


# Novel nonlinear wave transitions and interactions for (2+1)-dimensional generalized fifth-order KdV equation

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Received 16 May 2024, revised 23 July 2024

Accepted for publication 19 August 2024

Published 14 October 2024



CrossMark

## Abstract

The (2 + 1)-dimensional generalized fifth-order KdV (2GKdV) equation is revisited via combined physical and mathematical methods. By using the Hirota perturbation expansion technique and via setting the nonzero background wave on the multiple soliton solution of the 2GKdV equation, breather waves are constructed, for which some transformed wave conditions are considered that yield abundant novel nonlinear waves including X/Y-Shaped (XS/YS), asymmetric M-Shaped (MS), W-Shaped (WS), Space-Curved (SC) and Oscillation M-Shaped (OMS) solitons. Furthermore, distinct nonlinear wave molecules and interactional structures involving the asymmetric MS, WS, XS/YS, SC solitons, and breathers, lumps are constructed after considering the corresponding existence conditions. The dynamical properties of the nonlinear molecular waves and interactional structures are revealed via analyzing the trajectory equations along with the change of the phase shifts.

Keywords: (2+1)-dimensional generalized fifth-order KdV equation, nonlinear molecular wave, resonance soliton

(Some figures may appear in colour only in the online journal)

## 1. Introduction

The study of nonlinear wave bounded state structures set off a new research upsurge in more and more fields, such as optical fiber communication, fluid mechanics, biophysics and information science [1–3]. Investigating the above problems are of great significance because they can provide valuable mathematical physics information and theoretical support for nonlinear phenomena and experimental results analysis in some scientific fields [4–7]. The types of nonlinear waves can be roughly divided into four categories: soliton, breather, lump and rogue wave. Solitons have many intriguing properties of particles and waves, reflecting common nonlinear characteristics in nature. It is a well known fact that periodicity is a very basic and important attribute of nonlinear wave propagation, originating from periodic coherence between atoms. The breather perfectly describes such a solution that is periodic in both time and space dimensions [8]. The rogue wave

depicts a steep wave localized in both space and time dimensions [9]. The lump solution is constructed in many integrable models, it is a type of rational exact solution and spatially localized in certain region [10, 11]. The aforementioned nonlinear wave structures can transform each other under the particular conditions, which produce many new wave structures.

Furthermore, the phenomenon of soliton resonance has also been widely explored. Soliton resonance can cause nonlinear excitation modes such as soliton fission and fusion. It is worth noting that the resonant Y-Shaped (YS) solitons actually describe the fusion and fission phenomenon of soliton, while the X-Shaped (XS) solitons can be used to describe different interaction modes under specific constraints [12, 13]. As a special case of soliton fission and fusion phenomena, Space-Curved (SC) solitons can be used to describe common curved waves in shallow water waves [14]. Researchers conducted in-depth studies on the nonlinear wave structure

and asymptotic analysis of  $n$ -component nonlinear Schrödinger ( $n$ -NLS) equation with non-zero boundary/mixed boundary conditions. These findings have vital significance for understanding multi-component nonlinear physics [15, 16, 17]. In addition, the propagation of light pulses in laser systems is effectively described by introducing the generalized fractional nonlinear Schrödinger equation and constructing its basic solitons through the variational approximation method [18].

Several bounded (coherent) states of solitons including soliton molecules (SMs) and breather molecules (BMs) have been a hot topic as they can provide application and extensive theory values in nonlinear physics fields [19–21]. The velocity resonance involved in this article specifically includes two conditions: the trajectory equations of nonlinear wave components are parallel, and further, their propagation velocities are equal. Thus, multifarious novel kinds of the SMs and BMs have been constructed successively via utilizing velocity resonance. Therefore, the interactional solutions among different types of nonlinear waves have been deeply investigated, because the collision behaviors can be utilized to accurately imitate nonlinear behaviors in nature, from this point of view, the elastic collision and inelastic collision modes, the emergence and degradation behaviors of solitons, and the oscillatory and non-oscillatory modes have been carefully studied. It can be seen from the above that investigating nonlinear waves and their interactions can increase the richness of analytical solutions for nonlinear integrable systems and offer vital insights into the understanding and application of complex nonlinear integrable systems and corresponding complex nonlinear phenomena. Nevertheless, there are few investigations on the interactional structures among novel kinds of nonlinear molecular waves, resonance soliton solutions and breathers, especially the collision-free interaction modes, which arouses our interest to explore this kind solution in another model.

In this paper, we consider the (2+1)-dimensional generalized fifth-order KdV (2GKdV) equation

$$u_t + \alpha u_{5x} + \beta u u_{xxx} + \gamma u_x u_{xx} + \delta u^2 u_x + u_y = 0, \quad (1)$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  are real constant coefficients of the dispersion term and higher-order derivative terms in 2GKdV equation, respectively. Equation (1) demonstrates the behaviors of the long wave under gravity and in a two-dimensional nonlinear lattice in shallow water [22–24]. Equation (1) could convert into the (1+1)-dimensional Sawada–Kotera equation [25, 26] via removing the derivative term  $u_y$  and setting  $\alpha = 1$ ,  $\beta = 15$ ,  $\gamma = 15$  and  $\delta = 45$ . There are numerous studies that have focused on the 2GKdV equation, such as the lump wave and interaction structures of equation (1) were acquired in [27], complexitons and periodic soliton solutions were also studied in [28]; nonlinear superposition phenomena between lump solitons and other forms of nonlinear localized waves were examined in [29] and the lump-periodic, breather and two-soliton solutions were presented in [30].

The article is structured as follows. In section 2, we first revisit the  $N$ -soliton solution for the 2GKdV equation via the Hirota perturbation expansion method, then obtain the first-

order breather, XS soliton and lump wave of the 2 GKdV equation. Next, we analyze the dynamics of the breather, XS soliton and lump wave. Moreover, we present a new class of transformed wave conditions to construct different nonlinear wave structures such as the asymmetric M-Shaped (MS) soliton, the XS/YS soliton, the Oscillation M-Shaped (OMS) soliton, the SC soliton and lattice periodic soliton, etc. In section 3, new types of molecule structures and interactional structures are constructed via the velocity resonance ansatz. We also demonstrate the dynamical properties of the above nonlinear wave structures. Finally, section 4 contains a brief summary.

## 2. The X-Shaped soliton, breather, lump and other types of nonlinear waves

According to [27, 31], we adopt the following bilinear transformation under the conditions that  $\beta = 15\alpha$ ,  $\gamma = 45\alpha$ ,  $\delta = 15\alpha$  and

$$u(x, y, t) = u_0 + u_x, \quad (2)$$

where  $u_0 = u|_{x,y=\pm\infty}$  is an arbitrary constant. By inserting equation (2) into the 2 GKdV equation and integrating once with respect to  $x$ , we derive

$$15\alpha(u_x^3 + u_x u_{xxx} + u_0 u_{xxx}) + 45\alpha(u_0 u_x^2 + u_0^2 u_x) + \alpha u_{5x} + u_t + u_y = 0. \quad (3)$$

Adopting the transformation  $u = 2(\ln f)_x$ , the 2 GKdV equation is described in the bilinear form using the Hirota D operator

$$(45u_0^2 \alpha D_x^2 + 15u_0 \alpha D_x^4 + \alpha D_x^6 + D_x D_y + D_x D_t) f \cdot f = 0, \quad (4)$$

where  $D_x$ ,  $D_y$  and  $D_t$  represent the bilinear derivatives [32].

