 Printed in China Communications in Theoretical Physics

Commun. Theor. Phys. 77 (2025) 035401 (7pp) https://doi.org/10.1088/1572-9494/ad89b5

On the transverse-traceless gauge condition when matter is presented

Yadong Xue[1,2], **Xiaokai He**[3] **and Zhoujian Cao**[1,2,4]

[1] Institute for Frontiers in Astronomy and Astrophysics, Beijing Normal University, Beijing 102206, China
[2] Department of Astronomy, Beijing Normal University, Beijing 100875, China
[3] School of Mathematics and Statistics, Hunan First Normal University, Changsha 410205, China
[4] School of Fundamental Physics and Mathematical Sciences, Hangzhou Institute for Advanced Study, UCAS, Hangzhou 310024, China

E-mail: zjcao@amt.ac.cn

Received 25 August 2024, revised 9 October 2024
Accepted for publication 22 October 2024
Published 4 December 2024

Abstract

The transverse-traceless gauge condition is an important concept in the theory of gravitational waves. It is well known that a vacuum is one of the key conditions to guarantee the existence of the transverse-traceless gauge. Although it is thin, the interstellar medium is ubiquitous in the Universe. Therefore, it is important to understand the concept of gravitational waves when matter is presented. Bondi–Metzner–Sachs theory has solved the gauge problem related to gravitational waves. But it does not help with cases when the gravitational wave propagates in matter. This paper discusses possible extensions of the transverse-traceless gauge condition to Minkowski perturbation with matter presented.

Keywords: gravitational wave, transverse-traceless, gauge condition

1. Introduction

The gauge issue is a long-standing problem in general relativity. Related to gravitational waves (GWs), the gauge issue ever confused people including Einstein for almost half a century [1]. For isolated GW sources which correspond to asymptotically flat spacetime, Bondi–Metzner–Sachs theory [2, 3] solves the gauge problem elegantly and gives a beautiful description of GWs. In the cases of asymptotically non-flat spacetimes, including the cosmological situations, metric perturbation with special gauge choice is used to describe GWs [4, 5].

Since asymptotically flat spacetime can be approximated as a Minkowski perturbation, the transverse-traceless (TT) gauge condition becomes a convenient tool to describe GWs. Many textbooks use the TT gauge condition as the basic theory to present GWs [6–8]. The TT gauge condition provides a fundamental picture of GWs to many researchers.

It is well known that the TT gauge condition is a special harmonic gauge. It is the vacuum Einstein equation that aids us to pick out the special one from harmonic gauges. Although many textbooks use plane GWs as examples to illustrate the choice procedure of the TT gauge condition, the plane wave is not a necessity, which we will clarify later in the current paper. In contrast, the vacuum condition must be satisfied to guarantee the existence of the TT gauge condition. Our analysis later will make this fact apparent.

In real situations, we need to consider that GWs propagates in matter [9, 10]. This is because that interstellar medium is ubiquitous in the Universe although it is thin. And we need to consider the interaction between GWs and matter including the Weber bar GW detectors [11], GW-excited lunar vibrations [12, 13] and others. In the existing literature, people neglect the matter completely when they describe GWs in these situations. Intuitively we can believe this treatment is reasonable and quantitatively accurate enough. But it is still interesting to ask the mathematical principle behind. In fact Prof. Yuan-Zhong Zhang ever discussed this problem with one of us several times.

In the current paper we aim to discuss the concept of GWS when matter is presented. More specifically we will discuss the generalization of the TT gauge condition to situations when matter is presented. In the next two sections we describe the mathematics needed by our analysis. Although the mathematical principle has been scattered in the existing literature, the description provided here is more systematic and comprehensive. Based on this mathematical tool, we briefly review the decomposition of the metric perturbation of Minkowsky in section 4. Besides the introduction of the well known gauge invariant quantities, we design three gauge sensitive variables which are useful to discuss the gauge conditions later. After that we give a clearer analysis of harmonic gauge conditions from a new viewpoint. Within this framework we discuss the TT gauge condition choice in section 6. Along with that a natural extension of the usual TT gauge condition to situations when matters are present is proposed. Alternatively we analyze the CZ gauge condition proposed by other authors in section 7. Intuitively people may think the CZ gauge condition is similar but different to harmonic gauge condition. Our analysis indicates that the CZ gauge is more similar to the usual TT gauge condition. The similarity is that harmonic gauge is a family of gauges while the usual TT gauge and the CZ gauge are unique. Our analysis shows that the CZ gauge in general is not harmonic. But the CZ gauge condition returns to the TT gauge condition when the spacetime is a vacuum. In this sense the CZ gauge condition can also be looked as a generalization of TT gauge condition to situations when matters are present. We summarize the paper with some discussions about the generalized TT gauge condition in the last section.

