisfies equation (6).

Consequently, a new variable separation solution of the (4 + 1)-dimensional BLMP equation (1) is obtained in the form of

$$u = \frac{6a(a_2 + a_3p)q_x}{a_0 + a_1p + a_2q + a_3pq} + u_0, \tag{9}$$

where u_0 is determined by equation (6), p is determined by equation (7) and q is an arbitrary function of (x,t), or p is an arbitrary function of (y, z, s) and q is determined by equation (8). The novelty of the above variable separation solution is remarked on as follows.

Remark 1. It is noted that the solution, equation (9), contains at least one arbitrary variable separated function $p \equiv p(y, z, s)$ or $q \equiv q(x, t)$ of the indicated arguments. Other types of arbitrary functions can also be found from equations (6) and (7) with special arguments $-y + z$ and $-y + s$, as stated above. It is also noted that in the second case, the four independent variables are fully separated into two variable separated functions.

Remark 2. The potential field $U \equiv u_y + u_z + u_s$, from equation (9), can be written as

$$U = \frac{6a(a_0a_3 - a_1a_2)q_x(p_y + p_z + p_s)}{b(a_0 + a_1p + a_2q + a_3pq)^2}, \tag{10}$$

which becomes zero due to equation (7) or nonzero with the condition in equation (8). It is obvious that the potential field U, equation (10), reduces to the same form of the universal quantity [20] when p is independent of the variables z and s.

Remark 3. Except for different variables in functions p and q, compared with the universal quantity [20], the more important and fundamental difference lies in the fact that the second part u_0 in equation (9), being an arbitrary seed solution of $-y + z$ and $-y + s$, allows us to explore interplays between various nonlinear waves and different background waves. It is noted that solutions with nonzero backgrounds have been obtained by various other methods, and the interplays between nonlinear excitations and backgrounds have been investigated in detail [21–23].

3. (4 + 1)-dimensional nonlinear waves and dynamics

Due to the difficulty in graphically displaying four-dimensional nonlinear waves in a three-dimensional coordinate system, abundant four-dimensional nonlinear excitations will be illustrated in a reduced spatial space, and thus one can find different structures of the whole wave patterns from different views of angles.

Let us first explore some possible wave excitations for the potential field U given by equation (10), with p arbitrary and q determined by equation (8). It is seen that equation (8) is a linear partial differential equation and possesses many special analytical solutions. The following gives two explicit special solutions of equation (8) and two specific expressions of the arbitrary function p, so that two new types of nonlinear wave excitations are generated and their dynamics are depicted at some special reduced spatial–temporal coordinates.

Type 1 Periodic breathing lumps. The first special solution of equation (8) reads

$$q = \sum_{i=0}^{N} A_i \cos(k_i x + ak_i^3 t + c_i), \tag{11}$$

where A_i, k_i and c_i are arbitrary constants, which means q is a linear combination of N periodic travelling waves in the x-direction with the period $2\pi/k_i$, amplitude A_i, velocity $-ak_i^2$ and initial phase c_i. In this case, we focus on a periodic lump-type wave structure, and thus take p as

$$p = \frac{1}{s^2 + y^2 + z^2 + C}, \tag{12}$$

to capture the property of a standing lump for simplicity, where C is an arbitrary positive constant introduced to avoid singularity. It is noted that more complicated behaviour of the lump-type wave can be excited as p depends on time t. In this case, we have the time t in the cosine function and thus breathing lumps will be generated. Evidently, the potential field, equation (10) with equations (11) and (12), decays algebraically in the s, y, z directions and periodically in the x directions.

To make it easy to graphically display the wave structures and their dynamics, we take

$$a = a_0 = a_1 = a_2 = b = 1, \quad a_3 = 2, \tag{13}$$

and

$$\begin{aligned}&N = 2,\ c_0 = -5,\ c_1 = 5,\ k_0 = 1,\ k_1 = 2,\\&C = 16,\ A_0 = \frac{1}{2},\ A_1 = \frac{3}{10}.\end{aligned} \tag{14}$$

