 Communications in Theoretical Physics

Commun. Theor. Phys. **77** (2025) 015102 (13pp) https://doi.org/10.1088/1572-9494/ad7942

Group frames via magic states with applications to SIC-POVMs and MUBs

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Received 19 July 2024, revised 10 September 2024
Accepted for publication 11 September 2024
Published 8 November 2024

Abstract

We connect magic (non-stabilizer) states, symmetric informationally complete positive operator valued measures (SIC-POVMs), and mutually unbiased bases (MUBs) in the context of group frames, and study their interplay. Magic states are quantum resources in the stabilizer formalism of quantum computation. SIC-POVMs and MUBs are fundamental structures in quantum information theory with many applications in quantum foundations, quantum state tomography, and quantum cryptography, etc. In this work, we study group frames constructed from some prominent magic states, and further investigate their applications. Our method exploits the orbit of discrete Heisenberg–Weyl group acting on an initial fiducial state. We quantify the distance of the group frames from SIC-POVMs and MUBs, respectively. As a simple corollary, we reproduce a complete family of MUBs of any prime dimensional system by introducing the concept of MUB fiducial states, analogous to the well-known SIC-POVM fiducial states. We present an intuitive and direct construction of MUB fiducial states via quantum T-gates, and demonstrate that for the qubit system, there are twelve MUB fiducial states, which coincide with the H-type magic states. We compare MUB fiducial states and SIC-POVM fiducial states from the perspective of magic resource for stabilizer quantum computation. We further pose the challenging issue of identifying all MUB fiducial states in general dimensions.

Keywords: Group frames, magic states, MUBs, SIC-POVMs, quantum T-gates

(Some figures may appear in colour only in the online journal)

1. Introduction

With the development of quantum information theory, symmetric informationally complete positive operator valued measures (SIC-POVMs) have received much attention [1–18], though their existence in general dimensions still remains unknown (Zauner's weak conjecture). Nowadays, almost all SIC-POVMs are constructed via the orbits of the discrete Heisenberg–Weyl group acting on the so called SIC-POVM fiducial states. It is demonstrated that SIC-POVM fiducial states exist in many dimensions [10, 11], but the general existence problem for all dimensions remains open (Zauner's strong conjecture) [18].

The above construction of SIC-POVMs is a special case of group frames [19]. A group frame, as an important example of a structured frame, is constructed as an orbit of a finite group with a unitary representation acting on an initial vector (state).

SIC-POVMs are associated with equiangular tight frames and form 2-designs, and another significant 2-design is the $d+1$ mutually unbiased bases (MUBs) of $\mathbb{C}^d$ [1]. It is well known that there exist no more than $d+1$ MUBs of $\mathbb{C}^d$, and when $d=p^n$ is a prime-power, the upper bound can be attained [20], and various constructions are available [20–28]. However, the existence and construction of MUBs achieving the upper bound in non-prime-power dimensional systems remain an outstanding and tantalizing open problem. For example, although it is known that there exist at least three MUBs of $\mathbb{C}^6$, it is still not known if there exist four MUBs of $\mathbb{C}^6$, let alone seven (i.e., $6+1$) MUBs of such a space [18]. Notably, Zauner conjectured that the maximal number of MUBs of $\mathbb{C}^6$ is three, considerably smaller than seven [1, 18]. Inspired by the construction of SIC-POVMs via fiducial states, one may want to enquire whether an analogous concept for an MUB fiducial state exists. This will involve constructing a complete family of $d+1$ MUBs through the Heisenberg–Weyl group frame.

In this work, we study the interplay between magic states in the stabilizer formalism of quantum computation and quantum measurements by incorporating SIC-POVMs and MUBs into a broader framework known as group frames. We focus on those group frames generated by the discrete Heisenberg–Weyl group and some special initial states. Depending on the nature of the initial states, two radically different situations arise: when the initial state is a stabilizer state, the Heisenberg–Weyl group frame actually consists of d orthonormal bases, which are all equivalent up to phases. However, when the initial state is a magic (non-stabilizer) state, the structures of the Heisenberg–Weyl group frame becomes much richer and more complex. We define two quantities to measure how closely a frame approximates SIC-POVMs and MUBs, respectively, and reveal relations between these quantities and magic of initial states when the frame is generated by the discrete Heisenberg–Weyl group. Analogous to SIC-POVM fiducial states, we introduce the concept of MUB fiducial states. While SIC-POVM fiducial states possess the global maximal magic among all quantum states, MUB fiducial states achieve local maximal value of magic. We further reveal some intrinsic relations between MUB fiducial states, SIC-POVM fiducial states, H-type and T-type magic states. We emphasize here the magic of a state is quantified via the L^1-norm of characteristic function (Weyl transform) of a quantum state [29]. There are several other quantifiers of magic, such as the robustness of magic [30–32], negativity of the discrete Wigner functions (mana) [33–36], relative entropy of magic [33, 37, 38], and the stabilizer rank [39–41], etc. Each quantifier has its own usage, and it is desirable to investigate whether the SIC-POVM fiducial states still possess the global maximal magic among all quantum states.

As a simple corollary of group frames via magic states, we provide a method to construct an MUB fiducial state in any prime dimensional system by invoking quantum T-gates and the maximal superposition of the computational basis states. We see that the Heisenberg–Weyl group frame in $\mathbb{C}^d$ generated by an MUB fiducial state leads to d^2 magic states, which can be partitioned into d MUBs, and together with the computational base (d stabilizer states), constitute $d+1$ MUBs. This method leads to a class of MUBs that essentially align with Alltop [21], and Klappenecker and Rötteler [24].

The remainder of the work is organized as follows. In section 2, we make some preparations by reviewing briefly some basic notions in the stabilizer formalism of quantum computation, including MUBs, discrete Heisenberg–Weyl group, stabilizer states, magic states, and a recently introduced quantifier of magic (non-stabilizerness) via characteristic function [29], which will play an important and interesting role in characterizing SIC-POVM and MUB fiducial states. In section 3, we recall the concept of frames, with a special focus on group frames. We introduce two quantities to evaluate how close a tight frame is to SIC-POVMs and MUBs, respectively. In section 4, motivated by the concept of SIC-POVM fiducial states, we introduce MUB fiducial states, and investigate their basic features. Moreover, we connect the quantum T-gates with the MUB fiducial states and H-type magic states. By invoking these concepts, we present a simple approach to constructing a complete family of MUBs of prime dimensional systems. Finally, we conclude with a summary in section 5 and highlight the open issue of determining all MUB fiducial states in general dimensions. In the appendix, we present a concise review of SIC-POVMs and MUBs for convenience of reference.

2. Magic states and stabilizer formalism

In this section, we recall some basic notions such as MUBs, discrete Heisenberg–Weyl group, and a quantifier of magic arising from stabilizer quantum computation [29, 42–45].

Definition 1. Two orthonormal bases $\{|\psi_j\rangle : j \in \mathbb{Z}_d\}$ and $\{|\phi_j\rangle : j \in \mathbb{Z}_d\}$ of $\mathbb{C}^d$ are called mutually unbiased if

$$|\langle\psi_j|\phi_k\rangle| = c, \qquad \forall\ j, k \in \mathbb{Z}_d,$$

where c is a constant independent of $j, k \in \mathbb{Z}_d = \{0, 1, \cdots, d-1\}$ (the ring of integers modulo d). A family of n orthonormal bases is called mutually unbiased if any two bases in the family are mutually unbiased. If $n = d + 1$ (the maximally possible number of MUBs of $\mathbb{C}^d$), then the family is called complete.

It turns out that the above overlap constant $c = 1/\sqrt{d}$ is uniquely determined since $1 = \sum_{k=0}^{d-1}|\langle\psi_j|\phi_k\rangle|^2 = dc^2$ for any j.