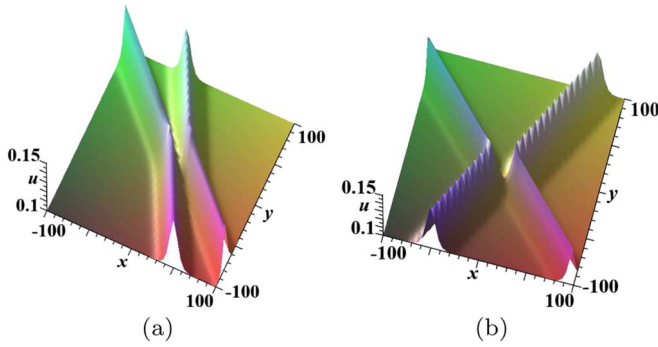
We first give the multiple soliton solutions of the 2 GKdV equation by the Hirota perturbation expansion method, then the first-order breather solution is constructed by setting partial parameters to be paired-complex conjugate forms [33], and the lump wave is created via the long-wave limit method [34, 35]. The  $N$ -soliton solution of the 2 GKdV equation is of the form

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \tau_i + \sum_{i<j<N} \mu_i \mu_j A_{ij}\right), \quad (5)$$

and

$$\begin{aligned} \tau_i &= k_i(x + l_i y + \omega_i t) + \phi_i, \\ \omega_i &= -(\alpha k_i^4 + 45\alpha u_0^2 l_i + 15\alpha u_0 k_i^2 + l_i), \\ e^{A_{ij}} &= \frac{(k_i^2 - k_j k_j + k_j^2)(k_i - k_j)^2 + 9u_0(k_i - k_j)^2}{(k_i^2 + k_j k_j + k_j^2)(k_i + k_j)^2 + 9u_0(k_i + k_j)^2}, \end{aligned} \quad (6)$$

where  $\tau_i$ ,  $A_{ij}$  represent phases and phase shifts of solitons, respectively. Besides,  $k_i$ ,  $l_i$ ,  $\phi_i$  ( $i = 1, 2, \dots, N$ ) are arbitrary real constants;  $\sum_{\mu=0,1}$  is the summation over possible combinations of  $\mu_i$ ,  $\mu_j = 0, 1$  ( $i, j = 1, 2, \dots, N$ ); the summation  $\sum_{i<j}^N$  should take all possible pairs selected from  $(i, j = 1, 2, \dots, N)$ .



**Figure 1.** (a) The XS soliton given by (12) with  $k_1 = 0.32, k_2 = 0.3, l_1 = 0.5, l_2 = 1, \alpha = 1, u_0 = 0.1, \phi_1 = 0, \phi_2 = 0, t = 0$ . (b) The XS soliton given by (12) with  $k_1 = -0.32, k_2 = 0.3, l_1 = -0.5, l_2 = 1, \alpha = 1, u_0 = 0.1, \phi_1 = 0, \phi_2 = 0, t = 0$ .

### 2.1. X-Shaped soliton and first-order breather

From the expression (5), the two-soliton of equation (1) is

$$u = 2(\ln f_2)_x, \quad f_2 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_1 + \tau_2 + A_{12}}, \quad (7)$$

where  $\tau_1, \tau_2, e^{A_{12}}$  are determined by equation (6).

To construct the XS solitons, we further take the above two-soliton solution (7) as an example to explore the generation mechanism of XS soliton solution. By setting  $A_{12} \gg 1$  or  $A_{12} \ll 1$  in solution (7), we derive some XS solitons. Figure 1 displays the XS solitons based on solution (7). In fact, these XS solitons can describe different mutual collision modes when  $A_{12} = O(1), A_{12} \gg 1$  or  $A_{12} \ll 1$ .

In order to better study the superposition mechanism among nonlinear waves, we start with the first-order breather solution and some parameters are chosen as

$$k_1 = k_2^* = m_1 + in_1, \quad l_1 = l_2^* = p_1 + iq_1, \quad \phi_1 = \phi_2^* = \gamma_1 + i\eta_1. \quad (8)$$

where the asterisk (\*) represents the complex conjugation and  $m_1, n_1, p_1, q_1, \gamma_1, \eta_1, \lambda_1$  denote arbitrary constants,  $i$  is the imaginary unit. Substituting equation (8) into equation (7), we suggest the function  $f=f(x, y, t)$  having the following appropriate form [36, 37]

$$f_2 \sim e^{\xi_1} + \lambda_1 \cos(\zeta_1) + \lambda_2 e^{-\xi_1}, \quad (9)$$

and

$$\begin{aligned} \xi_1 &= m_1x + (m_1p_1 - n_1q_1)y + \omega_{R1}t + \gamma_1, \\ \zeta_1 &= n_1x + (m_1q_1 + n_1p_1)y + \omega_{I1}t + \eta_1, \\ \omega_{R1} &= -\alpha m_1^5 + (10\alpha n_1^2 - 15\alpha u_0)m_1^3 \\ &\quad + (-5\alpha n_1^4 + 45\alpha u_0 l_1^2 \\ &\quad - p_1(45\alpha u_0^2 + 1)m_1 + q_1(45\alpha u_0^2 + 1)n_1, \\ \omega_{I1} &= -\alpha n_1^5 + 5(2\alpha m_1^2 + 3\alpha u_0)n_1^3 + ((-45\alpha u_0^2 - 1)p_1 \\ &\quad - 5\alpha m_1^2(m_1^2 + 9u_0))n_1 - q_1(45\alpha u_0^2 + 1)m_1, \\ \lambda_2 &= -\frac{n_1^2(m_1^2 - 3n_1^2 + 9u_0)}{4m_1^2(3m_1^2 - n_1^2 + 9u_0)}\lambda_1. \end{aligned} \quad (10)$$

Based on the definition of hyperbolic function, solution (9)

can be written as

$$f_2 \sim 2\sqrt{\lambda_2} \cosh\left(\xi_1 - \frac{1}{2} \ln \lambda_2\right) + \lambda_1 \cos \zeta_1. \quad (11)$$

Carrying equation (11) into the transformation  $u = u_0 + 2 \ln(f_2)_{xx}$ , the first-order breather solution of the 2 GKdV equation can be obtained

$$\begin{aligned} u &= u_0 \\ &+ \frac{r_1 + r_2 \cosh(\xi_1 - \frac{1}{2} \ln \lambda_2) \cos \zeta_1}{\left(2\sqrt{\lambda_2} \cosh(\xi_1 - \frac{1}{2} \ln \lambda_2) + \lambda_1 \cos \zeta_1\right)^2} \\ &+ \frac{r_3 \sinh(\xi_1 - \frac{1}{2} \ln \lambda_2) \sin \zeta_1}{\left(2\sqrt{\lambda_2} \cosh(\xi_1 - \frac{1}{2} \ln \lambda_2) + \lambda_1 \cos \zeta_1\right)^2}, \end{aligned} \quad (12)$$

with  $r_1 = 8\lambda_2 m_1^2 - 2\lambda_1^2 n_1^2, r_2 = 4\lambda_1 \sqrt{\lambda_2} (m_1^2 - n_1^2), r_3 = 8\lambda_1 \sqrt{\lambda_2} m_1 n_1$  and  $u_0$  being a constant background wave, which has also been presented in [38, 39].

From the above simplified expression, we find that the breather solution is composed of hyperbolic functions  $H = \left\{ \cosh(\xi_1 - \frac{1}{2} \ln \lambda_2), \sinh(\xi_1 - \frac{1}{2} \ln \lambda_2) \right\}$  and trigonometric functions  $T = \{ \cos \zeta_1, \sin \zeta_1 \}$ , which play essential roles in deciding the key properties of solution (12). The functions  $H$  and  $T$  reveal the localization and periodicity of the evolution process, respectively. Thus, the breather is actually a mixed solution component of the solitary-type wave and periodic-type wave. It describes the localized solutions with breathing oscillation behavior on the constant background  $u_0$ . Furthermore, the propagation velocity vector of the solitary wave  $v_{\xi_1}$  and propagation velocity vector of the periodic wave  $v_{\zeta_1}$  are expressed in the following forms

$$\begin{aligned} v_{\xi_1} &= (v_{\xi_{1x}}, v_{\xi_{1y}}) = \left( \frac{\omega_{R1}}{m_1}, \frac{\omega_{R1}}{m_1 p_1 - n_1 q_1} \right), \\ v_{\zeta_1} &= (v_{\zeta_{1x}}, v_{\zeta_{1y}}) = \left( \frac{\omega_{I1}}{n_1}, \frac{\omega_{I1}}{m_1 q_1 + n_1 p_1} \right), \end{aligned} \quad (13)$$

where  $v_{\xi_{1x}}, v_{\xi_{1y}}$  denote the disseminate speed of the solitary wave in the  $x$ -axis and  $y$ -axis;  $v_{\zeta_{1x}}, v_{\zeta_{1y}}$  denote the disseminate speed of period wave in the  $x$ -axis and  $y$ -axis, respectively.