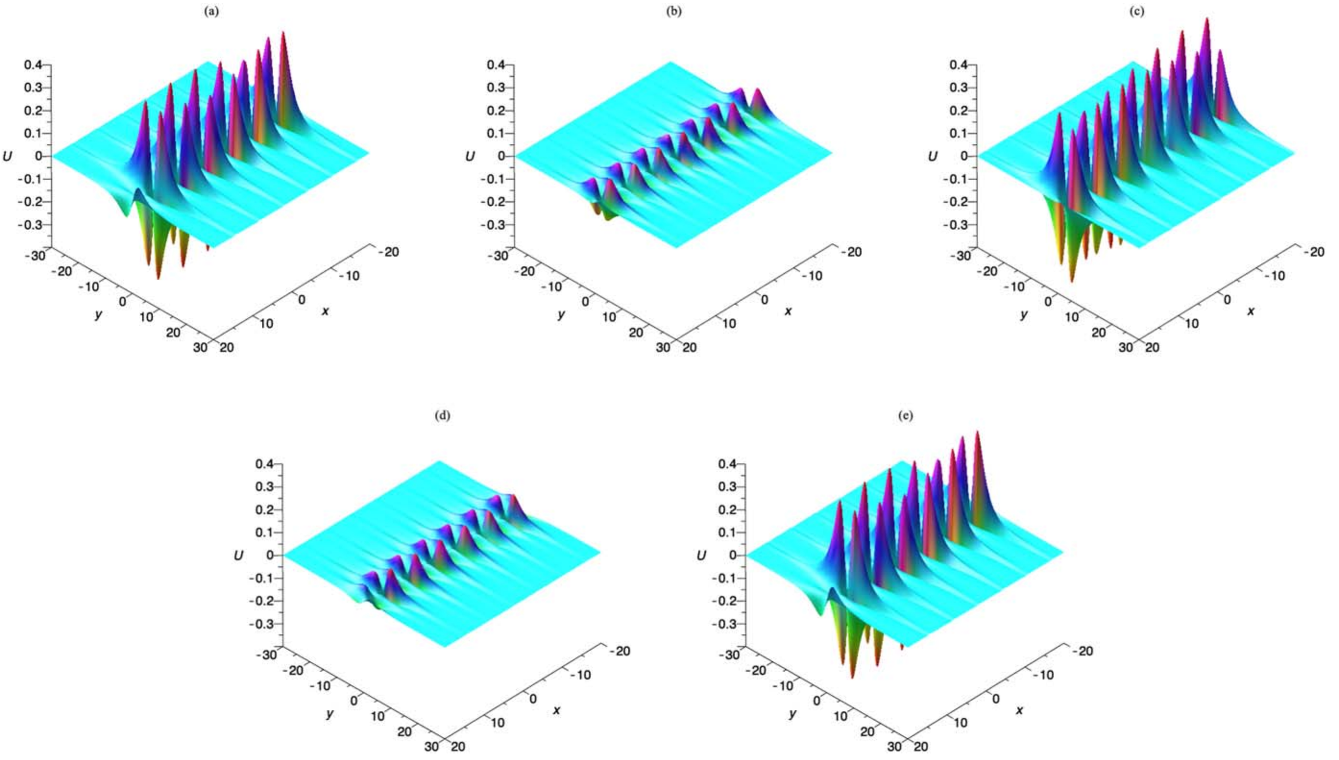

Figure 1. Two periodic lumps determined by equation (10) with equations (11)–(14) breathing at time (a) $t = 0$, (b) $t = \frac{\pi}{2}$, (c) $t = \pi$, (d) $t = \frac{3\pi}{2}$ and (e) $t = 2\pi$, respectively.

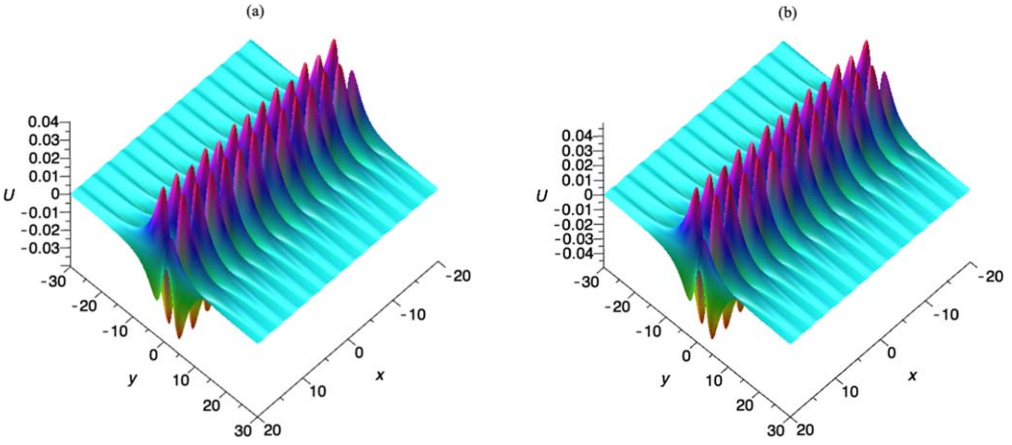

Figure 2. Two periodic lumps at time $t = 0$ determined by equation (10) with equations (11)–(14), except for (a) $A_0 = 0$, and (b) $A_1 = 0$, respectively.

In this situation, in the reduced spacetime $\{x, y, t\}$ coordinate system, an interacting breathing wave with its time period 2π is obtained, as shown in figure 1, which consists of two periodic lumps, as displayed in figure 2. The wave in the $y{-}s$ coordinate is shown in figure 3(a), exhibiting a dipole-lump structure, and its amplitude oscillates periodically, as illustrated in figure 3(b).

Type 2 Multi-dromion-ring-type instantons. The second special solution of equation (8) can be obtained as

$$q = \sum_{i=1}^{N} \frac{A_i}{\operatorname{sech}(k_i x - a k_i^3 t + c_i)}, \tag{15}$$

with arbitrary constants A_i, k_i and c_i, which leads to instantons with their existing time determined by ak_i^2. Instantons are important nonlinear excitations that are especially applied in field theory [24]. Here, we would like to show that instantons in higher dimensions can also possess nonlinear structures like dromions and ring solitons in different reduced dimensions. For instance, take p as

$$p = \sum_{j=0}^{M} B_j \operatorname{sech}(l_{1j}(s - s_j)^2 + l_{2j}(y - y_j)^2 + l_{3j}(z - z_j)^2 - R_j^2), \tag{16}$$