The geometric units with $c = G = 1$ are used throughout the paper.

2. Decomposition of a vector field

For any vector field $\vec{v}$ with asymptotic condition $\vec{v}(\vec{r} \to \infty) = 0$ we have the following decomposition

$$\vec{v} = \nabla\phi + \nabla \times \vec{u}, \tag{1}$$

where the scalar field ϕ is determined by

$$\nabla^2\phi = \nabla \cdot \vec{v}. \tag{2}$$

According to the partial differential equation theory for the Poisson equation, there is an unique solution for ϕ to the above equation. As the solution of the above mentioned Poisson equation, asymptotically we have $\phi(\vec{r} \to \infty) = 0$.

Then the decomposition condition equation (1) leads us to

$$\nabla \times \vec{u} = \vec{v} - \nabla\phi, \tag{3}$$

$$\nabla \times (\nabla \times \vec{u}) = \nabla \times \vec{v} - \nabla \times (\nabla\phi) = \nabla \times \vec{v}, \tag{4}$$

$$\nabla(\nabla \cdot \vec{u}) - \nabla^2\vec{u} = \nabla \times \vec{v}. \tag{5}$$

We assume $\nabla \cdot \vec{u} = 0$, thus the last equation becomes

$$\nabla^2\vec{u} = -\nabla \times \vec{v}. \tag{6}$$

According to the partial differential equation theory for the Poisson equation again, there is an unique solution for $\vec{u}$ to the above equation. And asymptotically we have $\vec{u}(\vec{r} \to \infty) = 0$.

Based on equation (6) we have

$$\nabla \cdot \nabla^2\vec{u} = -\nabla \cdot (\nabla \times \vec{v}) = 0 \tag{7}$$

$$\nabla^2(\nabla \cdot \vec{u}) = 0. \tag{8}$$

Using the partial differential equation theory for the Laplace equation, we confirm

$$\nabla \cdot \vec{u} = 0, \tag{9}$$

which satisfies the requirement to obtain equation (6).

In other words, any vector $\vec{v}$ with asymptotic condition $\vec{v}(\vec{r} \to \infty) = 0$ can be uniquely decomposed to a transverse part $\nabla \times \vec{u}$, where $\vec{u}$ is determined by equation (6) satisfying $\nabla \cdot \vec{u} = 0$, and longitudinal part $\nabla\phi$, where ϕ is determined by equation (2) .

The decomposition equation (1) admits an interesting property which is related to harmonic gauge condition involved in the GW theory. In order to illustrate this property we explain a theorem first. Considering a Poisson equation

$$\nabla^2 u = s, \tag{10}$$

if the source term s is an harmonic function which means $\Box s = 0$, then

$$\Box\nabla^2 u = \Box\, s = 0, \tag{11}$$

$$\nabla^2\, \Box\, u = 0. \tag{12}$$

Based on asymptotic boundary condition, the above equation means $\Box u = 0$. This is to say the solution of a Poisson equation with a harmonic source is a harmonic function.

The above mentioned theorem together with equations (2) and (6) tell us that if the original vector field $\vec{v}$ is harmonic $\Box\vec{v} = 0$, then the decomposition components ϕ and $\vec{u}$ are also harmonic satisfying $\Box\phi = 0$ and $\Box\vec{u} = 0$.

Actually the aforementioned property can even be stronger as follows. Assuming that ϕ and $\vec{u}$ are decomposition components of a given field $\vec{v}$, then $\Box\phi$ and $\Box\vec{u}$ are decomposition components of the vector field $\Box\vec{v}$.

3. Decomposition of a tensor field

Closely following the trick of the above decomposition of a vector field, for any tensor field h_{ij} with asymptotic condition $h_{ij}(\vec{r} \to \infty) = 0$ we have the following decomposition.

$$h_{ij} = \frac{1}{3}H\delta_{ij} + \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right]\lambda + \partial_i\epsilon_j + \partial_j\epsilon_i + h_{ij}^{\mathrm{TT}} \tag{13}$$

where the scalar fields H and λ are determined by

$$H = \delta^{ij}h_{ij}, \tag{14}$$

$$\nabla^2\tau = \frac{3}{2}\partial^j\partial^i\left(h_{ij} - \frac{1}{3}H\delta_{ij}\right) = \frac{3}{2}\partial^j\partial^i h_{ij} - \frac{1}{2}\nabla^2 H, \tag{15}$$

$$\nabla^2\lambda = \tau. \tag{16}$$

According to the partial differential equation theory for the Poisson equation, there are unique solutions for τ and λ to the above equations. Asymptotically we have $H(\vec{r} \to \infty) = 0$, $\tau(\vec{r} \to \infty) = 0$ and $\lambda(\vec{r} \to \infty) = 0$.