The concept of MUBs can be traced back to Schwinger's seminal work on unitary operator bases of $\mathbb{C}^d$ [46], where he introduced two maximally incompatible (complementary) unitary operators

$$X = \sum_{j=0}^{d-1}|j+1\rangle\langle j|, \qquad Z = \sum_{j=0}^{d-1}\omega^j|j\rangle\langle j|,$$

with $\omega = e^{2\pi i/d}$ and $\{|j\rangle : j \in \mathbb{Z}_d\}$ a computational basis of $\mathbb{C}^d$. Here, the arithmetic is modulo d, in particular, $|d\rangle = |0\rangle$. The eigenvectors of the phase operator Z are precisely the basis states $|j\rangle$, while the eigenvectors of the shift operator X, denoted as $|\tilde{j}\rangle$, are connected with $|j\rangle$ via the discrete Fourier transform $F_d = \frac{1}{\sqrt{d}}\sum_{j,k=0}^{d-1}\omega^{jk}|j\rangle\langle k|$ as

$$|\tilde{j}\rangle = F_d|j\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\omega^{jk}|k\rangle, \qquad j \in \mathbb{Z}_d.$$

This implies that

$$|\langle\tilde{j}|k\rangle| = \frac{1}{\sqrt{d}}, \qquad \forall\ j, k \in \mathbb{Z}_d.$$

Consequently, $\{|j\rangle : j \in \mathbb{Z}_d\}$ and $\{|\tilde{j}\rangle : j \in \mathbb{Z}_d\}$ are two MUBs.

The operators X and Z generate the discrete Heisenberg–Weyl group (also called the generalized Pauli group) [4]

$$\mathcal{P}_d = \{\tau^j D_{k,l}: j \in \mathbb{Z},\, k,\, l \in \mathbb{Z}_d\},$$

where the discrete Heisenberg–Weyl operators $D_{k,l}$ are defined as

$$D_{k,l} = \tau^{kl} X^k Z^l, \qquad \tau = -\mathrm{e}^{\pi \mathrm{i}/d}. \tag{1}$$

The normalizer of the discrete Heisenberg–Weyl group $\mathcal{P}_d$ in the full unitary group $\mathcal{U}_d$ of $\mathbb{C}^d$ is called the Clifford group [4]

$$\mathcal{C}_d = \{C \in \mathcal{U}_d: C\mathcal{P}_d C^\dagger = \mathcal{P}_d\}.$$

The discrete Heisenberg–Weyl group and the Clifford group are fundamental ingredients in the stabilizer formalism of quantum error correction and quantum computation.

In this formalism, a (pure) stabilizer state is defined as the common eigenstate of a maximal Abelian subgroup of $\mathcal{P}_d$ not containing $c\mathbf{1}$ for any $c \neq 1$. Alternatively, a pure stabilizer state $|\mathcal{A}\rangle$ is stabilized by the maximal Abelian subgroup $\mathcal{A}$ in the sense that $\mathcal{A} = \{D|\mathcal{A}\rangle = |\mathcal{A}\rangle: D \in \mathcal{P}_d\}$, and $|\mathcal{A}\rangle$ can be written in the projection operator form as

$$|\mathcal{A}\rangle\langle\mathcal{A}| = \frac{1}{d}\sum_{D\in\mathcal{A}} D.$$

For example, in a qubit system $\mathbb{C}^2$ with computational basis $\{|0\rangle, |1\rangle\}$, there are 6 stabilizer states:

$$|0_z\rangle = |0\rangle,\ |0_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ |0_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathrm{i}|1\rangle),$$
$$|1_z\rangle = |1\rangle,\ |1_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),\ |1_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - \mathrm{i}|1\rangle).$$

Here, $|0_\alpha\rangle$ and $|1_\alpha\rangle$ are the two eigenstates of the Pauli matrix σ_α, i.e. $\sigma_\alpha|0_\alpha\rangle = |0_\alpha\rangle$, $\sigma_\alpha|1_\alpha\rangle = -|1_\alpha\rangle$, $\alpha = x, y, z$.

When d is a prime, there are precisely $d(d+1)$ stabilizer states, which play the role of classical states, and can be partitioned into a complete family of $d+1$ MUBs of $\mathbb{C}^d$ [23]. In a qubit system $\mathbb{C}^2$, $\{|0_\alpha\rangle, |1_\alpha\rangle\}$, $\alpha = x, y, z$, are three MUBs of $\mathbb{C}^2$.

Any pure state other than the stabilizer states is called a magic state, a concept first introduced by Bravyi and Kitaev [47]. They identified two special classes of magic states: T-type magic states and H-type magic states (see section 4 for detailed definitions and expressions). These states are crucial for realizing universal quantum computation when combined with ideal Clifford gates and magic state distillation. In the stabilizer formalism of quantum computation, magic states are necessary elements for genuine quantum computation due to the celebrated Gottesman–Knill theorem [45]. In this work we are only concerned with pure magic states, although the notion of magic states can also be defined for mixed states [47]. We will see that both T-type magic states and H-type magic states have interesting applications in group frames, and in particular in constructing SIC-POVMs and MUBs.

Definition 2. For any quantum state ρ of a d-dimensional system $\mathbb{C}^d$, a quantifier of its magic is defined as [29]

$$M(\rho) = \sum_{k,l=0}^{d-1} |\mathrm{tr}(D_{k,l}\rho)|. \tag{2}$$

Some remarkable properties of this quantifier are summarized in [17, 29]. We emphasize that this quantity is only an indicator of magic, not a genuine measure of magic in the sense that it vanishes if and only if the state is a stabilizer state. Of course, one can simply subtract the minimal value to obtain a genuine measure of magic for pure states.

3. Group frames via stabilizer states and magic states

A set $\Phi = \{|\phi_\alpha\rangle: \alpha = 1, 2, \cdots\}$ of vectors (not necessary of unit norm) in $\mathbb{C}^d$ is called a frame, if there exist constants a, $b > 0$ such that [19, 48]

$$a\mathbf{1} \leqslant \sum_\alpha |\phi_\alpha\rangle\langle\phi_\alpha| \leqslant b\mathbf{1}.$$

When $a = b$, Φ is called a tight frame.

The frame potential of Φ is defined as [19]

$$P(\Phi) = \sum_{\alpha,\beta} |\langle\phi_\alpha|\phi_\beta\rangle|^2. \tag{3}$$

Specially, if Φ is a tight frame consisting of m unit vectors in $\mathbb{C}^d$, then $P(\Phi) = m^2/d$.

Two frames $\Phi = \{|\phi_\alpha\rangle: \alpha = 1, 2, \cdots\}$ and $\Psi = \{|\psi_\alpha\rangle: \alpha = 1, 2, \cdots\}$ of $\mathbb{C}^d$ are called unitarily equivalent if there exists a unitary operator U on $\mathbb{C}^d$ such that $U\Phi = \Psi$ (as sets), where $U\Phi = \{U|\phi_\alpha\rangle: \alpha = 1, 2, \cdots\}$. They are called equivalent up to phases if there exist $\theta_\alpha \in \mathbb{R}$ such that $\{\mathrm{e}^{\mathrm{i}\theta_\alpha}|\phi_\alpha\rangle: \alpha = 1, 2, \cdots\} = \{|\psi_\alpha\rangle: \alpha = 1, 2, \cdots\}$. A tight frame $\Psi = \{|\psi_\alpha\rangle: \alpha = 1, 2, \cdots, m\}$ consisting of unit vectors is called an equiangular tight frame if

$$|\langle\psi_\alpha|\psi_\beta\rangle|^2 = \frac{m-d}{d(m-1)}, \qquad \alpha \neq \beta. \tag{4}$$

Such frames with the same overlap for any two vectors play an important role in the field of signal processing. It is known that the number m of elements in an equiangular tight frame is constrained within the range $d \leqslant m \leqslant d^2$ [49]. When m reaches the upper bound d^2, from equation (4), we have

$$|\langle\psi_\alpha|\psi_\beta\rangle|^2 = \frac{1}{d+1}, \qquad \alpha \neq \beta,$$

and we come to a SIC-POVM

$$E = \left\{E_\alpha = \frac{1}{d}|\psi_\alpha\rangle\langle\psi_\alpha|: \alpha = 1, 2, \cdots, d^2\right\}.$$

Due to the significance of equiangular tight frames (in particular, SIC-POVMs), it is desirable to quantify how close a set of vectors is to be an equiangular tight frame. Inspired by the frame potential defined by equation (3) and the condition of equiangular tight frame given by equation (4), we introduce the following quantity.

Definition 3. Let $\Phi = \{|\phi_\alpha\rangle: \alpha = 1, 2, \cdots, m\}$ be a set of m ($d \leqslant m \leqslant d^2$) unit vectors in $\mathbb{C}^d$. A measure of how close Φ is to an equiangular tight frame consisting of m vectors is given by the function

$$P_{\mathrm{ETF}}(\Phi) = \sum_{\alpha<\beta}\left(|\langle\phi_\alpha|\phi_\beta\rangle| - \sqrt{\frac{m-d}{d(m-1)}}\right)^2.$$

Clearly, $P_{\mathrm{ETF}}(\Phi) = 0$ if and only if Φ is an equiangular tight frame with m elements.