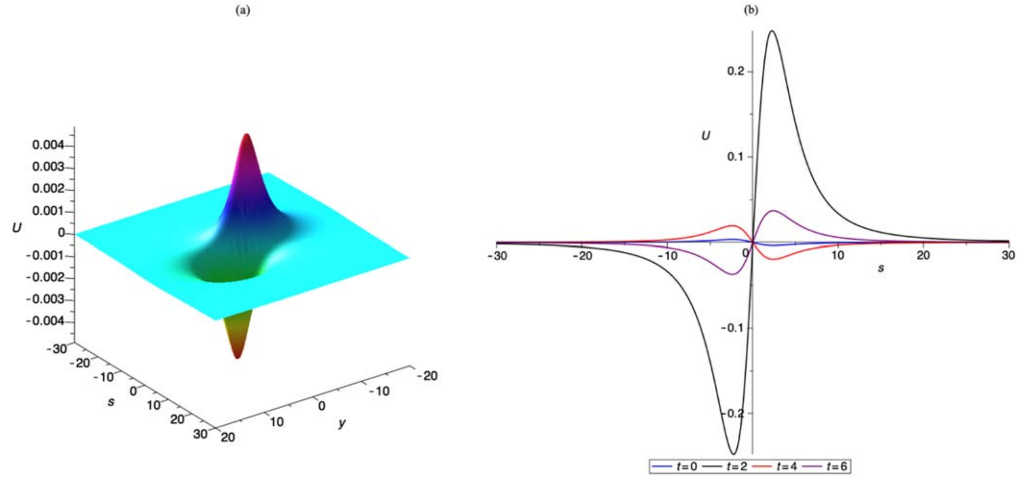

Figure 3. (a) The dipole-lump structure determined by equation (10) with equations (11)–(14): (a) in the $y-s$ space at time $t=0$, and (b) along the s direction at different times.

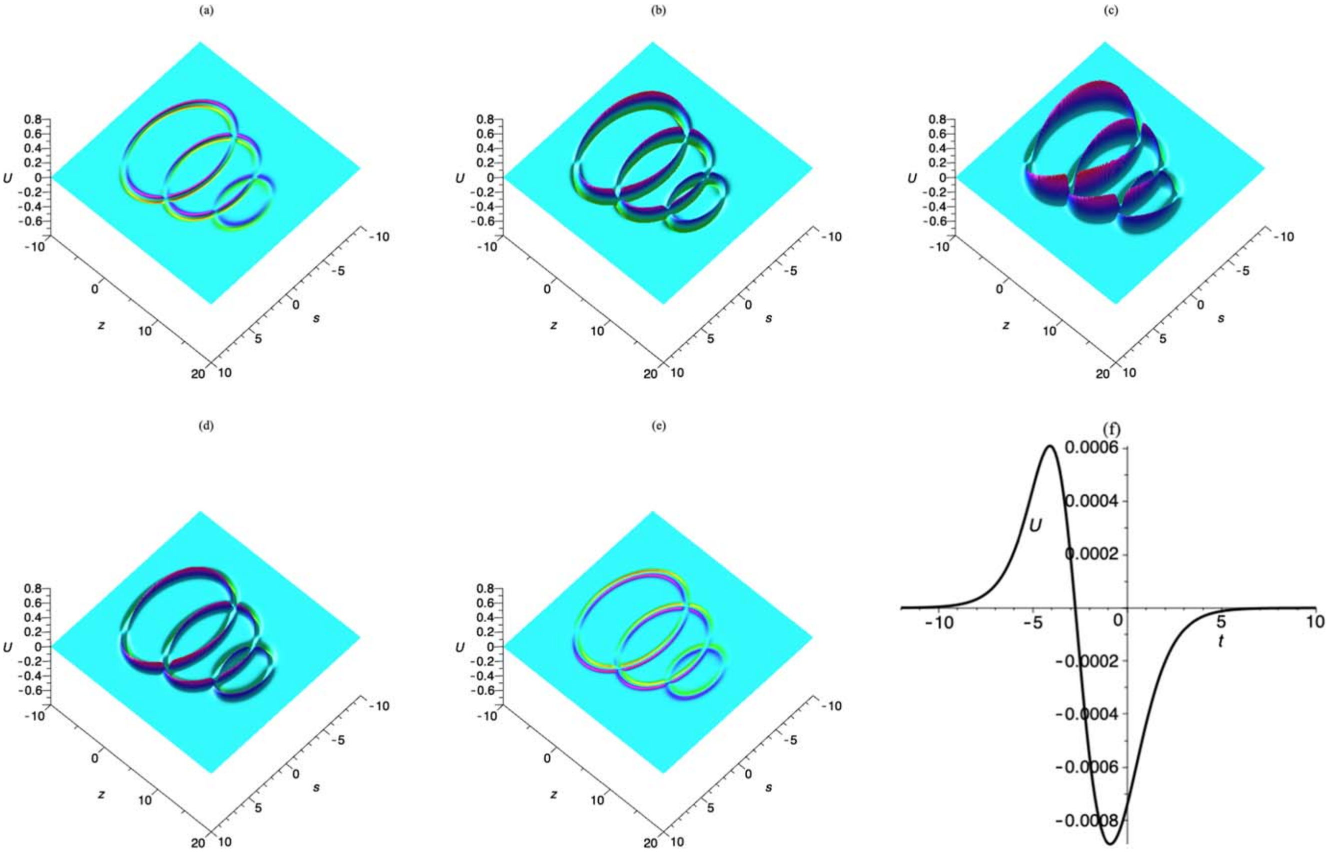

Figure 4. The dynamics of a three-ring-type instanton in the $z-s$ space at time (a) $t=-9$, (b) $t=-6$, (c) $t=-2$, (d) $t=2$ and (e) $t=6$. (f) The variation of the wave's amplitude along $s=5$ with respect to time t.