The transverse-traceless notation TT in equation (13) means

$$\partial^i h_{ij}^{\mathrm{TT}} = 0, \tag{17}$$

$$\delta^{ij}h_{ij}^{\mathrm{TT}} = 0. \tag{18}$$

Consequently, equation (13) leads us to

$$\partial_i\epsilon_j + \partial_j\epsilon_i = h_{ij} - \frac{1}{3}H\delta_{ij} - \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right]\lambda - h_{ij}^{\mathrm{TT}}, \tag{19}$$

$$\nabla^2\epsilon_j + \partial_j\partial^i\epsilon_i = \partial^i h_{ij} - \frac{1}{3}\partial_j H - \left[\nabla^2\partial_j - \frac{1}{3}\partial_j\nabla^2\right]\lambda, \tag{20}$$

$$\nabla^2\epsilon_j + \partial_j\partial^i\epsilon_i = \partial^i h_{ij} - \frac{1}{3}\partial_j H - \frac{2}{3}\partial_j\tau. \tag{21}$$

In addition, we require ϵ_i to be divergence free

$$\partial^i\epsilon_i = 0. \tag{22}$$

Consequently, equation (21) becomes

$$\nabla^2\epsilon_j = \partial^i h_{ij} - \frac{1}{3}\partial_j H - \frac{2}{3}\partial_j\tau. \tag{23}$$

According to the partial differential equation theory for the Poisson equation, there is a unique solution for ϵ_j to the above equation. In the mean time equation (23) leads to

$$\partial^i\nabla^2\epsilon_i = \partial^j\partial^i h_{ij} - \frac{1}{3}\nabla^2 H - \frac{2}{3}\nabla^2\tau, \tag{24}$$

$$\nabla^2\partial^i\epsilon_i = 0. \tag{25}$$

Using the partial differential equation theory for the Laplace equation, we know solution ϵ_i is automatically divergence free satisfying the assumption we made before. This equivalently means the tensor $\partial_i\epsilon_j$ is traceless.

Based on the determined H, λ and ϵ_i, we can determine h_{ij}^{TT} as

$$h_{ij}^{\mathrm{TT}} \equiv h_{ij} - \frac{1}{3}H\delta_{ij} - \left[\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right]\lambda - 2\partial_{(i}\epsilon_{j)}. \tag{26}$$

When h_{ij} is symmetric, h_{ij}^{TT} is symmetric, transverse and traceless [4, 14, 15].

Specifically for a planar GW

$$h_{ij} = A_{ij}\mathrm{e}^{I\vec{n}\cdot\vec{x}}, \tag{27}$$

thus equation (14) gives us

$$H = A\mathrm{e}^{I\vec{n}\cdot\vec{x}}, \tag{28}$$

$$A \equiv \delta^{ij}A_{ij}. \tag{29}$$

Here we have used I to denote $\sqrt{-1}$. Plugging the above relations into equation (15) we get

$$\nabla^2\tau = -\frac{3}{2}n^i n^j h_{ij} + \frac{1}{2}n^2 H, \tag{30}$$

$$\tau = \frac{3}{2n^2}n^i n^j h_{ij} - \frac{1}{2}H. \tag{31}$$

Continually plugging the above relations into equation (16) we get

$$\lambda = -\frac{3}{2n^4}n^i n^j h_{ij} + \frac{1}{2n^2}H. \tag{32}$$

Plugging the above relations into equation (23) we get

$$\nabla^2\epsilon_j = In^i h_{ij} - In_j\frac{1}{n^2}n^k n^l h_{kl}, \tag{33}$$

$$\epsilon_j = -I\frac{n^i}{n^2}h_{ij} + I\frac{n_j}{n^4}n^k n^l h_{kl}. \tag{34}$$

Collecting these results into equation (26) we get

$$h_{ij}^{\mathrm{TT}} \equiv h_{ij} - n_i n^k h_{kj} - n_j n^l h_{il} - \frac{1}{2}(H\delta_{ij} - Hn_i n_j - \delta_{ij}n^k n^l h_{kl} - n_i n_j n^k n^l h_{kl}). \tag{35}$$

This is to say h_{ij}^{TT} can also be alternatively expressed as [11]