Any k orthonormal bases of $\mathbb{C}^d$ form a tight frame of $\mathbb{C}^d$. Similarly, we introduce the following quantity to quantify how close a tight frame is to k MUBs.

Definition 4. Let $O_j = \{|\psi_{j,k}\rangle: k \in \mathbb{Z}_d\}$, $j = 1, 2, \cdots, n$, be n orthonormal bases of $\mathbb{C}^d$. A measure of how close $O_1, O_2, \cdots, O_n$ are to MUBs is given by

$$P_{\mathrm{MUB}}(O_1, O_2, \cdots, O_n) = \sum_{j_1<j_2}\sum_{k_1,k_2}\left(|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle| - \frac{1}{\sqrt{d}}\right)^2.$$

Clearly, $P_{\mathrm{MUB}}(O_1, O_2, \cdots, O_n) = 0$ if and only if $O_1, O_2, \cdots, O_n$ are mutually unbiased.

A simple straightforward approach to a tight frame is through the set

$$\{U|\psi\rangle: U \in G\}, \tag{5}$$

which is the orbit of a finite group G of unitary operators acting irreducibly on $\mathbb{C}^d$ with an initial state $|\psi\rangle$ [19]. Considering the equivalence of $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\theta}|\psi\rangle$ in describing a quantum state and avoiding trivial redundancy (they yield the same rank-one operator $|\psi\rangle\langle\psi|$), we may, without loss of generality, ignore the phases and equivalently consider the following frame

$$\mathcal{F}_{G,|\psi\rangle} = \{U|\psi\rangle: \tilde{U} \in G/\sim\}, \tag{6}$$

where $U \in \tilde{U}$ is any representative in the equivalence class $\tilde{U} = \{V \in G: V \sim U\}$, and $G/\sim = \{\tilde{U}: U \in G\}$. Here the equivalence relation $\sim$ in G is defined as $V \sim U$ if and only if $V = \mathrm{e}^{\mathrm{i}\theta}U$ for some $\theta \in \mathbb{R}$. In a word, among all vectors in the same complex line in the set (5), we only pick a single representative to form the group frame (6). In the sequel, we write

$$\mathcal{F}_{G,|\psi\rangle} = \{|\psi_\alpha\rangle = U_\alpha|\psi\rangle: \alpha = 1, 2, \cdots, m\}.$$

For instance, the discrete Heisenberg–Weyl group is irreducible, implying that $\mathcal{F}_{\mathcal{P}_d,|\psi\rangle}$ is a tight frame. Moreover, any group frame $\mathcal{F}_{G,|\psi\rangle}$ is equidistributed in the sense that for any $\alpha = 1, 2, \cdots, m$,

$$\{\langle\psi_\beta|\psi_\alpha\rangle: \beta = 1, 2, \cdots, m\} = \{\langle\psi_\beta|\psi\rangle: \beta = 1, 2, \cdots, m\}$$

This can be directly verified by

$$\langle\psi_\alpha|\psi_\beta\rangle = \langle\psi|U_\alpha^\dagger U_\beta|\psi\rangle = \langle\psi|\psi_\gamma\rangle,$$

where γ is uniquely determined by $U_\alpha^\dagger U_\beta = U_\gamma$.

Proposition 1. *Let $G = \{U_\alpha: \alpha = 1, 2, \cdots, m\}$ be a finite group irreducibly acting on $\mathbb{C}^d$ with $U_1 = \mathbf{1}$, and $|\psi\rangle$ be a pure state in $\mathbb{C}^d$. Let*

$$\mathcal{F}_{G,|\psi\rangle} = \{|\psi_\alpha\rangle = U_\alpha|\psi\rangle: \alpha = 1, 2, \cdots, m\}$$

be the associated group frame, then

$$P_{\mathrm{ETF}}(\mathcal{F}_{G,|\psi\rangle}) = \frac{m(m-d)}{d} + m\sqrt{\frac{m-d}{d(m-1)}} \times \left(1 - \sum_{\alpha=1}^{m}|\mathrm{tr}(U_\alpha|\psi\rangle\langle\psi|)|\right).$$

In particular, for $\mathcal{F}_{\mathcal{P}_d,|\psi\rangle} = \{|\psi_{k,l}\rangle = D_{k,l}|\psi\rangle: k, l \in \mathbb{Z}_d\}$, we have

$$P_{\mathrm{ETF}}(\mathcal{F}_{\mathcal{P}_d,|\psi\rangle}) = d^3 - d^2 + \frac{d^2}{\sqrt{d+1}}(1 - M(|\psi\rangle)). \tag{7}$$

Proof. By direct calculation, we have

$$\begin{aligned}P_{\mathrm{ETF}}(\mathcal{F}_{G,|\psi\rangle}) &= \frac{1}{2}\left(\sum_{\alpha,\beta}\left(|\langle\psi_\alpha|\psi_\beta\rangle| - \sqrt{\frac{m-d}{d(m-1)}}\right)^2 - \sum_\alpha\left(|\langle\psi_\alpha|\psi_\alpha\rangle| - \sqrt{\frac{m-d}{d(m-1)}}\right)^2\right)\\ &= \frac{1}{2}\sum_{\alpha,\beta}|\langle\psi_\alpha|\psi_\beta\rangle|^2 + \frac{m^2(m-d)}{2d(m-1)} - \sqrt{\frac{m-d}{d(m-1)}}\sum_{\alpha,\beta}|\langle\psi_\alpha|\psi_\beta\rangle| - \frac{m}{2}\left(1 - \sqrt{\frac{m-d}{d(m-1)}}\right)^2.\end{aligned}$$

Notice that

$$P(\mathcal{F}_{G,|\psi\rangle}) = \sum_{\alpha,\beta}|\langle\psi_\alpha|\psi_\beta\rangle|^2 = \frac{m^2}{d},$$

and $\mathcal{F}_{G,|\psi\rangle}$ is equidistributed, i.e.,

$$\sum_{\alpha,\beta}|\langle\psi_\alpha|\psi_\beta\rangle| = m\sum_\alpha|\langle\psi_\alpha|\psi\rangle| = m\sum_\alpha|\mathrm{tr}(U_\alpha|\psi\rangle\langle\psi|)|,$$

we obtain the desired result.

When $G=\mathcal{P}_d$, we have $m=d^2$ and

$$\sum_{k,l=0}^{d-1}|\mathrm{tr}(D_{k,l}|\psi\rangle\langle\psi|)|=M(|\psi\rangle),$$

which immediately yields equation (7).

Proposition 2. *Let $G=\{U_\alpha:\ \alpha=1,2,\cdots,m\}$ be a finite group acting irreducibly on $\mathbb{C}^d$ with $U_1=\mathbf{1}$, and $|\psi\rangle$ be a pure quantum state on $\mathbb{C}^d$. Let*

$$\mathcal{F}_{G,|\psi\rangle}=\{|\psi_\alpha\rangle=U_\alpha|\psi\rangle:\ \alpha=1,2,\cdots,m\}$$

be the associated group frame. If $\mathcal{F}_{G,|\psi\rangle}$ can be partitioned into n orthonormal bases $O_1,O_2,\cdots,O_n$ of $\mathbb{C}^d$, then (notice that $m=nd$)

$$P_{\mathrm{MUB}}(\mathcal{F}_{G,|\psi\rangle})=n^2d-nd+n\sqrt{d}\left(1-\sum_{\alpha=1}^{m}|\mathrm{tr}(U_\alpha|\psi\rangle\langle\psi|)|\right).$$

In particular, for $\mathcal{F}_{\mathcal{P}_d,|\psi\rangle}=\{|\psi_{k,l}\rangle=D_{k,l}|\psi\rangle:\ k,l\in\mathbb{Z}_d\}$, we have

$$P_{\mathrm{MUB}}(\mathcal{F}_{\mathcal{P}_d,|\psi\rangle})=d^3-d^2+d\sqrt{d}(1-M(|\psi\rangle)).\qquad(8)$$

Proof. By direct calculation, we have

$$\begin{aligned}P_{\mathrm{MUB}}(\mathcal{F}_{G,|\psi\rangle})&=\frac{1}{2}\left(\sum_{j_1,j_2,k_1,k_2}\left(|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle|-\frac{1}{\sqrt{d}}\right)^2-\sum_{j,k_1,k_2}\left(|\langle\psi_{j,k_1}|\psi_{j,k_2}\rangle|-\frac{1}{\sqrt{d}}\right)^2\right)\\&=\frac{1}{2}\sum_{j_1,j_2,k_1,k_2}|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle|^2+\frac{1}{2d}-\frac{1}{\sqrt{d}}\sum_{j_1,j_2,k_1,k_2}|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle|-n(d-\sqrt{d}).\end{aligned}$$

Notice that

$$P(\mathcal{F}_{G,|\psi\rangle})=\sum_{j_1,j_2,k_1,k_2}|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle|^2=\frac{(nd)^2}{d}=n^2d,$$

and

$$\begin{aligned}\sum_{j_1,j_2,k_1,k_2}|\langle\psi_{j_1,k_1}|\psi_{j_2,k_2}\rangle|&=nd\sum_{j,k}|\langle\psi_{j,k}|\psi\rangle|\\&=nd\sum_{\alpha=1}^{m}|\mathrm{tr}(U_\alpha|\psi\rangle\langle\psi|)|,\end{aligned}$$

we obtain the desired result.