The periods  $T_1$  of the breather solution (12) are given by

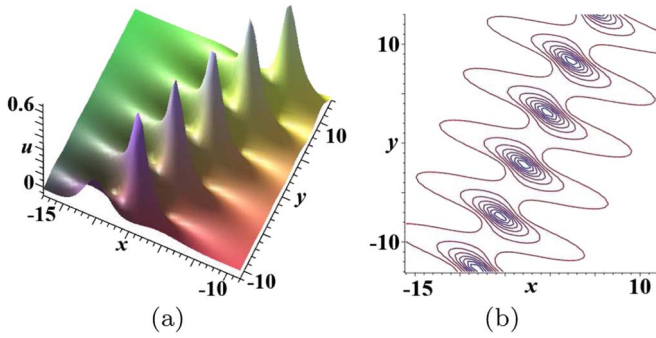
$$T_1 = (T_{1x}, T_{1y}) = \left( \frac{2\pi(n_1 q_1 - m_1 p_1)}{q_1(m_1^2 + n_1^2)}, \frac{2\pi m_1}{q_1(m_1^2 + n_1^2)} \right), \quad (14)$$

where  $T_{1x}, T_{1y}$  represent the periods of the breather solution (12) in the  $x$ -axis and  $y$ -axis directions, respectively.

From (14), we obtain the distance  $d_{\min}$  between the two nearest peaks of solution (12)

$$d_{\min} = |T_1| = \frac{2\pi}{m_1^2 + n_1^2} \sqrt{\frac{m_1^2 + (n_1 q_1 - m_1 p_1)^2}{q_1^2}}. \quad (15)$$

Additionally, the trajectory equations of the solitary-type wave member (SWM) and periodic-type wave member



**Figure 2.** The breather solution given by (12) with  $m_1 = 0.3, n_1 = 0.3, p_1 = 1.5, q_1 = 2, \lambda_1 = 2, \alpha = 1, u_0 = -0.01, \gamma_1 = 0, \eta_1 = 0, t = 0$ . (a) The spatial structure plot. (b) The contour plot of (a).

(PWM) in the breather solution (12) are:

$$\begin{aligned} L_{s_1}: m_1x + (m_1p_1 - n_1q_1)y + \omega_{R1}t + \gamma_1 &= 0, \\ L_{p_1}: n_1x + (m_1q_1 + n_1p_1)y + \omega_{I1}t + \eta_1 &= 0. \end{aligned}$$

Figure 2 shows the breather solution described by equation (12) in  $(x,y)$ -plane. In figure 2, the velocities of the solitary-type wave and the periodic-type wave are  $v_{\xi_1} = (0.5077, -1.0153)$  and  $v_{\zeta_1} = (-3.4564, -0.9875)$ , and the corresponding trajectory equations are  $0.3x - 0.15y + 0.1523t = 0$  and  $0.3x + 1.05y - 1.0369t = 0$  respectively. It should be clarified that the analytical condition to the solution (12) reads  $2\sqrt{\lambda_2} > |\lambda_1|$ .

### 2.2. Lump-type soliton and W-Shaped soliton solutions

In the previous section, we investigate the components of the breather solution and discuss the related dynamical properties. At present, we mainly consider the lump-type wave via the long wave limit technique, namely, when  $m_1^2 + n_1^2 \rightarrow 0$ , taking the limit of the infinity period. By analyzing equations (14) and (15), we find that if the condition  $m_1^2 + n_1^2 \rightarrow 0$  holds, the period  $T_1$  of solution (12) will tend to infinity and the breather solution as shown in figure 2 degenerates to a lump-type wave.

For the lump-type wave, we take

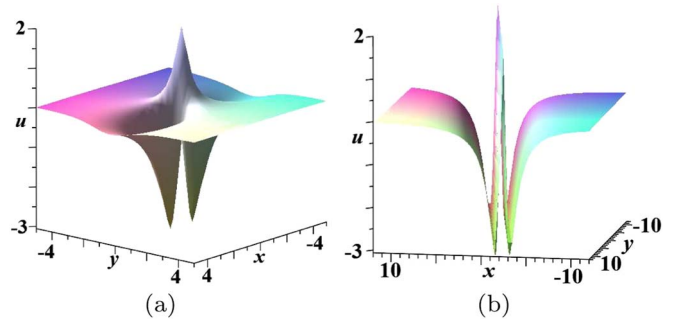
$$k_1 = k_2^* = m_1 + in_1, l_1 = l_2^* = p_1 + iq_1, \phi_1 = \phi_2^* = \gamma_1 + i\pi, \quad (16)$$

and put  $(m_1, n_1) = (\epsilon, \epsilon), \lambda_1 = -2, \lambda_2 = 1 + \epsilon^2$ . Then, taking the Taylor expansion of (9) when  $\epsilon = 0$ , we obtain

$$\begin{aligned} u_{\text{lump}} &= 2(\ln f_i)_{xx} + u_0 = [8 - 16(x + (p_1 + q_1)y \\ &\quad - (p_1 + q_1)t) \\ &\quad \times (x + (p_1 - q_1)y - (p_1 - q_1)t)]/f_i^2, \end{aligned}$$

where

$$\begin{aligned} f_i &= (x + (p_1 + q_1)y - (p_1 + q_1)t)^2 \\ &\quad + (x + (p_1 - q_1)y - (p_1 - q_1)t)^2 + 1. \end{aligned} \quad (17)$$



**Figure 3.** (a) The lump solution given by (17) with  $p_1 = 0.5, q_1 = -2, \alpha = 1, u_0 = 0, t = 0$ . (b) The WS soliton solution given by (19) with  $p_1 = 0.2, \alpha = 1, u_0 = 0, t = 0$ .

Figure 3(a) exhibits the lump-type wave with one wall-like peak and two small holes, which is localized in all directions in space [40, 41]. This solution can be viewed as a hybrid solution including a two-wave solution with variables  $\xi'_1 = x + (p_1 + q_1)y - (p_1 + q_1)t$  and  $\zeta'_1 = x + (p_1 - q_1)y - (p_1 - q_1)t$ . Similar to equation (13), we can express the speeds of the two-wave solution as the forms  $v_{\xi'_1} = (-(p_1 + q_1), -1), v_{\zeta'_1} = (-(p_1 - q_1), -1)$ . The trajectory equations ( $\xi'_1 = 0, \zeta'_1 = 0$ ) are determined accordingly. By utilizing the extremum value theorem of function, we calculate the maximum or minimum value points of the function  $u_{\text{lump}} = u(x, y, t_0)$ . It shows that the lump solution (17) possesses three extreme points at  $(0, t_0), (-0.8018, t_0)$  and  $(0.8018, t_0)$  with parameters being selected as  $p_1 = 0.5, q_1 = -2, \alpha = 1, u_0 = 0$ . The maximum value of lump solution (17) is 2 at  $(0, t_0)$ , while the minimum value is  $-3.0625$  at  $(\pm 0.8018, t_0)$ .

It is interesting that if we introduce a new class of parameter constraint condition that is known as the velocity resonance equation [42–44], a phenomena that the breather (12) converting into the rational WS solitary wave will occur when the following condition is satisfied, see figure 3(b).

$$v_{\xi'_1} = v_{\zeta'_1}, \quad i.e. \quad q_1 = 0. \quad (18)$$

Under this circumstance, a WS soliton solution can be expressed in the form

$$u_{ws} = 2(\ln f_w)_{xx} = \frac{4 - 8(x + p_1y - p_1t)^2}{(2(x + p_1y - p_1t)^2 + 1)^2}, \quad (19)$$

where

$$f_w = 2(x + p_1y - p_1t)^2 + 1. \quad (20)$$

As displayed in figure 3(b), we learn the WS soliton solution has single peak and double symmetrical valleys, of which the trajectory equation is  $x + 0.2(y - t) = 0$ .

### 2.3. Nonlinear wave structures from two-soliton solution (12)

In this section, we investigate the transformed wave conditions and the nonlinear wave structures including the WS solitary wave, the MS solitary wave, the OMS solitary wave and the Multi-Peak (MP) solitary wave, etc. The associated dynamic properties of the nonlinear wave structures will be studied exhaustively.