where the arbitrary constant B_i, together with A_i, determines the amplitude of the instanton on the varying background; s_j, y_i and z_j determine the sites where the dromion and ring soliton are situated; the positive constants $l_{ij}\,(i=1,2,3)$ and R_j determine the radius of the ring soliton. It is noted that the solution will exhibit the feature of a dipole dromion in the spatial domain with x and a ring soliton in the space without x. As an illustration, fixing

$$\begin{aligned}&M=N=A_1=c_2=2,\ A_2=3,\ c_1=-2,\ k_1=\frac{1}{2},\\&k_2=-1,\ l_{1j}=l_{2j}=l_{3j}=a=1,\\&s_j=y_j=z_0=0,\ z_1=6,\ z_2=12,\ R_1=6,\\&R_2=5,\ R_3=3,\quad j=0,1,2,\end{aligned}\tag{17}$$

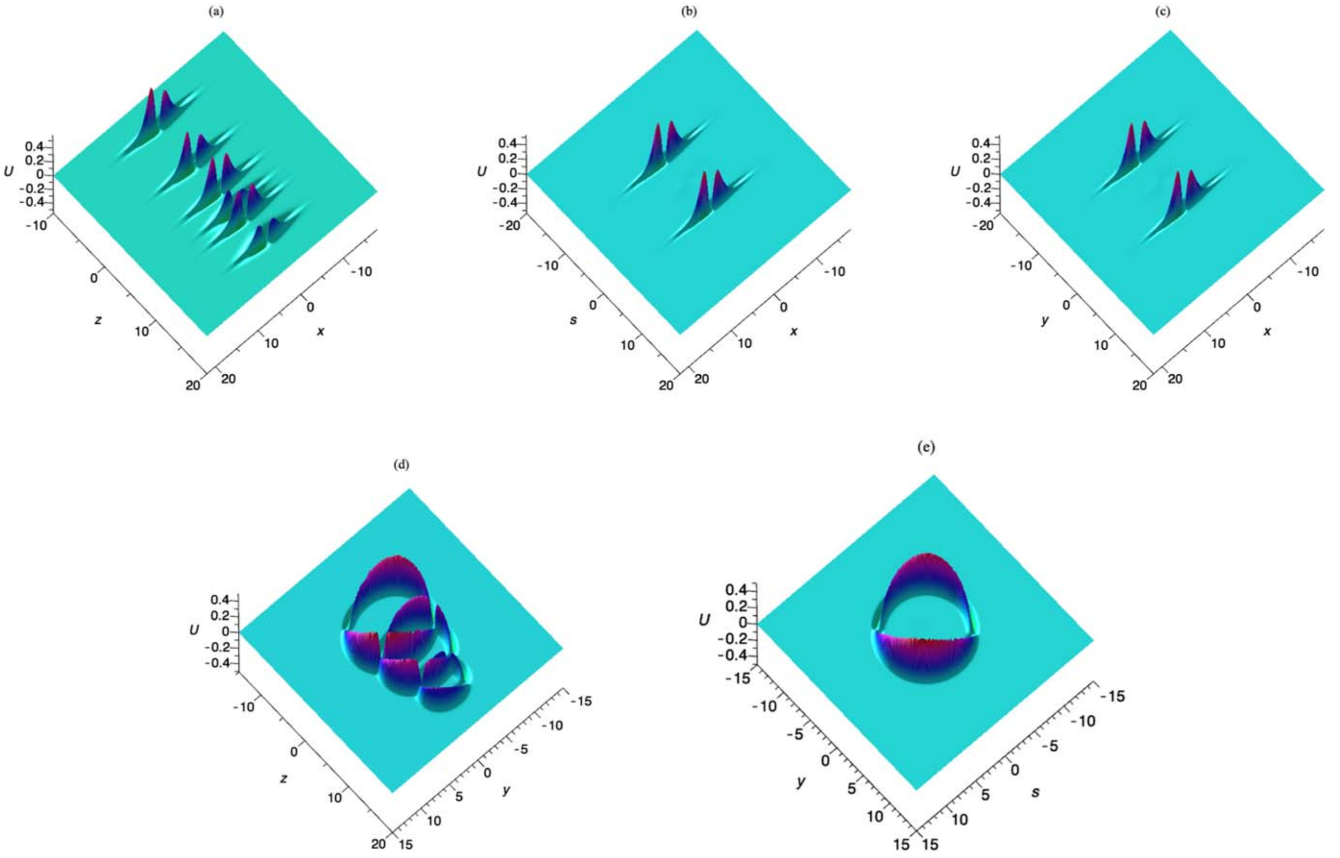

Figure 5. Profiles of a multi-dromion-ring-type instanton in other coordinate systems at time $t = 0$.

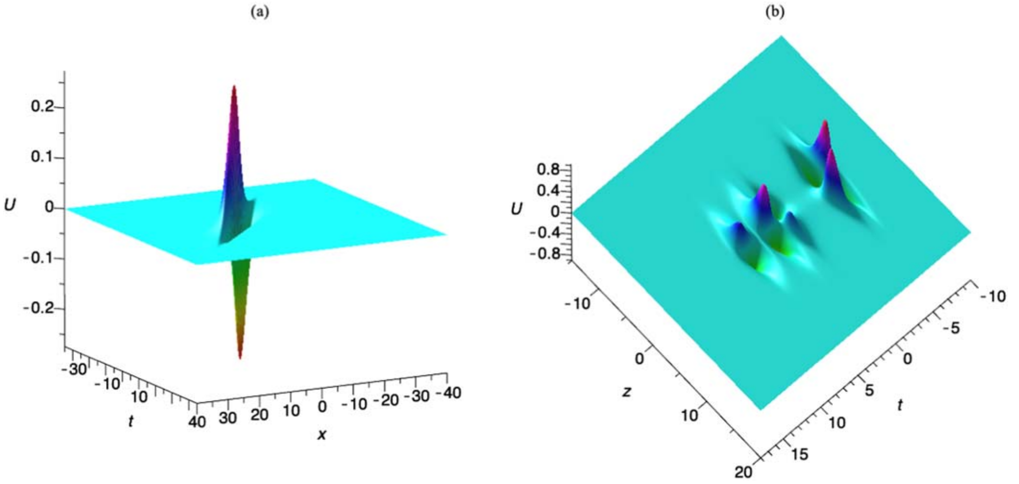

Figure 6. The lifetime of the instanton along the special lines: (a) in the $x - t$ coordinate with $s = -5,\ y = 0,\ z = 5$; and (b) in the $z - t$ coordinate with $s = -5,\ y = 0,\ x = 5$.