When $G=\mathcal{P}_d$, we have $n=d$ and

$$\sum_{k,l=0}^{d-1}|\mathrm{tr}(D_{k,l}|\psi\rangle\langle\psi|)|=M(|\psi\rangle),$$

which immediately yields equation (8).

Equations (7) and (8) reveal some intrinsic relations between magic of the fiducial states, SIC-POVMs and MUBs.

Considering the group frame generated by the discrete Heisenberg–Weyl group $\mathcal{P}_d$ acting on a stabilizer state, we have the following result.

Proposition 3. *For any stabilizer state $|\psi\rangle\in\mathbb{C}^d$, the group frame*

$$\mathcal{F}_{\mathcal{P}_d,|\psi\rangle}=\{|\psi_{k,l}\rangle=D_{k,l}|\psi\rangle:\ k,l\in\mathbb{Z}_d\}$$

can be partitioned into d orthonormal bases of $\mathbb{C}^d$, which are all equivalent up to phases, i.e., they all induce the same von Neumann measurement.

Proof. Since by definition, any stabilizer state in $\mathbb{C}^d$ is stabilized by a maximal Abelian subgroup of $\mathcal{P}_d$, we may assume that $|\psi\rangle$ is stabilized by the maximal Abelian subgroup $\mathcal{A}=\{A_j:\ j\in\mathbb{Z}_d\}\subseteq\mathcal{P}_d$, i.e., $A_j|\psi\rangle=|\psi\rangle, j\in\mathbb{Z}_d$. This implies

$$|\psi\rangle\langle\psi|=\frac{1}{d}\sum_{j=0}^{d-1}A_j.$$

Then, all common eigenstates $\{|\psi_0\rangle=|\psi\rangle,|\psi_1\rangle,\cdots,|\psi_{d-1}\rangle\}$ of operators in $\mathcal{A}$ (noting that $\mathcal{A}$ is Abelian) are stabilizer states and constitute an orthonormal basis of $\mathbb{C}^d$. For each $\alpha\in\mathbb{Z}_d$, there exists $D_\alpha\in\mathcal{P}_d$ such that $|\psi_\alpha\rangle$ is stabilized by $\{D_\alpha A_jD_\alpha^\dagger:\ j\in\mathbb{Z}_d\}$ and $|\psi_\alpha\rangle=D_\alpha|\psi\rangle$. Moreover, for each $k,l,j\in\mathbb{Z}_d$, there exists $c_{k,l,j}\in\mathbb{Z}_d$ such that

$$D_{k,l}|\psi\rangle=D_{k,l}A_jD_{k,l}^\dagger D_{k,l}|\psi\rangle=\omega^{c_{k,l,j}}A_jD_{k,l}|\psi\rangle.$$

This implies that $D_{k,l}|\psi\rangle$ is a common eigenstate of operators in $\mathcal{A}$, i.e., $D_{k,l}|\psi\rangle\in\{|\psi_0\rangle,|\psi_1\rangle,\cdots,|\psi_{d-1}\rangle\}$ (ignoring the overall phase). The above discussion demonstrates that $\mathcal{F}_{\mathcal{P}_d,|\psi\rangle}$ consists of d orthonormal bases of $\mathbb{C}^d$, which are equivalent up to phase.

Proposition 4. *In a d-dimensional quantum system, the group frame*

$$\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle}=\{|U_{k,l,\theta}\rangle=D_{k,l}|U_\theta\rangle:\ k,l\in\mathbb{Z}_d\}$$

generated by $\mathcal{P}_d$ acting on the initial state

$$|U_\theta\rangle=\sum_{j=0}^{d-1}\mathrm{e}^{\mathrm{i}\theta_j}|j\rangle,\qquad\theta_j\in[0,2\pi)$$

can be partitioned into d orthonormal bases $\{|U_{k,l,\theta}\rangle : l \in \mathbb{Z}_d\}$, $k \in \mathbb{Z}_d$.

Proof. The initial state $|U_\theta\rangle$ can be rewritten as $|U_\theta\rangle = U_\theta|+\rangle$, where

$$U_\theta = \sum_{j=0}^{d-1} e^{i\theta_j}|j\rangle\langle j|, \qquad |+\rangle = F_d|0\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle.$$

Noticing that U_θ is diagonal, which commutes with Z, we have, for any k, l, $l' \in \mathbb{Z}_d$,

$$\begin{aligned}\langle U_{k,l,\theta}|U_{k,l',\theta}\rangle &= \langle +|U_\theta^\dagger Z^{-l}X^{-k}X^kZ^{l'}U_\theta|+\rangle \\ &= \langle +|U_\theta^\dagger Z^{l'-l}U_\theta|+\rangle \\ &= \langle +|Z^{l'-l}|+\rangle = \frac{1}{d}\sum_{j=0}^{d-1}\omega^{j(l'-l)} = \delta_{l,l'}.\end{aligned}$$

Here $\delta_{l,l'}$ is the Kronecker delta function. This demonstrates that $\{|U_{k,l,\theta}\rangle : l \in \mathbb{Z}_d\}$, $k \in \mathbb{Z}_d$, are d orthonormal bases.

According to proposition 2, since the group frame $\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle}$ can be partitioned into d orthonormal bases of $\mathbb{C}^d$, the distance between $\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle}$ and d MUBs is given by

$$P_{\mathrm{MUB}}(\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle}) = d^3 - d^2 + d\sqrt{d}(1 - M(|U_\theta\rangle)),$$

which is determined by the magic $M(|U_\theta\rangle)$ of $|U_\theta\rangle$.

In the next section, we focus on the Heisenberg–Weyl group frames, where the initial states completely determine the orbit of the discrete Heisenberg–Weyl group on $\mathbb{C}^d$. If we take specific magic states as the initial states, we can obtain SIC-POVMs and MUBs. Specially, we introduce a method for systematically constructing $d+1$ MUBs of prime dimensional quantum systems with d^2 magic states via group frame and d computational basis states (which are stabilizer states).

4. SIC-POVM and MUB fiducial states

It is well known that most SIC-POVMs are constructed using group frames, e.g., via the orbits of the discrete Heisenberg–Weyl group on SIC-POVM fiducial states. SIC-POVM fiducial states possess many remarkable properties, though their existences in general dimensions remain open.

Proposition 5. *A pure state $|f_{\mathrm{SIC}}\rangle \in \mathbb{C}^d$ is a SIC-POVM fiducial state if and only if*

$$M(|f_{\mathrm{SIC}}\rangle) = 1 + (d-1)\sqrt{d+1}. \tag{9}$$

Here the quantifier of magic $M(\cdot)$ is defined by equation (2).