The previous investigation shows that if the direction of the propagation velocity vector  $v_{\xi_1}$  of the solitary wave and

the propagation velocity vector  $v_{\zeta_1}$  of the periodic wave are parallel, we can obtain different types of nonlinear wave structures. In order to study the structural properties of the breather (12), we take advantage of the following condition,

$$v_{\xi_1} \times v_{\zeta_1} = 0, \tag{21}$$

where the operator  $\times$  denotes the cross product of two velocity vectors, which is equivalent to

$$\begin{vmatrix} \frac{\omega_{R1}}{m_1} & \frac{\omega_{R1}}{m_1 p_1 - n_1 q_1} \\ \frac{\omega_{I1}}{n_1} & \frac{\omega_{I1}}{m_1 q_1 + n_1 p_1} \end{vmatrix} = 0. \tag{22}$$

That is to say, the subtle parameter condition should be met

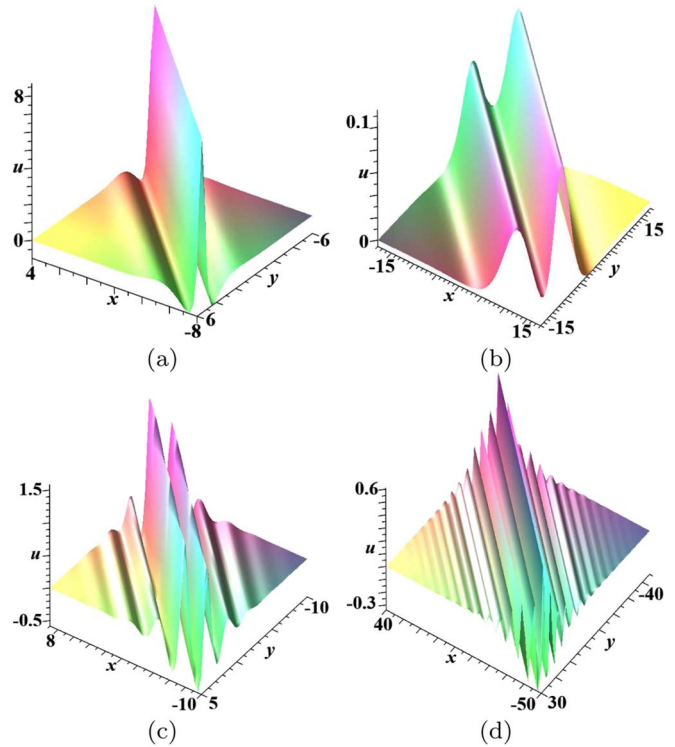
$$q_1 = 0. \tag{23}$$

From the above analysis, we learn that the breather solution (12) component of the functions  $H$  and  $T$  possesses uniform velocity direction. The tangent of the velocity direction is expressed in the form:  $\beta_1 = \arctan(p_1)$ , where  $\beta_1$  is the angle between the counterclockwise direction of two velocity vectors and the positive direction of the  $y$ -axis. Hence, it is summarized that the parameter  $p_1$  is responsible for the velocity direction of the different nonlinear wave structures. A series of nonlinear localized wave structures could be constructed via the specific mechanism (21)-(23). Figure 4 displays several types of nonlinear localized waves. The WS soliton solution with single peak and double symmetric valleys ( $L_{s_1} = x + y + 1.9725t = 0$  and  $L_{p_1} = 1.1x + 1.1y + 5.6704t = 0$ ) is shown in figure 4(a). Next, we demonstrate three cases:

- (i) When  $\frac{n_1}{m_1} \approx 1$ , an asymmetric MS soliton solution has double peaks and a valley ( $L_{s_1} = 0.3x + 0.3y - 0.2967t = 0$  and  $L_{p_1} = 0.2x + 0.2y - 0.1952t = 0$ ), exhibited in figure 4(b);
- (ii) With the increase of value  $\frac{n_1}{m_1}$  gradually, the MS solution will convert into the OMS soliton solution ( $\frac{n_1}{m_1} = 3.5$ ), the trajectory equations  $L_{s_1} = 0.5x + 0.6y + 6.5152t = 0$  and  $L_{p_1} = 1.75x + 2.1y + 1.065t = 0$ , presented in figure 4(c);
- (iii) When  $\frac{n_1}{m_1} = 10$ , an MP soliton solution will be obtained, as shown in figure 4(d), where the trajectory equations are  $L_{s_1} = 0.1x + 0.12y + 0.2495t = 0$  and  $L_{p_1} = x + 1.2y - 2.5955t = 0$ .

To proceed, we give an analysis of the essential physical quantity, the phase shift, which has a profound impact on the dynamic behaviors of the obtained nonlinear waves. In this paper, the concrete expressions of the phase shifts of solution (12) are given by

$$\begin{aligned} \phi_{\xi_1}(t) &= |v_{\xi_1}| t = \frac{\omega_{R1} t}{m_1(m_1 p_1 - n_1 q_1)} \sqrt{(m_1 p_1 - n_1 q_1)^2 + m_1^2}, \\ \phi_{\zeta_1}(t) &= |v_{\zeta_1}| t = \frac{\omega_{I1} t}{n_1(m_1 q_1 + n_1 p_1)} \sqrt{(m_1 q_1 + n_1 p_1)^2 + n_1^2}, \end{aligned} \tag{24}$$



**Figure 4.** Spatial structure plots of the different nonlinear waves described by (12) and  $\gamma_1 = 0, \eta_1 = 0, \alpha = 1, t = 0$ . (a) The WS soliton solution and  $m_1 = 1, n_1 = 1.1, p_1 = 1, \lambda_1 = 12, u_0 = -0.02$ . (b) The MS soliton solution and  $m_1 = 0.3, n_1 = 0.2, p_1 = 1, \lambda_1 = 2, u_0 = -0.01$ . (c) The OMS soliton solution and  $m_1 = 0.5, n_1 = 1.75, p_1 = 1.2, \lambda_1 = 2, u_0 = 0.5$ . (d) The MP soliton solution and  $m_1 = 0.1, n_1 = 1, p_1 = 1.2, \lambda_1 = 2, u_0 = 0.3$ .

where  $\phi_{\xi_1}(t), \phi_{\zeta_1}(t)$  represent the phase shifts of the SWM and PWM, respectively,  $|v_{\xi_1}|, |v_{\zeta_1}|$  denote the modulus of the velocity vectors of the SWM and PWM,  $\omega_{R1}, \omega_{I1}$  are determined by equation (10). According to equation (24), we conclude that the phase shifts of the SWM and PWM are not equal ( $\phi_{\xi_1}(t) \neq \phi_{\zeta_1}(t)$ ) and will alter with time varying. As a result, we obtain different nonlinear wave interactional structures at different times  $t$  because the interaction regions have changed that caused by the alteration of the trajectory equation.

If the solution  $f_2$  is expressed in the form

$$f_2 \sim e^{\tilde{\xi}_1} + \lambda_1 \cos(\tilde{\zeta}_1) + \lambda_2 e^{-\tilde{\xi}_1}, \tag{25}$$

where

$$\begin{aligned} \tilde{\xi}_1 &= a_1(x + b_1 y + c_1 t), & \tilde{\zeta}_1 &= a_2(x + b_2 y + c_2 t), \\ c_1 &= (-45b_1 u_0^2 + (-15a_1^2 + 45a_2^2)u_0 - a_1^4 + 10a_1^2 a_2^2 - 5a_2^4)\alpha - b_1, \\ c_2 &= (-45b_2 u_0^2 + (-45a_1^2 + 15a_2^2)u_0 - a_2^4 + 10a_1^2 a_2^2 - 5a_1^4)\alpha - b_2, \\ \lambda_2 &= -\frac{a_2^2(a_1^2 - 3a_2^2 + 9u_0)}{4a_1^2(3a_1^2 - a_2^2 + 9u_0)} \lambda_1, \end{aligned} \tag{26}$$

and  $a_i, b_i (i = 1, 2)$  are arbitrary real constants.

The solution (12) is readily rewritten as the following form

$$\tilde{u} = u_0 + \frac{r_1 + r_2 \cosh(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2) \cos \tilde{\zeta}_1}{(2\sqrt{\lambda_2} \cosh(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2) + \lambda_1 \cos \tilde{\zeta}_1)^2} + \frac{r_3 \sinh(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2) \sin \tilde{\zeta}_1}{(2\sqrt{\lambda_2} \cosh(\tilde{\xi}_1 - \frac{1}{2} \ln \lambda_2) + \lambda_1 \cos \tilde{\zeta}_1)^2}, \quad (27)$$

with  $r_1 = 8\lambda_2 a_1^2 - 2\lambda_1^2 a_2^2$ ,  $r_2 = 4\lambda_1 \sqrt{\lambda_2} (a_1^2 - a_2^2)$ ,  $r_3 = 8\lambda_1 \sqrt{\lambda_2} a_1 a_2$  and  $u_0$  being a constant background wave.