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij}{}^{kl}h_{kl}, \tag{36}$$

$$\Lambda_{ij}{}^{kl} \equiv P_i{}^k P_j{}^l - \frac{1}{2}P_{ij}P^{kl}, \tag{37}$$

where the projector operator P_i^k is defined as

$$P_i{}^k\tilde{h}_{kj}(\vec{n}) \equiv \tilde{h}_{ij}(\vec{n}) - n_i n^k\tilde{h}_{kj}(\vec{n}) \tag{38}$$

with respect to the Fourier component $\tilde{h}_{kj}(\vec{n})$

$$h_{ij} \equiv \int\tilde{h}_{ij}(\vec{n})e^{I\vec{n}\cdot\vec{x}}d^3\vec{n}. \tag{39}$$

4. Decomposition of four dimensional rank 2 tensor

Given a Minkowski perturbation $h_{\mu\nu}$ satisfying $h_{\mu\nu}(\vec{r}\to\infty)\to 0$, we can decompose its components with the techniques shown above as follows [4]

$$h_{tt} = 2\phi, \tag{40}$$

$$h_{ti} = \beta_i + \partial_i\gamma, \tag{41}$$

$$h_{ij} = h_{ij}^{\mathrm{TT}} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\epsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda. \tag{42}$$

Different to section 3 we have absorbed the factor 2 into ϵ_i here.

Similarly we have decomposition for infinitely small gauge transformation ξ_μ satisfying $\xi_\mu(\vec{r}\to\infty)\to 0$

$$\xi_t = A, \tag{43}$$

$$\xi_i = B_i + \partial_i C. \tag{44}$$

Similar to the property of vector decomposition shown in the previous section, if $h_{\mu\nu}$ is harmonic, then all the decomposition components are harmonic. If the infinitely small gauge transformation ξ_μ is harmonic, the components A, B_i and C are also harmonic.

Under such gauge transformation ξ_μ, the aforementioned decomposition components of $h_{\mu\nu}$ will change as

$$\phi \to \phi - \dot{A}, \tag{45}$$

$$\beta_i \to \beta_i - \dot{B}_i, \tag{46}$$

$$\gamma \to \gamma - A - \dot{C}, \tag{47}$$

$$H \to H - 2\nabla^2 C, \tag{48}$$

$$\lambda \to \lambda - 2C, \tag{49}$$

$$\epsilon_i \to \epsilon_i - 2B_i, \tag{50}$$

$$h_{ij}^{\mathrm{TT}} \to h_{ij}^{\mathrm{TT}}. \tag{51}$$

Based on the above transformation rules for the decomposition components, we can easily construct gauge invariant quantities [4]

$$\Phi \equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda}, \tag{52}$$

$$\Theta \equiv \frac{1}{3}(H - \nabla^2\lambda), \tag{53}$$

$$\Xi_i \equiv \beta_i - \frac{1}{2}\dot{\epsilon}_i, \tag{54}$$

$$h_{ij}^{\mathrm{TT}}. \tag{55}$$

Straightforward calculation shows that the linearized Einstein tensor can be expressed with the above gauge invariant quantities as [4]

$$G_{tt} = -\nabla^2\Theta, \tag{56}$$

$$G_{ti} = -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta}, \tag{57}$$

$$G_{ij} = -\frac{1}{2}\,\Box\, h_{ij}^{\mathrm{TT}} + \delta_{ij}\left[\frac{2}{3}\nabla^2(\Phi + \frac{1}{2}\Theta) - \ddot{\Theta}\right] - \partial_{(i}\dot{\Xi}_{j)} - \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)(\Phi + \frac{1}{2}\Theta). \tag{58}$$

It can be seen more that the above form is also the decomposition form of the Einstein tensor.

We decompose the stress-energy tensor $T_{\mu\nu}$ also as

$$T_{tt} = \rho, \tag{59}$$

$$T_{ti} = S_i + \partial_i S, \tag{60}$$

$$T_{ij} = T_{ij}^{\mathrm{TT}} + P\delta_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma. \tag{61}$$

One interesting point is that $T_{\mu\nu}$ is gauge invariant up to high order approximation. This is because $T_{\mu\nu}$ is a first order small quantity in our cases and the gauge transformation matrix of infinitely small gauge transformation is the sum of a identity matrix and a first order small quantity.