Proof. According to definition 3, we know that $\mathcal{F}_{\mathcal{P}_d,|f_{\mathrm{SIC}}\rangle}$ is an equiangular tight frame with d^2 elements, i.e., $|f_{\mathrm{SIC}}\rangle$ is a SIC-POVM fiducial state if and only if $P_{\mathrm{ETF}}(\mathcal{F}_{\mathcal{P}_d,|f_{\mathrm{SIC}}\rangle}) = 0$. By equation (7), $P_{\mathrm{ETF}}(\mathcal{F}_{\mathcal{P}_d,|f_{\mathrm{SIC}}\rangle}) = 0$ implies that

$$M(|f_{\mathrm{SIC}}\rangle) = 1 + (d-1)\sqrt{d+1}.$$

It is interesting to notice that from [17, 29], we know that for any pure state $|\psi\rangle \in \mathbb{C}^d$, it holds that

$$d \leqslant M(|\psi\rangle) \leqslant 1 + (d-1)\sqrt{d+1},$$

which implies that $|\psi\rangle$ is a SIC-POVM fiducial state if and only if the upper bound (which is also the maximal value of the magic if there exists a SIC-POVM fiducial state) is achieved. Moreover, the lower bound (minimal value) is achieved if and only if $|\psi\rangle$ is a stabilizer state.

It is remarkable that the SIC-POVM fiducial states for a qubit system $\mathbb{C}^2$ are precisely the eight T-type magic states [47, 50, 51]

$$|T_{j,k}\rangle = \cos\frac{\theta_j}{2}|0\rangle + e^{i(2k+1)\pi/4}\sin\frac{\theta_j}{2}|1\rangle, \quad j = 0, 1;\ k = 0, 1, 2, 3.$$

Here

$$\theta_0 = \arccos\left(\frac{1}{\sqrt{3}}\right) \in \left(0, \frac{\pi}{2}\right), \qquad \theta_1 = \pi - \theta_0,$$

and $\{|0\rangle, |1\rangle\}$ is the computational basis of $\mathbb{C}^2$ consisting of the eigenstates of σ_z. The density matrices of the T-type magic states can be conveniently expressed as

$$\frac{1}{2}\left(\mathbf{1} + \frac{1}{\sqrt{3}}(\pm\sigma_x \pm \sigma_y \pm \sigma_z)\right), \tag{10}$$

which may be compared with the subsequent equation (14). The T-type magic states possess the maximum value of magic

$$M(|T_{j,k}\rangle) = 1 + (d-1)\sqrt{d+1} = 1 + \sqrt{3}, \qquad \forall\ j, k$$

among all qubit states, i.e.,

$$M(|T_{j,k}\rangle) = \max_{|\psi\rangle} M(|\psi\rangle). \tag{11}$$

Inspired by the idea of SIC-POVM fiducial states [1, 2, 17], we introduce the concept of MUB fiducial states as follows.

Definition 5. A pure state $|f_{\mathrm{MUB}}\rangle \in \mathbb{C}^d$ is called an MUB fiducial state if the group frame

$$\mathcal{F}_{\mathcal{P}_d,|f_{\mathrm{MUB}}\rangle} = \{D_{k,l}|f_{\mathrm{MUB}}\rangle : k, l \in \mathbb{Z}_d\},$$

which consists of d^2 states, can be partitioned into d MUBs of $\mathbb{C}^d$. Here $D_{k,l}$ are the discrete Heisenberg–Weyl operators defined by equation (1).

It is known that the maximal number $N(d)$ of MUBs of $\mathbb{C}^d$ is either $N(d) = d+1$ or $N(d) \leqslant d-1$ [52]. Hence, if an MUB fiducial state exists, the induced d MUBs can always be extended to a complete family of $d+1$ MUBs by adding an orthonormal basis. Now, two fundamental questions arise naturally:

(1) Does there exist an MUB fiducial state in any dimension?
(2) How to construct an MUB fiducial state?

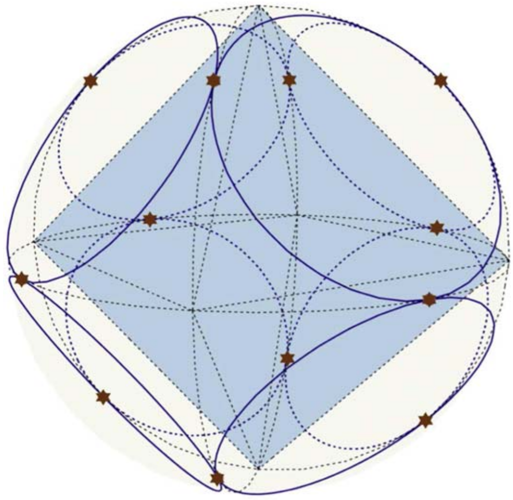

Figure 1. For the qubit system $\mathbb{C}^d$ with $d = 2$, the states, as represented by the points on the eight circles in the Bloch sphere, all have the magic value $1 + (d-1)\sqrt{d} = 1 + \sqrt{2}$. The twelve H-type magic states (as represented by the red stars) are the MUBs fiducial states.

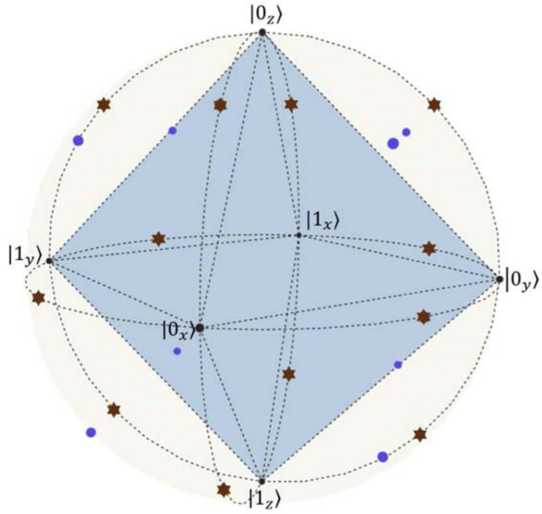

Figure 2. For the qubit system $\mathbb{C}^2$, there are six stabilizer states $|0_\alpha\rangle$, $|1_\alpha\rangle$, $\alpha = x, y, z$ (black dots), twelve H-type magic states (red stars), and eight T-type magic states (blue dots).

Before addressing these questions, we first reveal some general properties of MUB fiducial states.

Proposition 6. *The magic of any MUB fiducial state* $|f_{\rm MUB}\rangle$ *(assuming its existence) in* $\mathbb{C}^d$ *is*

$$M(|f_{\rm MUB}\rangle) = 1 + (d-1)\sqrt{d}. \tag{12}$$

Here, the quantifier of magic $M(\cdot)$ *is defined by equation (2).*

Proof. According to definitions 4 and 5, we know that if $|f_{\rm MUB}\rangle$ is an MUB fiducial state, then we have $P_{\rm MUB}(\mathcal{F}_{\mathcal{P}_d,|f_{\rm MUB}\rangle}) = 0$. By equation (8), $P_{\rm MUB}(\mathcal{F}_{\mathcal{P}_d,|f_{\rm MUB}\rangle}) = 0$ implies that $M(|f_{\rm MUB}\rangle) = 1 + (d-1)\sqrt{d}$.

It is desirable to determine all MUB fiducial states. In view of proposition 6, it is of interest to first determine states $|\psi\rangle$ which possess the magic value

$$M(|f_{\rm MUB}\rangle) = 1 + (d-1)\sqrt{d}.$$

For a qubit system, these states are represented by the eight circles in the Bloch sphere, as shown in figure 1.

It is remarkable that for a qubit system $\mathbb{C}^2$, the MUB fiducial states coincide with the twelve H-type magic states, which arise naturally from the stabilizer formalism of quantum computation and magic state distillation [47]. Recall that in a qubit system, the H-type magic states are defined as [47, 50, 51]

$$|H_{\alpha,k}\rangle = \frac{1}{\sqrt{2}}(|0_\alpha\rangle + {\rm e}^{{\rm i}(2k+1)\pi/4}|1_\alpha\rangle),$$
$$k = 0, 1, 2, 3;\ \alpha = x, y, z, \tag{13}$$

where $|0_\alpha\rangle$ and $|1_\alpha\rangle$ are the two eigenstates of the Pauli matrix σ_α, i.e., $\sigma_\alpha|0_\alpha\rangle = |0_\alpha\rangle$, $\sigma_\alpha|1_\alpha\rangle = -|1_\alpha\rangle$, Consequently, there are $3 \times 4 = 12$ H-type magic states. In the density matrix form, the H-type magic states defined by equation (13) can be equivalently expressed as

$$\frac{1}{2}\left(\mathbf{1} + \frac{1}{\sqrt{2}}(\pm\sigma_\alpha \pm \sigma_\beta)\right), \qquad \alpha \neq \beta \in \{x, y, z\}, \tag{14}$$

which should be compared with equation (10) and will be convenient for later use.