Similarly to expression (13), the propagation velocity vector  $v_{\tilde{\xi}_1}$  of the solitary wave and the propagation velocity vector  $v_{\tilde{\zeta}_1}$  of the periodic wave can be expressed in the following forms

$$v_{\tilde{\xi}_1} = \left( a_1 c_1, \frac{a_1 c_1}{b_1} \right), \quad v_{\tilde{\zeta}_1} = \left( a_2 c_2, \frac{a_2 c_2}{b_2} \right). \quad (28)$$

If the condition  $v_{\tilde{\xi}_1} \times v_{\tilde{\zeta}_1} = \mathbf{0}$  holds, namely,  $b_1 = b_2$ ,  $c_1 = c_2$ , and further letting the parameter  $a_1$  vanish, the corresponding periodic wave solution is given in the form

$$u'' = u_0 - \frac{2\lambda_1 a_2^2 (\lambda_1 + (1 + \lambda_2) \cos \tau)}{(1 + \lambda_2 + \lambda_1 \cos \tau)^2}, \quad (29)$$

where

$u_0 = \frac{2a_2^2}{15}$ ,  $\tau = a_2(x + b_2 y + (a_2^4 \alpha - \frac{4}{5} \alpha b_2 a_2^4 - b_2)t)$ . The existence condition for the periodic wave is  $\left| \frac{\lambda_1}{\lambda_2 + 1} \right| < 1$ . We notice that solution (29) will convert into different types of periodic structures under variant constraints about the parameters  $\lambda_1, \lambda_2$ .

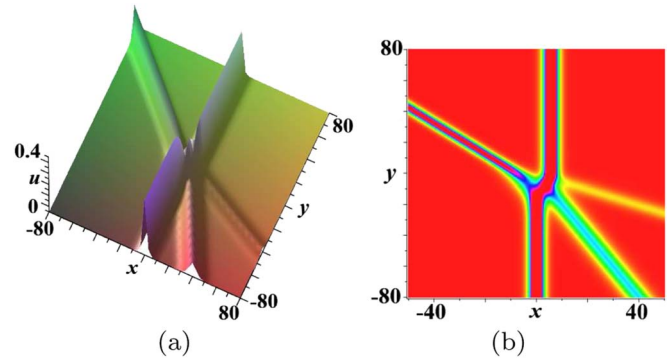
- (i) When  $0 < \left| \frac{\lambda_1}{\lambda_2 + 1} \right| < 1$ , a sine-like periodic wave is exhibited;
- (ii) when  $\left| \frac{\lambda_1}{\lambda_2 + 1} \right| = \frac{1}{2}$ , a soliton lattice with U-Shaped bottom is derived;
- (iii) when  $\frac{1}{2} < \left| \frac{\lambda_1}{\lambda_2 + 1} \right| < 1$ , a soliton lattice with W-Shaped bottom is constructed.

### 3. Mixed solutions including YS and SC solitons and other nonlinear molecular waves

In this section, the aim is to investigate interaction structures and nonlinear molecular waves based on the mixed solution and the fourth-order soliton under the specific velocity resonance mechanism. We discuss the elastic and inelastic collision modes among the nonlinear localized wave structures. Then, we give further explanations for the inelastic collisions by the trajectory equation and the phase shift analysis.

#### 3.1. Interaction structures from the mixed solution

Next, we focus on the Y-type soliton of the 2 GKdV equation. Taking  $N = 3$ ,  $A_{12} = 0$  in equations (5) and (6), the relevant



**Figure 5.** Spatial structure plots of the mixed solution given by (30) with  $\alpha = 1$ ,  $u_0 = -0.05$ ,  $t = 0$ . (a) The interactional structure between the Y-type soliton and the single soliton with  $k_1 = 0.5$ ,  $k_2 = 0.3$ ,  $l_1 = 0.5$ ,  $l_2 = 1$ ,  $l_3 = 0$ ,  $\phi_1 = 0$ ,  $\phi_2 = 0$ ,  $\phi_3 = 0$ . (b) The contour plot of (a).

resonant Y-type soliton is presented through the following expression,

$$u = 2(\ln f_3)_x, \quad f_3 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_3} + e^{\tau_1 + \tau_3 + A_{13}} + e^{\tau_2 + \tau_3 + A_{23}}, \quad (30)$$

and the constraints  $k_2 = \frac{1}{2}k_1 \pm \frac{1}{2}\sqrt{-3k_1^2 - 36u_0}$ ,  $u_0 < -\frac{1}{12}k_1^2$  should be met. Figure 5 shows the mutual collision between the YS soliton and the single soliton. The YS solitons are formed because of the resonance effect between solitons, which describes the fusion and fission phenomena between solitons. When  $A_{12} = 0$ , the fission phenomenon represents the splitting of one soliton into two solitons with the same structure. The fusion phenomenon represents two structurally similar solitons merging into one single soliton.

By introducing complex conjugation relationships to partial parameters, the well known Hirota multiple soliton solutions can be expanded to obtain the following form of mixed solution. Based on this type of solution, we can construct rich nonlinear localized waves and their interaction structures [45]. Let the detailed mixed solution expression  $f_3$  be of the form

$$f_3 \sim 1 - \lambda_1 \cos(\zeta_1) + \lambda_2 e^{2\zeta_1} + e^{\tau_3} (1 - \lambda_1 e^{\zeta_1} (L_{R1} \cos(\zeta_1) - L_{I1} \sin(\zeta_1))) + \lambda_2 e^{\tau_3} e^{2\zeta_1} (L_{R1}^2 + L_{I1}^2), \quad (31)$$

with

$$\tau_3 = k_3(x + l_3 y + \omega_3 t) + \phi_3, \quad \omega_3 = -(\alpha k_3^4 + 45\alpha u_0^2 l_3 + 15\alpha u_0 k_3^2 + l_3), \quad \lambda_2 = -\frac{n_1^2(m_1^2 - 3n_1^2 + 9u_0)}{4m_1^2(3m_1^2 - n_1^2 + 9u_0)} \lambda_1, \quad L_{R1} = \text{Re}(e^{A_{13}}), \quad L_{I1} = \text{Im}(e^{A_{13}}), \quad (32)$$

where  $\xi_1$  and  $\zeta_1$  are described in expression (26). Substituting equation (31) into equation (2) leads to the interactional structure consisting of breather and soliton, which presents an elastic collision mode due to the shape of the interactional structure remaining unchanged. By applying  $(m_1, n_1) = (\epsilon, \epsilon)$ ,  $\lambda_1 = 2$ ,  $\lambda_2 = 1 + \epsilon^2$  and performing the Taylor expansion on

equation (31) as  $\epsilon = 0$ , we obtain an interactional wave structure including a lump and a soliton via appropriate parameter selections.

### 3.2. Interaction structures from the fourth-order solution

For the aim of constructing the space-curved resonant solitons, we assume that  $\exp(A_{js}) \rightarrow 0$  is true if, and only if,  $k_j = k_s$ , ( $0 < j < s \leq 4$ ) [46]. Introducing  $k_1 = k_2, k_3 = k_4$  to the multiple soliton solutions (5) and (6) when  $N = 4$  leads to the elimination of all or part of the corresponding terms  $\exp(A_{js}) \rightarrow 0$ , ( $0 < j < s \leq 4$ ), and  $f_4$  in  $u = 2(\ln f_4)_x$  is of the form

$$f_4 = 1 + e^{\tau_1} + e^{\tau_2} + e^{\tau_3} + e^{\tau_4} + e^{\tau_1 + \tau_3 + A_{13}} + e^{\tau_1 + \tau_4 + A_{14}} + e^{\tau_2 + \tau_3 + A_{23}} + e^{\tau_2 + \tau_4 + A_{24}}. \tag{33}$$

If equation (33) additionally satisfies the following parameters paired-complexification relationships

$$k_3 = k_4^* = m_2 + in_2, \quad l_3 = l_4^* = p_2 + iq_2, \tag{34}$$

$$\phi_3 = \phi_4^* = \gamma_2 + i\eta_2,$$

where  $m_2, n_2, p_2, q_2, \gamma_2, \eta_2$  are real constants, the interaction between a space-curved soliton and a breather can be obtained. Figure 6 shows the space-curved resonant soliton interaction including the other space-curved resonant soliton/breather.