could lead to a three-ring-type instanton, as shown in figure 4 for the quantity U, equation (10) with equations (13), (15) and (17), in the $z - s$ space. In other coordinate systems, the wave will exhibit the structures of multi-dromion, three-ring and single-ring solutions, as displayed in figure 5. The lifetime of the instanton along two special lines is presented in figure 6, which clearly shows that the wave does exist during a particular time period. Specifically, it is also localized in the time variable.

As mentioned in remark 1, the solution, equation (9), also allows q to be an arbitrary function of (x, t), and p and u_0 to be arbitrary functions involving special arguments $(z - y)$ and $(s - y)$. Therefore, in this situation, in contrast to the excited waves describing the potential field U, we can explore nonlinear excitations situated on various background waves governed by u_0 for the original physical field u. As an illustration, a hybrid wave interacting on a doubly periodic wave background is exhibited below.

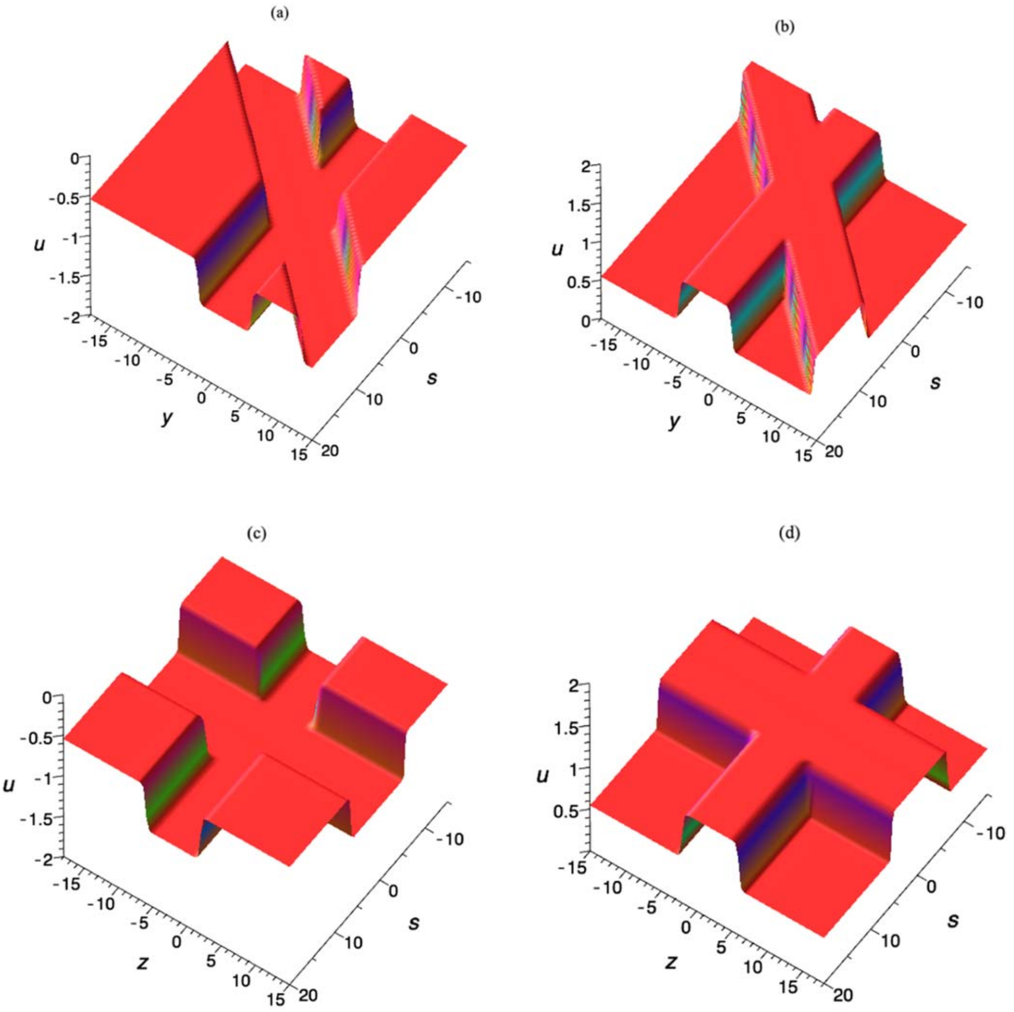

Figure 7. Flat-top and flat-bottom crossed line solitons without background waves in different coordinates at time $t = -1$ in (a), (c), and $t = 1$ in (b), (d), respectively.