Proposition 7. *In a qubit system* $\mathbb{C}^2$*, there are twelve MUB fiducial states, which are precisely the twelve H-type magic states defined by equation (13), or equivalently, equation (14).*

Proof. It can be easily checked that all H-type magic states are MUB fiducial states. On the other hand, noting that for a qubit system, $X = \sigma_x$, $Z = \sigma_z$, and thus in view of equation (1), the $2^2 = 4$ discrete Heisenberg–Weyl operators are

$$D_{0,0} = \mathbf{1}, \quad D_{0,1} = \sigma_z, \quad D_{1,0} = \sigma_x, \quad D_{1,1} = -\sigma_y.$$

Noting that the density matrix form of any pure qubit state $|\psi\rangle\langle\psi|$ can be expressed as

$$|\psi\rangle\langle\psi| = \frac{1}{2}(\mathbf{1} + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z),$$

with the real coefficients satisfying $r_x^2 + r_y^2 + r_z^2 = 1$, it follows that

$$|\langle\psi|\sigma_\alpha|\psi\rangle| = |r_\alpha|, \qquad \alpha = x, y, z.$$

If $|\psi\rangle$ is an MUB fiducial state, then the orbit

$$\{|\psi\rangle, \sigma_x|\psi\rangle, -\sigma_y|\psi\rangle, \sigma_z|\psi\rangle\}$$

can be partitioned into two MUBs, which implies that one of the numbers $|r_\alpha|$, $\alpha = x, y, z$, is zero and the other two are $1/\sqrt{2}$. Consequently, $|\psi\rangle$ is an H-type magic state in view of equation (14).

It is interesting to compare the MUB fiducial states and SIC-POVM fiducial states on one hand, and the H- and T-type magic states on the other hand.

Geometrically, there is a nice representation of the stabilizer states, H- and T-type states in the Bloch sphere, as illustrated in figure 2.

The next result shows that among a special class of states, the MUB fiducial states achieve the maximum value of magic. This will play a crucial role in our explicit construction of MUB fiducial states.

Proposition 8. *For* $\mathbb{C}^d$ *with* $\{|j\rangle: j\in\mathbb{Z}_d\}$ *a computational basis, let* $|U_\theta\rangle = U_\theta|+\rangle$ *with*

$$U_\theta=\sum_{j=0}^{d-1}\mathrm{e}^{\mathrm{i}\theta_j}|j\rangle\langle j|,\qquad |+\rangle=F_d|0\rangle=\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle.$$

Then the magic of the states $|U_\theta\rangle$ *achieves the maximum value* $1+(d-1)\sqrt{d}$ *if and only if* $|U_\theta\rangle$ *is an MUB fiducial state. More explicitly,*

$$M(|U_\theta\rangle)\leqslant 1+(d-1)\sqrt{d},\ \forall\ \boldsymbol{\theta}=(\theta_1,\theta_2,\cdots,\theta_d)\in\mathbb{R}^d,$$

and the equality is achieved if and only if $|U_\theta\rangle$ *is an MUB fiducial state.*

Proof. Put $|U_{k,l,\theta}\rangle = D_{k,l}|U_\theta\rangle$, then clearly $|U_{0,0,\theta}\rangle=|U_\theta\rangle$. Since $|\langle U_\theta|U_{0,l,\theta}\rangle|=\delta_{0,l}$, we have

$$\begin{aligned}M(|U_\theta\rangle)&=\sum_{k,l=0}^{d-1}|\mathrm{tr}(D_{k,l}|U_\theta\rangle\langle U_\theta|)|\\&=\sum_{k,l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|\\&=1+\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|.\end{aligned}\qquad(15)$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left(\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|\right)^2&\leqslant\left(\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}1\right)\left(\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|^2\right)\\&=d(d-1)\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|^2.\end{aligned}\qquad(16)$$

Moreover, since $\{\frac{1}{\sqrt{d}}D_{k,l}: k,l\in\mathbb{Z}_d\}$ is an orthonormal basis of the operator ($d\times d$ matrix) space $\mathbb{C}^{d\times d}$, we obtain that

$$\sum_{k,l=0}^{d-1}|\mathrm{tr}(D_{k,l}|U_\theta\rangle\langle U_\theta|)|^2=\sum_{k,l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|^2=d.$$

Combined with $|\langle U_\theta|U_{0,l,\theta}\rangle|=\delta_{0,l}$, we have

$$\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\langle U_\theta|U_{k,l,\theta}\rangle|^2=d-1.$$

Consequently, inequality (16) becomes

$$\left(\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}|\mathrm{tr}(D_{k,l}U_\theta|+\rangle\langle+|U_\theta^\dagger)|\right)^2\leqslant d(d-1)^2,$$

where the equality is achieved if and only if

$$|\langle U_\theta|U_{k,l,\theta}\rangle|=\frac{1}{\sqrt{d}}$$

are equal for all $k\in\mathbb{Z}_d\backslash\{0\}$ and $l\in\mathbb{Z}_d$. This implies, by equation (15) and inequality (16), that

$$M(|U_\theta\rangle)\leqslant 1+(d-1)\sqrt{d}.$$

By proposition 4, the group frame $\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle}=\{|U_{k,l,\theta}\rangle: k,l\in\mathbb{Z}_d\}$ can be partitioned into d orthonormal bases of $\mathbb{C}^d$, and by proposition 2, we known that

$$P_{\mathrm{MUB}}(\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle})=d^3-d^2+d\sqrt{d}(1-M(|U_\theta\rangle)),$$

from which $|U_\theta\rangle$ is an MUB fiducial state, i.e., $P_{\mathrm{MUB}}(\mathcal{F}_{\mathcal{P}_d,|U_\theta\rangle})=0$, if and only if its magic achieves the maximum. This completes the proof of proposition 8.

Now, noting that

$$|\langle j|U_{k,l,\theta}\rangle|=\frac{1}{\sqrt{d}}|\tau^{kl}\omega^{(j-k)l}\mathrm{e}^{\mathrm{i}\theta_{j-k}}|=\frac{1}{\sqrt{d}},$$

by augmenting with the computational basis, we obtain a complete family of $d+1$ MUBs of $\mathbb{C}^d$ as

$$\{|j\rangle: j\in\mathbb{Z}_d\},\qquad\{D_{k,l}|U_\theta\rangle: l\in\mathbb{Z}_d\},\qquad k\in\mathbb{Z}_d.$$

In a d-dimensional quantum system $\mathbb{C}^d$ with d prime, the d^2-1 Heisenberg–Weyl operators (excluding the identity operator $D_{0,0}=\mathbf{1}$) can be partitioned into the following $d+1$ sets

$$\begin{aligned}&\mathcal{D}_j=\{D_{1,j}^k: k=1,2,\cdots,d-1\},\quad j\in\mathbb{Z}_d,\\&\mathcal{D}_d=\{D_{0,k}: k=1,2,\cdots,d-1\}.\end{aligned}\qquad(17)$$

A pure quantum state $|\psi\rangle$ can take the form $U_\theta|+\rangle$ if and only if $\mathrm{tr}(D_{0,k}|\psi\rangle\langle\psi|)=0$ for all $k\in\mathbb{Z}_d\backslash\{0\}$. According to the partition of Heisenberg–Weyl operators given by (17), we obtain the following result, extending proposition 8.

Proposition 9. *For any pure state* $|\psi\rangle\in\mathbb{C}^d$ *(d is prime) satisfying*

$$\mathrm{tr}(D|\psi\rangle\langle\psi|)=0,\qquad\forall\ D\in\mathcal{D}_j$$

for some $j\in\{0,1,\cdots,d\}$*, we have*

$$M(|\psi\rangle)\leqslant 1+(d-1)\sqrt{d}.$$

Moreover, the equality is achieved if and only if $|\psi\rangle$ *is an MUB fiducial state.*

Proof. This can be obtained from the fact that for each j, there exists a Clifford operator $C\in\mathcal{C}_d$ such that $\mathcal{D}_j=C\mathcal{D}_dC^\dagger$.

The above proposition demonstrates that MUB fiducial states are locally optimal under certain restrictions, whereas SIC-POVM fiducial states are globally optimal.

The problem remains to construct MUB fiducial states. When the quantum system dimension d is a prime number, we find a simple approach to constructing MUB fiducial states via a class of special diagonal quantum T-gate. We remark that the quantum T-gate plays a key role in promoting the stabilizer quantum circuits based on Clifford gates to genuine quantum computation [53–59]. In particular, it is shown that the quantum T-gate is optimal in generating magic resources among the class of diagonal unitary gates for qubit and qutrit systems [58, 59].