Similar to the way of obtaining the first-order breather, we can derive the second-order breather solution with condition (34). The fourth-order soliton solution is given as

$$f_4 \sim 1 + \lambda_1 e^{\xi_1} \cos(\zeta_1) + \lambda_2 e^{2\xi_1} + \lambda_3 e^{\xi_2} \cos(\zeta_2) + \lambda_4 e^{2\xi_2} + (L_{R1}^2 + L_{I1}^2)(L_{R2}^2 + L_{I2}^2)\lambda_2\lambda_4 e^{2\xi_1 + 2\xi_2} + \frac{\lambda_1\lambda_3}{2} e^{\xi_1 + \xi_2} (L_{R1} \cos(\zeta_1 + \zeta_2) - L_{I1} \sin(\zeta_1 + \zeta_2)) + \frac{\lambda_1\lambda_3}{2} e^{\xi_1 + \xi_2} (L_{R2} \cos(\zeta_1 - \zeta_2) - L_{I2} \sin(\zeta_1 - \zeta_2)) + ((L_{R1}L_{R2} + L_{I1}L_{I2})\cos(\zeta_2) - (L_{I1}L_{R2} - L_{R1}L_{I2})\sin(\zeta_2))\lambda_2\lambda_3 e^{2\xi_1 + \xi_2} + ((L_{R1}L_{R2} - L_{I1}L_{I2})\cos(\zeta_1) - (L_{I1}L_{R2} + L_{R1}L_{I2})\sin(\zeta_1))\lambda_1\lambda_4 e^{\xi_1 + 2\xi_2}, \tag{35}$$

and

$$\xi_2 = m_2x + (m_2p_2 - n_2q_2)y + \omega_{R2}t + \gamma_2, \tag{36}$$

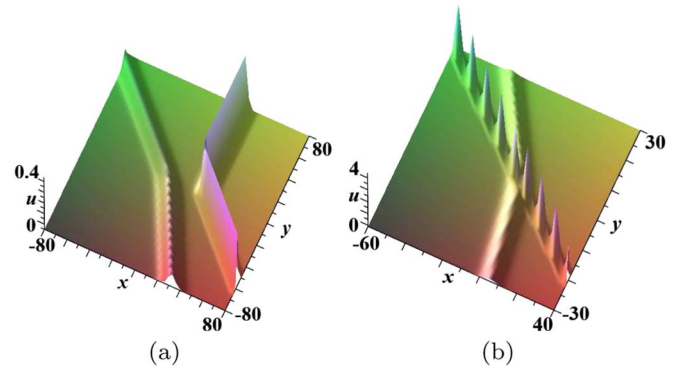
$$\zeta_2 = n_2x + (m_2q_2 + n_2p_2)y + \omega_{I2}t + \eta_2,$$

$$\omega_{R2} = -\alpha m_2^5 + (10\alpha n_2^2 - 15\alpha u_0)m_2^3 + (-5\alpha n_2^4 + 45\alpha u_0 l_2^2 - p_2(45\alpha u_0^2 + 1))m_2 + q_2(45\alpha u_0^2 + 1)n_2,$$

$$\omega_{I2} = -\alpha n_2^5 + 5(2\alpha m_2^2 + 3\alpha u_0)n_2^3 + ((-45\alpha u_0^2 - 1)p_2 - 5\alpha m_2^2(m_2^2 + 9u_0))n_2 - q_2(45\alpha u_0^2 + 1)m_2,$$

$$\lambda_4 = -\frac{n_2^2(m_2^2 - 3n_2^2 + 9u_0)}{4m_2^2(3m_2^2 - n_2^2 + 9u_0)}\lambda_3,$$

$$L_{R2} = \text{Re}(e^{A_{14}}), \quad L_{I2} = \text{Im}(e^{A_{14}}),$$



**Figure 6.** (a) The interactional structure between two space-curved resonance line soliton  $k_1 = k_2 = 0.5, k_3 = k_4 = -1, l_1 = 0.5, l_2 = 1, l_3 = 0, l_4 = 1, \alpha = 1, u_0 = -0.05, \phi_1 = \phi_2 = 0, \phi_3 = \phi_4 = 20$ . (b) The interactional structure between a space-curved resonance line soliton and a breather  $k_1 = k_2 \approx 0.667, l_1 = 0.25, l_2 = 1, m_2 = 3, n_2 = 0.005, p_2 = 2, q_2 = 0.005, \alpha = 1, u_0 = -0.1, \phi_1 = \phi_2 = 1, \gamma_2 = 2, \eta_2 = 0.1$ .

where  $\xi_1, \zeta_1, \lambda_2$  are defined by equation (26), and  $L_{R1}, L_{I1}$  are defined by equation (32). The essential difference between the second-order breather and the first-order breather is that the former has two sets of velocity vectors  $\{(\mathbf{v}_{\xi_1}, \mathbf{v}_{\zeta_1}), (\mathbf{v}_{\xi_2}, \mathbf{v}_{\zeta_2})\}$ . The expressions of  $(\mathbf{v}_{\xi_2}, \mathbf{v}_{\zeta_2})$  are similar to equation (13), which are given by

$$\mathbf{v}_{\xi_2} = \left( \frac{\omega_{R2}}{m_2}, \frac{\omega_{R2}}{m_2 p_2 - n_2 q_2} \right), \tag{37}$$

$$\mathbf{v}_{\zeta_2} = \left( \frac{\omega_{I2}}{n_2}, \frac{\omega_{I2}}{m_2 q_2 + n_2 p_2} \right).$$

For the convenience of discussion, we define the slope of the trajectory equation where the two breathers are located as follows

$$\Lambda_i = -\frac{m_i}{m_i p_i - n_i q_i}, \quad i = 1, 2. \tag{38}$$

The ordinary second-order breather solution can be obtained under the condition that  $\xi_1 \neq \xi_2, \zeta_1 \neq \zeta_2$  and  $\Lambda_1 \neq \Lambda_2$ . We can also construct the particular second-order breather structure, i.e. the breather molecule [47–49] under the conditions  $\xi_1 \neq \xi_2, \zeta_1 \neq \zeta_2$  and  $\Lambda_1 = \Lambda_2$ .

If we consider the following fixed conditions, i.e. let the parameters satisfy the conditions

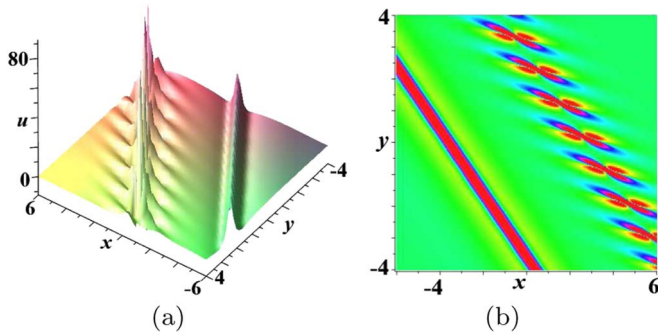
$$\mathbf{v}_{\xi_1} \times \mathbf{v}_{\zeta_1} = 0, \quad \mathbf{v}_{\xi_2} \times \mathbf{v}_{\zeta_2} \neq 0, \quad \Lambda_1 = \Lambda_2, \tag{39}$$

namely,

$$q_1 = 0, \quad -\frac{1}{p_1} = -\frac{m_2}{m_2 p_2 - n_2 q_2}, \tag{40}$$

then the second-order breather solution (35) transforms into a coherent structure (also called collision-free modes) of breather and soliton [50]. From figure 7, it is observed that the propagation direction of the breather and soliton are parallel, and the breather and soliton do not interfere with each other.

On the other hand, we consider the following case, say, the



**Figure 7.** The breather-soliton molecule (BSM) with  $m_1 = n_1 = 1$ ,  $p_1 = 1$ ,  $q_1 = 0$ ,  $m_2 = n_2 = 1.5$ ,  $p_2 = 3$ ,  $q_2 = 2$ ,  $\alpha = 1$ ,  $\lambda_1 = 0.75$ ,  $\lambda_3 = 2$ ,  $\delta_1 = 5$ ,  $\delta_2 = 0$ ,  $\eta_1 = \eta_2 = 0$ ,  $u_0 = 0$ ,  $t = 0$ . (a) The spatial structure plot. (b) The density plot of (a).