Based on the decomposition form of $G_{\mu\nu}$ and $T_{\mu\nu}$, the Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ can be expressed as

$$\nabla^2\Theta = -8\pi\rho, \tag{62}$$

$$\nabla^2\Xi_i = -16\pi S_i, \tag{63}$$

$$\dot{\Theta} = -8\pi S, \tag{64}$$

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi T_{ij}^{\mathrm{TT}}, \tag{65}$$

$$\frac{2}{3}\nabla^2(\Phi + \frac{1}{2}\Theta) - \ddot{\Theta} = 8\pi P, \tag{66}$$

$$\dot{\Xi}_i = -8\pi\sigma_i, \tag{67}$$

$$\Phi + \frac{1}{2}\Theta = -8\pi\sigma. \tag{68}$$

Combining equations (62), (64) and (66) we get

$$\nabla^2\Phi = 4\pi(\rho + 3P - 3\dot{S}). \tag{69}$$

Regarding $T_{\mu\nu}$ the conservation law $\nabla^\mu T_{\mu\nu} = 0$ leads us to [4]

$$\dot{\rho} - \nabla^2 S = 0, \tag{70}$$

$$\frac{3}{2}P - \frac{3}{2}\dot{S} + \nabla^2\sigma = 0, \tag{71}$$

$$2\dot{S}_i - \nabla^2\sigma_i = 0. \tag{72}$$

These three relations guide us to introduce three gauge sensitive variables

$$\Pi \equiv \nabla^2\gamma - \dot{\phi} - \frac{1}{2}\dot{H}, \tag{73}$$

$$\Lambda \equiv -\frac{1}{4}H + \frac{3}{2}\phi + \nabla^2\lambda - \frac{3}{2}\dot{\gamma}, \tag{74}$$

$$\Sigma_i \equiv \dot{\beta}_i - \frac{1}{2}\nabla^2\epsilon_i \tag{75}$$

$$= \dot{\Xi}_i + \frac{1}{2}\ddot{\epsilon}_i - \frac{1}{2}\nabla^2\epsilon_i \tag{76}$$

$$= \dot{\Xi}_i - \frac{1}{2}\,\Box\,\epsilon_i \tag{77}$$

$$= -8\pi\sigma_i - \frac{1}{2}\,\Box\,\epsilon_i. \tag{78}$$

The second line of the above equation about Σ_i is due to equation (54). The last line of the above equation about Σ_i is due to equation (67). Under the gauge transformation defined by ξ_μ, these gauge sensitive variables will change as

$$\Pi \to \Pi - \Box\, A, \tag{79}$$

$$\Lambda \to \Lambda - \frac{3}{2}\,\Box\, C, \tag{80}$$

$$\Sigma_i \to \Sigma_i + \Box\, B_i. \tag{81}$$

Given Ξ_i and Σ_i we can construct ϵ_i and β_i as follows. Based on equation (77) we have

$$\Box\epsilon_i = 2(\dot{\Xi}_i - \Sigma_i). \tag{82}$$

Up to initial data of ϵ_i and $\dot{\epsilon}_i$ we can get ϵ_i. Then based on equation (54) we have

$$\beta_i = \Xi_i + \frac{1}{2}\dot{\epsilon}_i. \tag{83}$$

Given Θ, Φ and Λ we can construct λ through

$$\Box\lambda = \Theta + 2\Phi + \frac{4}{3}\Lambda. \tag{84}$$

Then λ is determined up to initial data of λ and $\dot{\lambda}$. Based on the determined λ and the given Θ we can determine H as

$$H = 3\Theta + \nabla^2\lambda. \tag{85}$$

Based on the determined λ and H and the given Π and Φ we can construct γ through

$$\Box\gamma = \Pi - \dot{\Phi} + \frac{1}{2}\dot{H} - \frac{1}{2}\dddot{\lambda}. \tag{86}$$

Then γ is determined up to initial data of γ and $\dot{\gamma}$. Finally we can determine ϕ through

$$\phi = \dot{\gamma} - \frac{1}{2}\ddot{\lambda} - \Phi. \tag{87}$$

5. Harmonic gauge conditions

The linearized Einstein equation under the harmonic gauge condition reads as

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu}, \tag{88}$$

here $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$ means the corresponding trace-reversed metric perturbation of $h_{\mu\nu}$. The relation between the four dimensional trace h and three dimensional trace H is $h = H - 2\phi$. Corresponding to equations (40)–(42) we have

$$\bar{h}_{tt} = \phi + \frac{1}{2}H, \tag{89}$$

$$\bar{h}_{ti} = \beta_i + \partial_i\gamma, \tag{90}$$

$$\bar{h}_{ij} = h_{ij}^{\rm TT} + \frac{1}{3}(3\phi - \frac{1}{2}H)\delta_{ij} + \partial_{(i}\epsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda. \tag{91}$$