For number-theoretic reasons, the quantum T-gate on $\mathbb{C}^d$ for the cases $d = 2$, 3, and prime $d \geqslant 5$ need to be treated separately.

(1) $d = 2$

In the qubit system, the quantum T-gate is defined as [45, 47, 54, 55]

$$T_2 = |0\rangle\langle 0| + e^{\pi i/4}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi i/4} \end{pmatrix}.$$

(2) $d = 3$

In the qutrit system, a version of quantum T-gate is given by Howard and Vala as [55]

$$T_3 = \sum_{j=0}^{2} \zeta^{j^3}|j\rangle\langle j| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/9} & 0 \\ 0 & 0 & e^{-2\pi i/9} \end{pmatrix}, \qquad \zeta = e^{2\pi i/9}.$$

(3) $d \geqslant 5$ with d a prime number

In this case, a version of quantum T-gate is defined as [57]

$$T_d = \sum_{j=0}^{d-1} \omega^{j^3}|j\rangle\langle j|, \qquad \omega = e^{2\pi i/d},$$

which satisfies $T_d^d = \mathbf{1}$.

We summarize the above quantum T-gates as follows.

$$T_d = \begin{cases} |0\rangle\langle 0| + e^{\pi i/4}|1\rangle\langle 1|, & \text{for } d = 2, \\ \sum_{j=0}^{2} \zeta^{j^3}|j\rangle\langle j|,\ \zeta = e^{2\pi i/9}, & \text{for } d = 3, \\ \sum_{j=0}^{d-1} \omega^{j^3}|j\rangle\langle j|,\ \omega = e^{2\pi i/d}, & \text{for prime } d \geqslant 5. \end{cases}$$

Proposition 10. *For any prime dimensional system $\mathbb{C}^d$ with $\{|j\rangle: j \in \mathbb{Z}_d\}$ a computational basis, let the quantum T-gate T_d be defined as above and $|+\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}|j\rangle$ be the maximal superposition of the computational basis states. Then, $|T_d\rangle = T_d|+\rangle$ is an MUB fiducial state in $\mathbb{C}^d$.*

Proof. It can be straightforwardly checked that

$$M(|T_2\rangle) = 1 + \sqrt{2}, \qquad M(|T_3\rangle) = 1 + 2\sqrt{3},$$

and for prime $d \geqslant 5$,

$$\begin{aligned} M(|T_d\rangle) &= 1 + \frac{1}{d}\sum_{k=1}^{d-1}\sum_{l=0}^{d-1}\left|\sum_{j=0}^{d-1}\omega^{jl + j^3 - (j+k)^3}\right| \\ &= 1 + (d-1)\left|\sum_{j=0}^{d-1}\omega^{j^2}\right| = 1 + (d-1)\sqrt{d}, \end{aligned}$$

where the last equality is obtained from the quadratic Gauss sum [60]

$$\sum_{j=0}^{d-1}\omega^{j^2} = \begin{cases} \sqrt{d}, & \text{if } d = 1 \pmod 4, \\ i\sqrt{d}, & \text{if } d = 3 \pmod 4). \end{cases}$$

In view of proposition 9, we immediately conclude that $|T_d\rangle$ is an MUB fiducial state. Moreover, for $n = 1, 2, \cdots, d-1$, it can be checked in the same way that $T_d^n|+\rangle$ are also MUB fiducial states.

Based on the above proposition, for any prime dimensional system $\mathbb{C}^d$ $(d \geqslant 5)$ with $\{|j\rangle: j \in \mathbb{Z}_d\}$ a computational basis, the MUB fiducial state takes the form

$$|f_{\mathrm{MUB}}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{j^3}|j\rangle, \qquad \omega = e^{2\pi i/d}.$$

The d orthonormal bases

$$\left\{D_{k,l}|f_{\mathrm{MUB}}\rangle = \frac{1}{\sqrt{d}}\tau^{kl}\sum_{j=0}^{d-1}\omega^{(j-k)^3 + l(j-k)}|j\rangle: l \in \mathbb{Z}_d\right\}, \quad k \in \mathbb{Z}_d,$$

together with $\{|j\rangle: j \in \mathbb{Z}_d\}$, constitute a complete family of $d + 1$ MUBs of $\mathbb{C}^d$, which are essentially equivalent to those obtained in [21, 24].

It is desirable to compare the MUB fiducial states and the SIC-POVM fiducial states from the perspective of magic resources.

Proposition 11. *Let $|f_{\mathrm{MUB}}\rangle$ and $|f_{\mathrm{SIC}}\rangle$ be an MUB fiducial state and a SIC-POVM fiducial state in $\mathbb{C}^d$, respectively. Then*

$$\sqrt{\frac{d}{d+1}} < \frac{M(|f_{\mathrm{MUB}}\rangle)}{M(|f_{\mathrm{SIC}}\rangle)} < 1.$$

In particular,

$$\lim_{d\to\infty}\frac{M(|f_{\mathrm{MUB}}\rangle)}{M(|f_{\mathrm{SIC}}\rangle)} = 1.$$

The proof follows readily from equations (9) and (12).

As we already demonstrated, in a qubit system, the SIC-POVM fiducial states and MUB fiducial states correspond respectively to the T- and H-type magic states. There are, at present, no higher dimensional extensions of H- and T-type magics states. Now in view of proposition 5, equation (11), propositions 7 and 8, we may intuitively regard SIC-POVM fiducial states and MUB fiducial states as high dimensional analogues of T-type magic states and H-type magic states, respectively. This correspondence highlights the significance and distinctiveness of these two types of fiducial states.

5. Discussion and summary

In this work, we have integrated SIC-POVMs and MUBs into the comprehensive framework of group frames, with a specific focus on those generated by the discrete Heisenberg–Weyl group. By examining the effects of different initial states, such as stabilizer states and magic states, we have revealed some intrinsic structures of these group frames.

We have introduced two quantities to measure the approximation of tight frames to SIC-POVMs and MUBs, which provide insights into the relationship between group frames and stabilizerness or magic of initial states. This leads naturally to the notion of MUB fiducial states, which possess local maximum magic, in contrast to the global maximum magic possessed by SIC-POVM fiducial states, with the magic quantified by the L^1-norm of characteristic function.

As a straightforward application, by introducing the concept of MUB fiducial states and exploiting the discrete Heisenberg–Weyl group, we have presented a simple and constructive approach to MUBs of prime dimensional systems via MUB fiducial states. A key aspect of our approach is its simplicity and transparency. An MUB fiducial state acts as a starting point, on which the discrete Heisenberg–Weyl group generates an orbit which, together with the computational basis, form a complete family of MUBs. In any prime dimensional system, we have provided a method for constructing an MUB fiducial state by invoking the quantum T-gate and the maximal superposition of the computational basis states. It is remarkable that the quantum T-gate, which is a fundamental gate in the stabilizer formalism of quantum computation, comes to play an interesting role here in constructing MUBs.

We have revealed some intrinsic connections between the SIC-POVM fiducial states and MUB fiducial states. In contrast to the fact that magic of a SIC-POVM fiducial state $|f_{\rm SIC}\rangle$ can attain the maximum value $M(|f_{\rm SIC}\rangle) = 1 + (d-1)\sqrt{d+1}$ in a d-dimensional system, we have proved that the magic of an MUB fiducial state $|f_{\rm MUB}\rangle$ is $M(|f_{\rm MUB}\rangle) = 1 + (d-1)\sqrt{d}$. When d is sufficiently large, these two values converge to the same limit. According to the Gottesman–Knill theorem [42–45], magic is an essential quantum resource for universal quantum computation. Although the SIC-POVM fiducial state possesses the maximal magic resource, its existence in a general dimensional quantum system still remains an open problem. In contrast, MUB fiducial states exist and can be simply constructed at least in any prime-power dimensional system, and when the prime d is large enough, they can be considered as approximately optimal magic states.

We remark that our method for constructing MUBs via fiducial states can be extended to prime-power dimensional systems with $d = p^n$ for prime $p \geqslant 5$. The cases for $p = 2, 3$ and $n > 3$ need separate treatments and further investigations.

Finally, it is desirable to identify all MUB fiducial states in general dimensions (in case of their existence), which seems a difficult issue.