directions of the two sets of propagation velocities are parallel

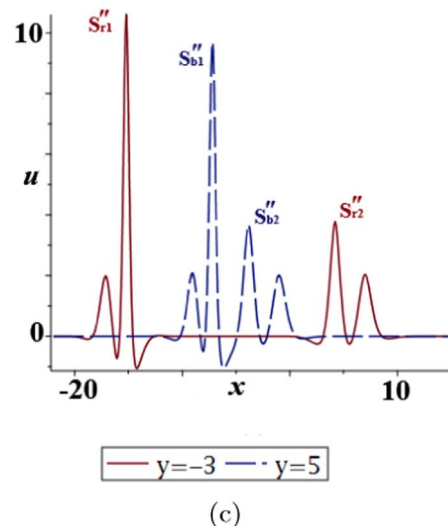
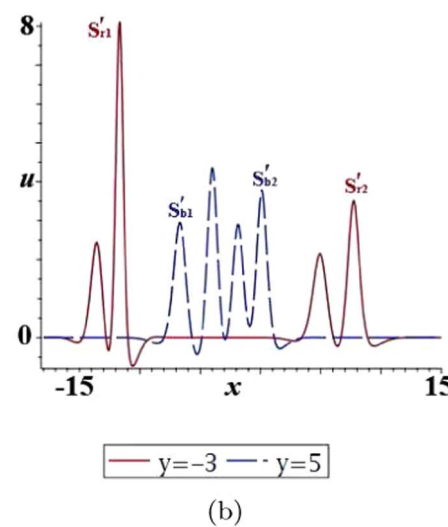
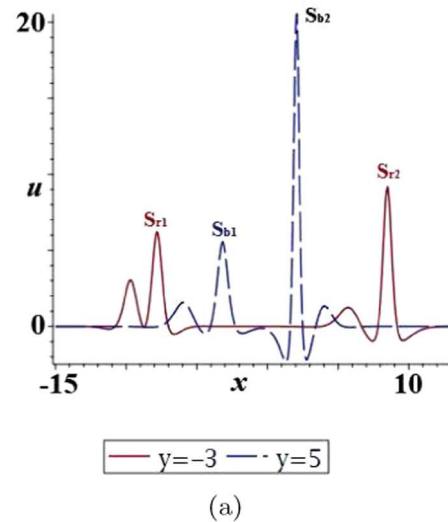
$$\mathbf{v}_{\xi_1} \times \mathbf{v}_{\xi_2} = \mathbf{0}, \quad \mathbf{v}_{\xi_1} \times \mathbf{v}_{\xi_2} = \mathbf{0}, \quad (41)$$

Under this case, if  $\Lambda_1 \neq \Lambda_2 (p_1 \neq p_2)$ , the molecule structure between two M-Shaped soliton (2MS) solutions is given in figure 7. On the contrary, if  $\Lambda_1 = \Lambda_2 (p_1 = p_2)$ , together with the above conditions (41), we can obtain a variety of nonlinear wave molecules that are composed of a bounded state of different kinds of nonlinear waves and are viewed as collision-free modes. The components of the molecules do not affect each other during the process of propagation. We show four kinds of molecule structures including the W-Shaped soliton and M-Shaped soliton (WSMS) molecule, the M-Shaped soliton and M-Shaped soliton (MSMS) molecule, the W-Shaped solitary and W-Shaped solitary (WSWS) molecule and the M-Shaped solitary and periodic soliton (MSP) molecule.

Different from the previous expressions of phase shifts (24), the phase shifts of the various interaction structures in section 3 are determined not only by the time change but also the collision of different types waves, which are given as follows

$$\begin{aligned} \phi_{\xi_i}(t) &= |\mathbf{v}_{\xi_i}| t + \text{Re}(Z_{ij}) = \frac{\omega_{Ri} t \sqrt{(m_i p_i - n_i q_i)^2 + m_i^2}}{m_i(m_i p_i - n_i q_i)} \\ &\quad + \text{Re}(Z_{ij}), \\ \phi_{\xi_i}(t) &= |\mathbf{v}_{\xi_i}| t + \text{Im}(Z_{ij}) = \frac{\omega_{Ii} t \sqrt{(m_i q_i + n_i p_i)^2 + n_i^2}}{n_i(m_i q_i + n_i p_i)} \\ &\quad + \text{Im}(Z_{ij}), \\ Z_{ij} &= \frac{(k_i^2 - k_i k_j + k_j^2)(k_i - k_j)^2 + 9u_0(k_i - k_j)^2}{(k_i^2 + k_i k_j + k_j^2)(k_i + k_j)^2 + 9u_0(k_i + k_j)^2}, \\ &\quad (i = 1, 2, j = 3, 4). \end{aligned} \quad (42)$$

Figure 8 displays an evolution process of the 2MS molecule solution under the combined action of time and collision. The phase shift of the 2MS molecule solution is different in the evolution process, and their trajectory equations also change, which results in the change of the external waveform. When  $t = -0.01$ , the WS soliton ( $S_{r2}$ ) converts into WS soliton ( $S_{b2}$ ) with a higher amplitude after interaction. When  $t = 0$ , the amplitude of the MS soliton ( $S'_{r1}$ ,  $S'_{r2}$ ) becomes larger, and the



**Figure 8.** The evolution diagram of the 2MS molecule interactional structure with time and  $m_1 = 1.8$ ,  $n_1 = 1.6$ ,  $p_1 = -1$ ,  $q_1 = 0$ ,  $m_2 = 1.5$ ,  $n_2 = 1$ ,  $p_2 = 1$ ,  $q_2 = 0$ ,  $\lambda_1 = 5$ ,  $\lambda_3 = 2$ ,  $\delta_1 = 10$ ,  $\delta_2 = 0$ ,  $\eta_1 = 10$ ,  $\eta_2 = 0$ ,  $\alpha = 1$ ,  $u_0 = -0.1$ . (a)  $t = -0.01$ . (b)  $t = 0.1$ . (c)  $y = -3$ ,  $y = 5$ ,  $t = 0.2$  respectively.

waveforms of both change accordingly. When  $t=0.2$ , compared with  $t=0.1$ , the amplitudes of the two peaks of MS solitons ( $S''_{r2}$ ) exchanged. The amplitudes of the peaks of MS solitons ( $S''_{b1}$ ) are greatly increased, and the amplitudes of the peaks of MS solitons ( $S''_{b2}$ ) are slightly decreased.

#### 4. Conclusion

We investigate the 2 GKdV equation to unveil its abundant dynamic properties of related nonlinear waves while setting on a constant background wave. By introducing the paired-complex parameter restrictions and the velocity resonance conditions, we derive the breather solutions (first/second-order), lump solution, and various kinds of nonlinear waves. Furthermore, the analytic expressions of the obtained nonlinear wave structures are given, and the associated dynamic characteristics are analyzed. By using the concept of the trajectory equation of breather members (SWM and PWM) and investigating the expressions of the phase shifts, we point out the reason why the shape and amplitude of the nonlinear wave change with time. For the mixed solution and the fourth-order soliton solution, we obtain abundant interactional structures and nonlinear wave molecule solutions composed of the WSMS, MSMS, WWS and MSP molecules. We also unearth the essence of the inelastic collision among different types of localized waves and nonlinear wave molecules via decomposing the phase shifts into a time variation part and collision part. The combined methods for investigating nonlinear wave structures to the 2 GKdV equation are also suitable to study the related nonlinear wave phenomena in other nonlinear models.

#### Acknowledgments

Funding was provided by the National Natural Science Foundation of China (Grant No. 12271324), the Natural Science Basic Research Program of Shaanxi Province (Grant No. 2024JC-YBQN-0069), the China Postdoctoral Science Foundation (Grant No. 2024M751921), the 2023 Shaanxi Province Postdoctoral Research Project (Grant No. 2023BSHEDZZ186), the Fundamental Research Funds for the Central Universities (Grant No. 1301032598).