The harmonic gauge condition $\eta^{\mu\sigma}\partial_\mu\bar{h}_{\sigma\nu} = 0$ can be expressed as

$$\nabla^2\gamma - \dot{\phi} - \frac{1}{2}\dot{H} = 0, \tag{92}$$

$$\frac{2}{3}\nabla^2\partial_i\lambda + \frac{1}{2}\nabla^2\epsilon_i + \partial_i\phi - \frac{1}{6}\partial_i H - \dot{\beta}_i - \partial_i\dot{\gamma} = 0. \tag{93}$$

The linearized Einstein equation under the harmonic gauge condition becomes

$$\Box\phi = -4\pi(\rho + 3P), \tag{94}$$

$$\Box\beta_i = -16\pi S_i, \tag{95}$$

$$\Box\gamma = -16\pi S, \tag{96}$$

$$\Box H = 24\pi(P - \rho), \tag{97}$$

$$\Box\lambda = -16\pi\sigma, \tag{98}$$

$$\Box\epsilon_i = -16\pi\sigma_i, \tag{99}$$

$$\Box h_{ij}^{\rm TT} = -16\pi T_{ij}^{\rm TT}. \tag{100}$$

Interestingly the harmonic gauge condition equations (92) and (93) can be expressed with these gauge sensitive variables as

$$\Pi = 0, \tag{101}$$

$$\frac{2}{3}\partial_i\Lambda - \Sigma_i = 0. \tag{102}$$

And more within harmonic gauge conditions, equation (99) tells us $\Sigma_i = 0$. Consequently equation (102) becomes

$$\partial_i\Lambda = 0. \tag{103}$$

Because Λ goes to zero when $\vec{r} \to \infty$, the solution of the above equation is

$$\Lambda = 0. \tag{104}$$

In another word, harmonic gauge condition is equivalent to

$$\Pi = 0, \tag{105}$$

$$\Lambda = 0, \tag{106}$$

$$\Sigma_i = 0. \tag{107}$$

Together with the gauge invariant variables Φ, Θ, Ξ_i and $h_{ij}^{\rm TT}$ determined by equations (69), (62) and (63), the perturbation metric components can be determined up to initial data of ϵ_i, $\dot{\epsilon}_i$, λ, $\dot{\lambda}$, γ and $\dot{\gamma}$ under harmonic gauge condition. Alternatively we can check equations (94)–(100) which can determine the solutions up to initial data. But we need to notice that the initial data for $h_{ij}^{\rm TT}$ correspond to initial GW content. The initial data for ϵ_i, λ, and γ correspond to the gauge freedom (propagating freedom) within the harmonic gauge conditions. While the initial data for ϕ, H and β_i should satisfy constrain equations (105)–(107) given by the harmonic gauge conditions.

6. TT gauge condition

Equations (94)–(100) tell us that all the decomposition components ϕ, β_i, γ, H, λ, ϵ_i and $h_{ij}^{\rm TT}$ are harmonic functions for the vacuum case. Equations (62), (63) and (69) tell us that $\Theta = \Xi_i = \Phi = 0$ for the vacuum case. This fact results in relations

$$H = \nabla^2\lambda, \tag{108}$$

$$\beta_i = \frac{1}{2}\dot{\epsilon}_i, \tag{109}$$

$$\phi = \dot{\gamma} - \frac{1}{2}\ddot{\lambda}, \tag{110}$$

for any harmonic gauge condition if only $T_{\mu\nu} = 0$. Starting from any harmonic gauge condition, we can choose

$$C = \frac{1}{2}\lambda, \tag{111}$$

$$B_i = \frac{1}{2}\epsilon_i, \tag{112}$$

$$A = \gamma - \frac{1}{2}\dot{\lambda}, \tag{113}$$

to form ξ_μ and perform a gauge transformation. Since λ, ϵ_i and γ are harmonic functions, this gauge transformation will result in a harmonic gauge. At the mean time λ, ϵ_i and γ become zero in the new gauge condition. Equations (108)–(110) guarantee that H, β_i and ϕ become zero automatically. Consequently $h_{\mu\nu}$ has only $h_{ij}^{\rm TT}$ left. This new gauge condition is nothing but the TT gauge condition.

When matter is presented, equations (94)–(100) tell us that all the decomposition components ϕ, β_i, γ, H, λ, ϵ_i and $h_{ij}^{\rm TT}$ cannot be zero, and cannot even be harmonic functions. So the usual TT gauge condition $h_{\mu\nu}$ equals $h_{ij}^{\rm TT}$ does not exist in general. But we can still use the gauge transformation equations (111)–(113) to set initial data of ϵ_i, λ, and γ to zero. Then the initial data for ϕ, H and β_i are determined by matter through

$$\phi = -\Phi, \tag{114}$$

$$H = 3\Theta, \tag{115}$$

$$\beta_i = \Xi_i. \tag{116}$$

So when matter is presented, we can call the above initial data choice as generalized TT gauge condition.