Acknowledgments

This work was supported by the National Key R&D Program of China, Grant No. 2020YFA0712700, and the National Natural Science Foundation of China 'Mathematical Basic Theory of Quantum Computing' special project, Grant No. 12341103.

Appendix

Here we review briefly some basic features of SIC-POVMs and MUBs for the convenience of their comparison and extensions.

A1. SIC-POVMs

In the modern formalism of quantum mechanics, a measurement in a quantum system described by a d-dimensional Hilbert space $\mathbb{C}^d$ is mathematically represented by a positive operator valued measure (POVM) [45]

$$E = \{E_\alpha: \alpha = 1, 2, \cdots\},$$

which consists of a set of non-negative operators E_α summing to the identity operator, that is, each $E_\alpha \geqslant 0$, and $\sum_\alpha E_\alpha = \mathbf{1}$.

The notion of symmetric informationally complete positive operator valued measure (SIC-POVM), which was introduced by Renes *et al* [2], can be traced back to Zauner's work on quantum design [1]. This special class of quantum measurements plays a significant role in quantum information theory. More precisely, a SIC-POVM on $\mathbb{C}^d$ is a POVM $E = \{E_\alpha: \alpha = 1, 2, \cdots, m\}$ such that [2]:

(1) (Informational completeness) $m = d^2$, and the measurement operators E_α span the whole state (pure or mixed) space.
(2) (Equal-overlap) ${\rm tr}(E_\alpha E_\beta) = b$ is a constant independent of $\alpha \neq \beta$.
(3) (Equal-trace) ${\rm tr}E_\alpha = t$ is a constant independent of α.
(4) (Rank-one) E_α are of rank-one in the sense that $E_\alpha = t|\psi_\alpha\rangle\langle\psi_\alpha|$ for some pure states $|\psi_\alpha\rangle$ in $\mathbb{C}^d$.

It turns out that the parameters b and t in a SIC-POVM are uniquely determined by the system dimension d as [2]

$$b = \frac{1}{d^2(d+1)}, \qquad t = \frac{1}{d}.$$

Nowadays, the existence of SIC-POVMs for arbitrary dimensions remains an elusive and outstanding conjecture (Zauner's conjecture). However, this existence is widely believed and supported by many analytical and numerical evidences [10, 11].

Recently, new classes of POVMs have been introduced by relaxing certain conditions of SIC-POVMs. For instance,

Table 1. Various constructions of complete MUBs of $\mathbb{C}^d$ with d prime.

	Complete MUBs	Dimension d
Alltop [21] Klappenecker and Rötteler [24]	$\{\|j\rangle : j \in \mathbb{Z}_d\}$ $\{\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{(j+k)^3\ +\ l(j+k)}\|j\rangle : l \in \mathbb{Z}_d\}$, $k \in \mathbb{Z}_d$	any prime $d \geqslant 5$
Ivonovic [22] Wootters and Fields [20]	$\{\|j\rangle : j \in \mathbb{Z}_d\}$ $\{\frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{kj^2+lj}\|j\rangle : l \in \mathbb{Z}_d\}$, $k \in \mathbb{Z}_d$	any prime $d \geqslant 3$
Bandyopadhyay *et al* [23]	$\{\|j\rangle : j \in \mathbb{Z}_d\}$ { Eigenstates of XZ^k}, $k \in \mathbb{Z}_d$	any prime d
MUBs via fiducial states	$\{\|j\rangle : j \in \mathbb{Z}_d\}$, $\{D_{k,l}\|f_{\mathrm{MUB}}\rangle : l \in \mathbb{Z}_d\}$, $k \in \mathbb{Z}_d$ where $\|f_{\mathrm{MUB}}\rangle = T_d\| + \rangle$	any prime d

Wootters and Fields [20], Bandyopadhyay *et al* [23], Klappenecker and Rötteler [24] further constructed MUBs of prime-power dimensional systems, which are not listed here.

semi-SIC-POVMs were developed to relax the equal-trace condition in SIC-POVMs, leading to some interesting new phenomena [61]. By simultaneously dropping $m = d^2$ and the equal-trace condition, the notion of equioverlapping measurement was introduced and studied in [62–64]. Equioverlapping measurements include von Neumann measurements, SIC-POVMs, equiangular measurements, and semi-SIC-POVMs as special cases. Another significant generalization, which is known as generalized SIC-POVM [65–68], is proposed by relaxing the rank-one condition for measurement operators. It is remarkable that generalized SIC-POVMs have been proved to exist in every dimension. Furthermore, Siudzińska introduced two broad classes of symmetric measurements, (N, M)-POVMs [69] and generalized symmetric measurements [70], which are collections of N POVMs with certain symmetric condition and include the mutually unbiased measurements and general SIC-POVMs as special cases. Just as a SIC-POVM corresponds to an equiangular tight frame, Siudzińska's construction lead to the concept of generalized equiangular tight frames [71].

A2. MUBs

MUBs have their origin in the seminal work of Schwinger concerning unitary operator bases [46]. In an important contribution to signal processing involving sequences with low periodic correlations, Alltop actually constructed a special class of complete MUBs of prime dimensional systems [21], although it remains unnoticed for sometime. For quantum state estimation, measurements along MUBs have the ability to maximize information extraction and to minimize redundancy [20, 22, 82]. With the emergence of quantum information, the quest for MUBs has attracted much attention from both theoretical and practical perspectives due to their special features in capturing quantum complementarity and representing information [1, 3, 18, 23–28, 72–81]. These unusual bases play an interesting and significant role in foundational aspects of quantum measurement, as well as in quantum tomography [20, 22, 82], quantum key distribution [83–85], uncertainty relations [86–89], detection of quantum entanglement [90–92], etc.

It is well known that in a d-dimensional system $\mathbb{C}^d$, there exist no more than $d+1$ MUBs, and when $d = p^n$ is a prime-power, this upper bound can be attained [20]. Apart from the pioneering and inexplicit construction of Alltop [21], the first complete family of MUBs of an odd prime dimensional system was explicitly constructed by Ivonovic [22], who considered the problem of quantum state determination. The computational basis $\{|j\rangle : j \in \mathbb{Z}_d\}$ of $\mathbb{C}^d$ together with the following d bases

$$\left\{|\psi_{k,l}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{kj^2+lj}|j\rangle : l \in \mathbb{Z}_d\right\}, \qquad k \in \mathbb{Z}_d$$

constitute $d+1$ MUBs of $\mathbb{C}^d$ with $d > 2$ a prime. Here $\omega = \mathrm{e}^{2\pi \mathrm{i}/d}$. This result was later generalized by Wootters and Fields to the case of $d = p^n$ for any prime p and natural number n by exploiting the Galois field $\mathbb{Z}_{p^n}$ [20]. Bandyopadhyay *et al* reformulated the previous construction in the stabilizer formalism of quantum computation by identifying the $d(d+1)$ states in the above complete MUBs as stabilizer states in $\mathbb{C}^d$, which can be partitioned into precisely $d+1$ MUBs [23]. Another class of complete family of MUBs was presented by Klappenecker and Rötteler [24]: the computational basis $\{|j\rangle : j \in \mathbb{Z}_d\}$ together with other d bases

$$\left\{|\phi_{k,l}\rangle = \frac{1}{\sqrt{d}}\sum_{j=0}^{d-1}\omega^{(j+k)^3+l(j+k)}|j\rangle : l \in \mathbb{Z}_d\right\}, \qquad k \in \mathbb{Z}_d$$

also constitute $d+1$ MUBs of $\mathbb{C}^d$ when $d > 5$ is a prime. Here $\omega = \mathrm{e}^{2\pi \mathrm{i}/d}$. This can be extended to the case of any prime-power dimension $d = p^n$ for prime $p \geqslant 5$. This construction, in its mathematical essence, can be traced back to Alltop in the realm of signal processing [21], although it gained limited attention in quantum information theory until later times.

These various classes of complete MUBs provide constructive proofs of existence of $d+1$ MUBs of prime-power dimensional systems. The constructions rely on subtle number-theoretic properties. There are some further constructions of MUBs by invoking other methods such as orthogonal spreads and extraspecial groups [72], orthogonal decomposition of Lie algebras [74], spin models [77], symplectic spreads [79], positive definite functions [80], and special functions on finite fields [81], etc. For illustration and comparison, we summarize various constructions of complete MUBs of prime dimensional systems in table 1.

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