Type 3 A hybrid wave on a doubly periodic wave. Localized nonlinear waves on variable backgrounds in higher-dimensional models have attracted considerable interest [21–23]. It is revealed that localized waves might be greatly affected by background waves. For instance, it is shown in [21] that a lump might turn into a wave with a well-localized spike and declining structures on a constant background or may result in the coexistence, interacting wave and breather formation due to periodic and localized-type arbitrary spatial backgrounds, and a soliton exhibited kink and anti-kink wave patterns along different spatio-temporal backgrounds. In this case, we assume the background wave is governed as

$$u_0 = 0.01\,\mathrm{sn}(z - y, m_1) + 0.01\,\mathrm{dn}(s - y, m_2) + 0.02\,\mathrm{dn}(z - y, m_3), \tag{18}$$

where sn and dn are the Jacobi elliptic functions with associated moduli m_1, m_2 and m_3, respectively, showing a non-moving wave that is periodic in two particular directions along $z - y$ and $s - y$. It is noted that backgrounds determined by Jacobi elliptic functions have also been applied to investigate the influence of arbitrary periodic, localized and combined background waves on single-kink solitons and rogue waves [22]. Here, we just concentrate on the variation of nonlinear waves due to different moduli, and thus only the moduli remain free and the other parameters are fixed.

The arbitrary constants in the solution, equation (9), are fixed as

$$a = 20,\ a_0 = a_1 = a_2 = 1,\ a_3 = 2,\ b = \frac{1}{10}, \tag{19}$$

and the other two arbitrary lower-dimensional variable separated functions are simply taken as exponential functions of the indicated arguments with fixed parameters,

$$q = 5\mathrm{e}^{-(x-t)^2-9}, \tag{20}$$

and

$$p = \mathrm{e}^{-(z-y-2t)^2+16} + \mathrm{e}^{-(s-y+0.5t)^2+36}, \tag{21}$$

respectively, which can be responsible for the flat-top and flat-bottom crossed line solitons, as shown in figure 7. It is clearly seen that with the time evolution, the crossed line solitons shift from a flat top to a flat bottom.

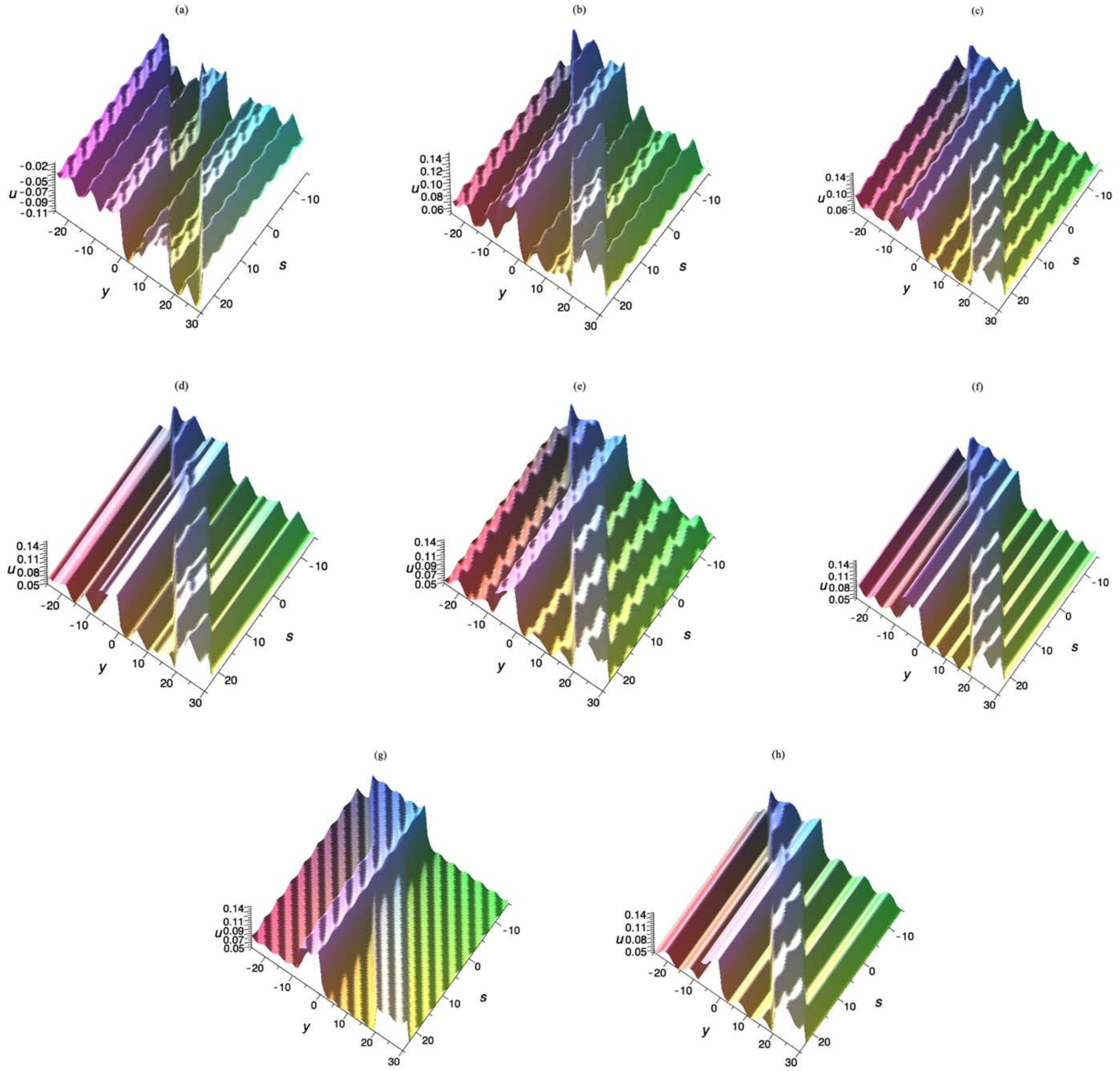

Figure 8. The interactions between the flat-top/bottom crossed line soliton and the wave background in the $y-s$ coordinate with the moduli $m_1=0.9$, $m_2=0.94$, $m_3=0.98$ at time (a) $t=-2$, and (b) $t=2$. The corresponding partially limited behaviour of the nonlinear wave in (b) when one or two moduli approach one is displayed in (c) $m_1=1$, (d) $m_2=1$, (e) $m_3=1$, (f) $m_1=m_2=1$, (g) $m_1=m_3=1$ and (h) $m_2=m_3=1$, respectively.