7. CZ gauge condition

In [15, 16] CZ gauge condition is proposed

$$\partial^i h_{i\mu} - \frac{1}{2}\partial_\mu h^i{}_i = 0. \tag{117}$$

With the decomposition components of perturbation metric, the above condition can be equivalently expressed as

$$\nabla^2\gamma = \frac{1}{2}\dot{H}, \tag{118}$$

$$\nabla^2\epsilon_i = \frac{1}{3}\partial_i H - \frac{4}{3}\nabla^2\partial_i\lambda. \tag{119}$$

Take divergence of equation (119) we get

$$0 = \frac{1}{3}\nabla^2 H - \frac{4}{3}\nabla^2\nabla^2\lambda, \tag{120}$$

$$H = 4\nabla^2\lambda. \tag{121}$$

Consequently equation (119) becomes

$$\nabla^2\epsilon_i = 0, \tag{122}$$

$$\epsilon_i = 0. \tag{123}$$

Under the CZ gauge condition, the linearized Einstein equations reduce to [15]

$$\nabla^2 h_{0\mu} = -16\pi S_{0\mu}, \tag{124}$$

$$\nabla^2(\partial_i h^i{}_j) = -16\pi\partial_j T_{00}, \tag{125}$$

$$\nabla^2(h^i{}_i) = -32\pi T_{00}, \tag{126}$$

$$\Box\hat{h}_{ij} = -16\pi\hat{S}_{ij}, \tag{127}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$ is the trace-reversed stress-energy tensor with T the trace of $T_{\mu\nu}$. The quantities $\hat{S}_{\mu\nu}$ and $\hat{h}_{\mu\nu}$ are defined by the Poisson equations

$$\nabla^2(S_{\mu\nu} - \hat{S}_{\mu\nu}) = \partial_\mu\partial_i S^i{}_\nu + \partial_\nu\partial_i S^i{}_\mu - \partial_\mu\partial_\nu S^i{}_i, \tag{128}$$

$$\nabla^2(h_{\mu\nu} - \hat{h}_{\mu\nu}) = \partial_\mu\partial_k h^k{}_\nu + \partial_\nu\partial_k h^k{}_\mu - \partial_\mu\partial_\nu h^k{}_k. \tag{129}$$

Due to equation (117), we have

$$\partial_\nu\partial^i h_{i\mu} - \frac{1}{2}\partial_\nu\partial_\mu h^i{}_i = 0. \tag{130}$$

Equivalently we have

$$\partial_\mu\partial^i h_{i\nu} - \frac{1}{2}\partial_\mu\partial_\nu h^i{}_i = 0. \tag{131}$$

The summation of equations (130) and (131) gives us

$$\partial_\mu\partial^i h_{i\nu} + \partial_\nu\partial^i h_{i\mu} - \partial_\mu\partial_\nu h^i{}_i = 0. \tag{132}$$

Then equation (129) becomes

$$\nabla^2(h_{\mu\nu} - \hat{h}_{\mu\nu}) = 0, \tag{133}$$

which means

$$\hat{h}_{\mu\nu} = h_{\mu\nu}, \tag{134}$$

under the CZ gauge condition.

Equation (124) can be written as

$$\nabla^2\phi = -4\pi(\rho + 3P), \tag{135}$$

$$\nabla^2\beta_i = -16\pi S_i, \tag{136}$$

$$\nabla^2\gamma = -16\pi S, \tag{137}$$

which determines ϕ, β_i and γ completely.

Equation (126) can be written as

$$\nabla^2 H = -32\pi\rho, \tag{138}$$

which determines H completely. Based on the determined H, equation (121) determined λ completely.

Comparing equations (135)–(138) and equations (62)–(69) we have relations

$$\nabla^2(\phi + \Phi) = \frac{3}{2}\ddot{\Theta}, \tag{139}$$

$$\beta_i = \Xi_i, \tag{140}$$

$$H = 4\Theta. \tag{141}$$

The left $h_{ij}^{\rm TT}$ is controlled by equation (127). The initial data of $h_{ij}^{\rm TT}$ is determined by the initial GW content. That is to say the CZ gauge is determined completely by matter content. There is no more gauge freedom (non-propagating freedom) in the CZ gauge condition.

In the vacuum case, equations (135)–(138) and (121) determine $\phi = \beta_i = \gamma = H = \lambda = \epsilon_i = 0$ which correspond to the usual TT gauge. So the CZ gauge condition can be viewed as another generalization of the TT gauge condition.