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#### References

- [1] Bailung H, Sharma S K and Nakamura Y 2011 Observation of peregrine solitons in a multicomponent plasma with negative ions *Phys. Rev. Lett.* **107** 255005
- [2] Liu X M, Yao X K and Cui Y D 2018 Real-time observation of the buildup of soliton molecules *Phys. Rev. Lett.* **121** 023905
- [3] Rao J G, Cheng Y and Kuppuswamy P 2020 PT-symmetric nonlocal Davey-Stewartson I equation: Soliton solutions with nonzero background *Physica D* **401** 132180
- [4] Kartashov Y V, Malomed B A and Torner L 2011 Solitons in nonlinear lattices *Rev. Mod. Phys.* **83** 248–305
- [5] Yang J W, Gao Y T and Feng Y J 2017 Solitons and dromion-like structure in an inhomogeneous optical fiber *Nonlinear Dyn.* **87** 851–62
- [6] Ma Y L 2019 Interaction and energy transition between the breather and rogue wave for a generalized nonlinear Schrödinger system with two higher-order dispersion operators in optical fibers *Nonlinear Dyn.* **97** 95–105
- [7] Yuan F 2020 The dynamics of the smooth positon and b-positon solutions for the NLS-MB equations *Nonlinear Dyn.* **102** 1761–71
- [8] Tao Y S and He J S 2012 Multisolitons, breathers, and rogue waves for the Hirota equation generated by the Darboux transformation *Phys. Rev. E* **85** 026601
- [9] Zhang X E, Chen Y and Tang X Y 2018 Rogue wave and a pair of resonance stripe solitons to KP equation *Comput. Math. Appl.* **76** 1938–49
- [10] Feng Y X and Zhao Z L 2024 New lump solitons and several interaction solutions and their dynamics of a generalized (3+1)-dimensional nonlinear differential equation *Commun. Theor. Phys.* **76** 025001
- [11] Yue Y F, Huang L L and Chen Y 2019 Localized waves and interaction solutions to an extended (3+1)-dimensional Jimbo-Miwa equation *Appl. Math. Lett.* **89** 70–77
- [12] Shen Y, Tian B, Zhou T Y and Gao X T 2022 Shallow-water-wave studies on a (2+1)-dimensional Hirota-Satsuma-Ito system: X-type soliton, resonant Y-type soliton and hybrid solutions *Chaos Soliton Fract.* **157** 111861
- [13] Zhao Z L and He L C 2021 Resonance Y-type soliton and hybrid solutions of a (2+1)-dimensional asymmetrical Nizhnik-Novikov-Veselov equation *Appl. Math. Lett.* **122** 107497
- [14] Zhao Z L, Zhang C F, Feng Y X and Yue J 2023 Space-curved resonant solitons and interaction solutions of the (2+1)-dimensional Ito equation *Appl. Math. Lett.* **38** 060501
- [15] Weng W F, Zhang G Q, Zhang M H, Zhou Z J and Yan Z Y 2022 Semi-rational vector rogon-soliton solutions and asymptotic analysis for any n-component nonlinear Schrödinger equation with mixed boundary conditions *Physica D* **432** 133150
- [16] Peng W Q and Chen Y 2023 Bound-state soliton and rogue wave solutions for the sixth-order nonlinear Schrödinger equation via inverse scattering transform method *Math. Meth. Appl.* **46** 126–41
- [17] Zhang G Q, Ling L M and Yan Z Y 2021 Multi-component nonlinear Schrödinger equations with nonzero boundary conditions: higher-order vector Peregrine solitons and asymptotic estimates *J. Nonlinear Sci.* **31** 81
- [18] Zhong M, Chen Y, Yan Z Y and Malomed B A 2024 Suppression of soliton collapses, modulational instability and rogue-wave excitation in two-Lévy-index fractional Kerr media *P. Roy. Soc. A* **480** 20230765
- [19] Zhuang J H, Chen X and Chu J Y 2023 Line solitons, lumps, and lump chains in the (2+1)-dimensional generalization of the Korteweg-de Vries equation *Results Phys.* **52** 106759
- [20] Chen Z T and Jia M 2021 Novel travelling wave structures: few-cycle-pulse solitons and soliton molecules *Commun. Theor. Phys.* **73** 025003
- [21] Li B Q and Ma Y L 2023 Optical soliton resonances and soliton molecules for the Lakshmanan-Porsezian-Daniel system in nonlinear optics *Nonlinear Dyn.* **111** 6689–99
- [22] Sun F W and Gao W 2014 N-soliton solution of the (2+1)-dimensional generalized fifth-order KdV equation (in Chinese) *J. North Chin. Univer. Tech.* **26** 47–52

- [23] Wazwaz A M 2007 The extended tanh method for new solitons solutions for many forms of the fifth-order KdV equations *Appl. Math. Comput.* **184** 1002–14
- [24] Bilige S D and Temuer C L 2010 An extended simplest equation method and its application to several forms of the fifth-order KdV equation *Appl. Math. Comput.* **216** 3146–52
- [25] Bilge A H 1996 Singular travelling wave solutions of the fifth-order KdV, Sawada-Kotera and Kaup equations *J. Phys. A: Math. Gen.* **29** 4967–75
- [26] Wang W, Yao R X and Lou S Y 2020 Abundant traveling wave structures of (1+1)-dimensional Sawada-Kotera equation: few cycle solitons and soliton molecules *Chin. Phys. Lett.* **37** 100501
- [27] Lü J Q, Bilige S and Chaolu T 2018 The study of lump solution and interaction phenomenon to (2+1)-dimensional generalized fifth-order KdV equation *Nonlinear Dyn.* **91** 1669–76
- [28] Tahir M and Awan A U 2019 The study of complexitons and periodic solitary-wave solutions with fifth-order KdV equation in (2+1)-dimensions *Mod. Phys. Lett. B* **33** 1950411
- [29] Tan W and Liu J 2020 Superposition behaviour between lump solutions and different forms of N-solitons ( $N \rightarrow \infty$ ) for the fifth-order Korteweg-de Vries equation *Pramana* **94** 36
- [30] Yusuf A and Sulaiman T A 2021 Dynamics of lump-periodic, breather and two-wave solutions with long wave in shallow water under gravity and 2D nonlinear lattice *Commun. Nonlinear Sci. Numer. Simulat.* **99** 105846
- [31] Liu J G 2018 Lump-type solutions and interaction solutions for the (2+1)-dimensional generalized fifth-order KdV equation *Appl. Math. Lett.* **86** 36–41
- [32] Hirota R 1971 Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons *Phys. Rev. Lett.* **27** 1192–4
- [33] Li Y, Yao R X, Xia Y R and Lou S Y 2021 Plenty of novel interaction structures of soliton molecules and asymmetric solitons to (2+1)-dimensional Sawada-Kotera equation *Commun. Nonlinear Sci. Numer. Simulat.* **100** 105843
- [34] Ablowitz M J and Satsuma J 1978 Solitons and rational solutions of nonlinear evolution equations *J. Math. Phys.* **19** 2180–6
- [35] Chen Q, Qi Z and Chen J 2021 Resonant line wave soliton solutions and interaction solutions for (2+1)-dimensional nonlinear wave equation *Results Phys.* **2021** 104480
- [36] Dai Z D, Liu Z J and Li D L 2018 Exact periodic solitary-wave solution for KdV equation *Chin. Phys. Lett.* **25** 1531–3
- [37] Wang C J, Fang H and Tang X X 2019 State transition of lump-type waves for the (2+1)-dimensional generalized KdV equation *Nonlinear Dyn.* **95** 2943–61
- [38] Wang X B, Tian S F and Qin C Y 2017 Dynamics of the breathers, rogue waves and solitary waves in the (2+1)-dimensional Ito equation *Appl. Math. Lett.* **68** 40–7
- [39] Zhao J F, Wang L and Zhou S 2023 Transition mechanisms of breather molecules in the Hirota equation *Phys. Scr.* **98** 055509
- [40] Ma W X 2018 Abundant lumps and their interaction solutions of (3.1)-dimensional linear PDEs *J. Geom. Phys.* **133** 10–6
- [41] Yao R X, Shen Y L and Li Z B 2020 Lump solutions and bilinear Bäcklund transformation for the (4.1)-dimensional Fokas equation *Math. Sci.* **14** 301–8
- [42] Lou S Y 2020 Soliton molecules and asymmetric solitons in three fifth order systems via velocity resonance *J. Phys. Commun.* **4** 041002
- [43] Yang X Y, Fan R and Li B 2020 Soliton molecules and some novel interaction solutions to the (2+1)-dimensional B-type Kadomtsev-Petviashvili equation *Phys. Scr.* **95** 045213
- [44] Yan Z W and Lou S Y 2020 Special types of solitons and breather molecules for a (2+1)-dimensional fifth-order KdV equation *Commun. Nonlinear Sci. Numer. Simulat.* **91** 105425
- [45] Yuan F, Cheng Y and He J S 2020 Degeneration of the breathers in the Kadomtsev-Petviashvili I equation *Commun. Nonlinear Sci. Numer. Simulat.* **83** 105027
- [46] Qi Z Q, Zhang Z and Li B 2021 Space-Curved resonant line solitons in a generalized (2+1)-dimensional fifth-order KdV system *Chin. Phys. Lett.* **38** 060501
- [47] Xu G, Gelash A, Chabchoub A and Zakharov V 2019 Breather wave molecules *Phys. Rev. Lett.* **122** 084101
- [48] Jia M, Lin J and Lou S Y 2020 Soliton and breather molecules in few-cycle-pulse optical model *Nonlinear Dyn.* **100** 3745–57
- [49] Cui Y P, Wang L and Gegen H 2021 M-breather, M-lump, breather molecules and their interaction solutions for a (2+1)-dimensional KdV equation *Phys. Scr.* **96** 095211
- [50] Zhang Z, Yang X Y and Li B 2020 Novel soliton molecules and breather-positon on zero background for the complex modified KdV equation *Nonlinear Dyn.* **100** 1551