The interactions between the flat-top/bottom crossed line solitons and the doubly periodic wave background viewed in the $y-s$ space are displayed in figures 8(a) and (b) at time $t=-2$ and $t=2$, respectively. It is known that as the moduli approach one, the Jacobi elliptic functions exhibit limiting behaviour, namely, sn(x, 1) goes to $\tanh(x)$, while dn(x, 1) becomes $\mathrm{sech}(x)$. Hence, the corresponding partially limiting behaviour of the nonlinear wave in figure 8(b) is displayed in figures 8(c)–(h) when one or two moduli approach one. If all the moduli are fixed as one, i.e. $m_1=m_2=m_3=1$, then a hybrid wave on a nonzero wave background may experience abundant interaction dynamics, as exhibited in figure 9. Finally, nonlinear structures of the hybrid wave in other coordinates under various particular moduli are illustrated in figure 10, where one can observe two dipole dromions in the $x-y$ space, and a line segment overlying a line soliton in the $x-s$ coordinate, and both can be excited on a periodic wave background.

4. Summary and discussion

In summary, the main features of this work are two-fold. First, it is demonstrated that new variable separation solutions with

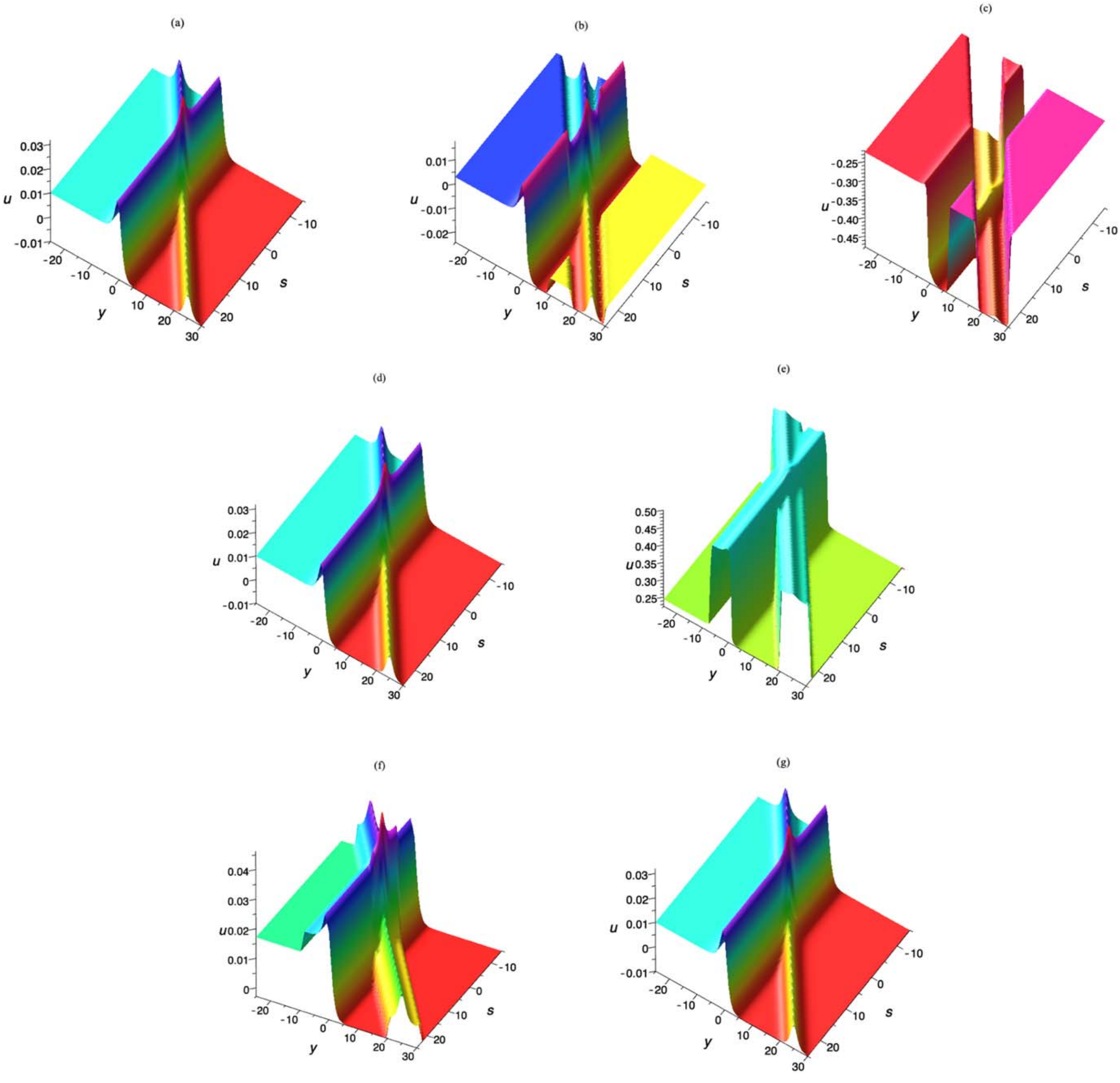

Figure 9. The dynamics of a hybrid wave on a nonzero wave background at time (a) $t=-4$, (b) $t=-2.5$, (c) $t=-1.5$, (d) $t=0$, (e) $t=1.5$, (f) $t=2.5$ and (g) $t=4$, respectively.

arbitrary lower-dimensional functions are obtainable for nonlinear integrable systems in $(4+1)$ dimensions. It is revealed that the $(4+1)$-dimensional BLMP equation possesses nonlinear variable separation solutions with two types of lower-dimensional arbitrary functions. Additionally, it is interesting to find that an arbitrary lower-dimensional wave background coming from the seed solution exists in the original field, while it disappears in a special potential field; thus, the wave background has no effect on this particular potential field.

Second, by taking advantage of the arbitrary functions, versatile nonlinear wave structures and dynamics can be obtained. In particular, three types of complex waves are graphically displayed to show that higher-dimensional nonlinear systems can excite more interesting waves and dynamics. However, the nonlinear mechanism of higher-dimensional waves needs further investigation. Moreover, considering that an arbitrary function responsible for varying backgrounds exists in a variable separation solution, the salient features of the spatially varying backgrounds can be unearthed. Here, just a type of wave on the background determined by Jacobi elliptic functions is presented, and the results with different moduli of Jacobi elliptic functions are exhibited simply and graphically. In fact, the effects of background waves on nonlinear excitations need more elaborate exploration.

As mentioned above, there are more kinds of combinations of the independent arguments for the variable separated

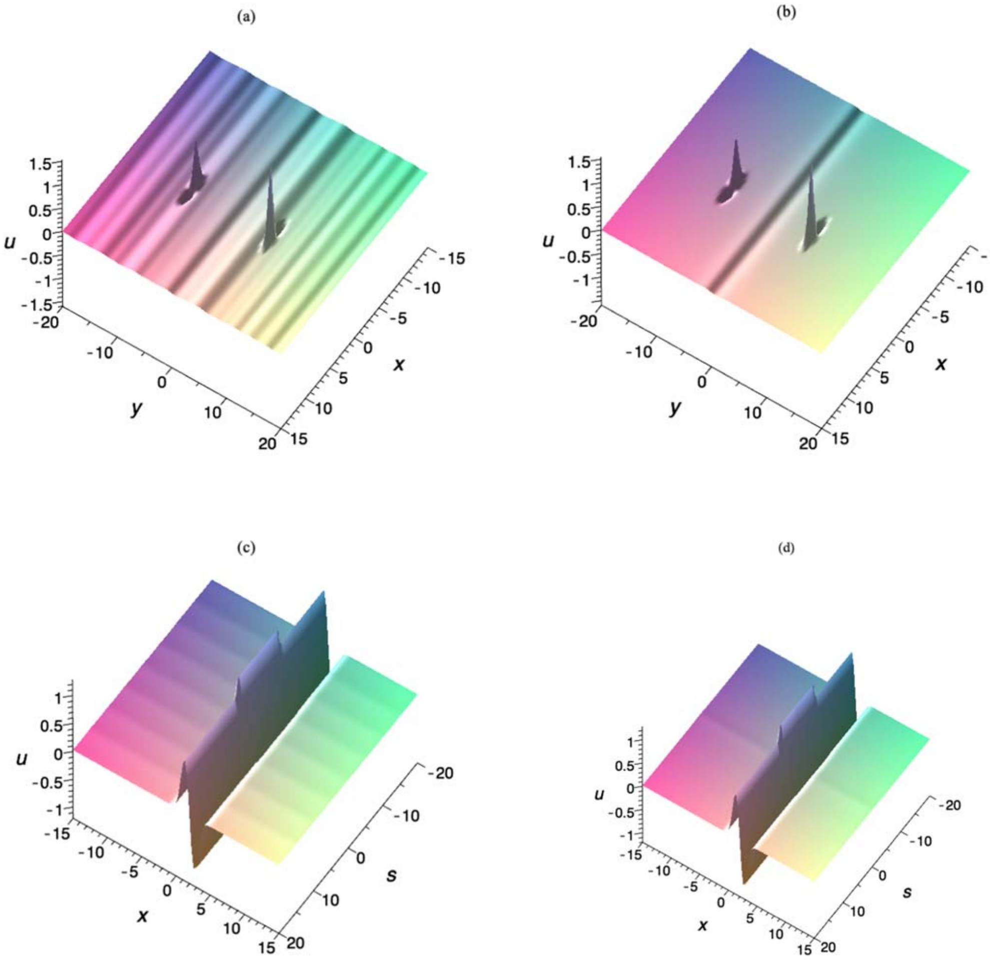

Figure 10. Nonlinear structures of the hybrid wave in different coordinates at time $t = 2$ when the moduli are given by $m_1 = 0.9$, $m_2 = 0.94$, $m_3 = 0.98$ for (a), (c), and $m_1 = m_2 = m_3 = 1$ for (b), (d), respectively.

functions in higher-dimensional nonlinear equations; however, only one of them is considered, which leads to two types of solutions. Therefore, new solutions under different separations of the independent variables and novel nonlinear localized waves under different selections of new arbitrary variable separated functions and background waves are worthy of further investigation, which might be useful in understanding nonlinear phenomena in higher dimensions.

Acknowledgments

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