It is interesting to ask if the CZ gauge condition is identical to the generalized TT gauge condition defined in the previous section. In order to answer this question we can investigate the behavior of the gauge sensitive variables Π, Λ and Σ_i under the CZ gauge condition. Due to equations (118), (121) and (123) we have

$$\Pi = -\dot{\phi}, \tag{142}$$

$$\Lambda = \frac{3}{2}(\phi - \dot{\gamma}), \tag{143}$$

$$\Sigma_i = \dot{\beta}_i. \tag{144}$$

equations (135) and (137) tell us

$$\nabla^2(\phi - \dot{\gamma}) = -4\pi(\rho + 3P - 4\dot{S}). \tag{145}$$

From the above equation and equations (135) and (136), we can see Π, Λ and Σ_i cannot be zero in general when matter is presented, which means the CZ gauge is harmonic if and only if it is a vacuum. This is to say that the CZ gauge condition is different to the generalized TT gauge condition defined in the previous section when matter is presented.

8. Conclusion and discussion

Along with metric perturbation, the TT gauge condition provides a clear physical picture for GWs. It is well known the TT gauge condition is not valid any more when matter is present. In the current paper we use scalar-vector-tensor decomposition to analyze harmonic gauge conditions systematically and in depth. Based on our analysis it becomes clear why the TT gauge condition cannot be satisfied when matter is present. Accordingly, we generalize the usual TT condition within the harmonic gauge condition framework.

Alternative to the harmonic gauge condition, we also analyzed the CZ gauge condition. We find out that the CZ gauge condition is different to the harmonic in general. The CZ gauge is harmonic when and only when the spacetime is a vacuum.

Mathematically, the harmonic gauge conditions are controlled by the wave equation while the CZ gauge conditions are controlled by the Poisson equation. Since the boundary conditions for the equations are always given by physical conditions, the CZ gauge condition is uniquely determined while the harmonic gauge conditions can be different to each other up to the initial condition for the equation. When the spacetime is a vacuum, the uniquely determined CZ gauge condition is nothing but the TT gauge condition. Consequently the CZ gauge condition can be looked at as the generalized TT gauge condition when matter is present.

Within the TT gauge condition, the perturbation metric components except h_{ij}^{TT} vanishes. This is why people are familiar with the gGW description with h_{ij}^{TT}.

When matters are present, the gauge invariant quantity h_{ij}^{TT} is controlled by a wave equation. The propagating property makes $h_{ij}^{\mathrm{TT}} \sim \frac{1}{r}$ near null infnity. In contrast, the rest gauge invariant quantities Φ, Θ and Ξ_i are all controlled by Poisson equations. The multiple expansion of the source results in

$$\Theta = \frac{2M}{r} + O\left(\frac{1}{r^2}\right),$$
$$\Phi = -\frac{M}{r} + O\left(\frac{1}{r^2}\right),$$
$$\Xi = \frac{4P_i}{r} + O\left(\frac{1}{r^2}\right),$$

where $M \equiv \int \rho d^3x$ is the total mass of the source and $P_i \equiv \int S_i d^3x$ is the total linear momentum of the source. Due to the conservation of mass and linear momentum, if we just keep accuracy as $\frac{1}{r}$ order near null infnity, we can say there is only h_{ij}^{TT} admits wave behavior. Once again it is consistent to our familiar picture that h_{ij}^{TT} describes gravitational waves.

If we use the generalized TT gauge within harmonic gauge conditions when matter is present, all the perturbation metric components are controlled by wave equations. Consequently, all of these components behave as $\sim\frac{1}{r}$ near null infinity.

Differently, if we use the CZ gauge when matter is present, the perturbation metric component h_{ij}^{TT} is controlled by a wave equation and other components are controlled by Poisson equations. Consequently if we just keep accuracy as $\frac{1}{r}$ order near null infinity, the metric behaves as $h_{\mu\nu} = h_{ij}^{\mathrm{TT}} \sim \frac{1}{r}$.

Acknowledgments

This work was supported in part by the National Key Research and Development Program of China Grant No. 2021YFC2203001 and in part by the NSFC (Grant Nos. 11920101003, 12021003 and 12005016). ZC was supported by 'the Fundamental Research Funds for the Central Universities' of Beijing Normal University. XH is supported by the NSF of Hunan province (Grant No. 2023JJ30179).

References

[1] Traveling D 2007 *Traveling at the Speed of Thought: Einstein and the Quest for Gravitational Waves* (Princeton University Press)
[2] Bondi H, Van der Burg M G J and Metzner A W K 1962 Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems *Proc. R. Soc. Lon. Ser. A. Math. Phys. Sci.* **269** 21–52
[3] Sachs R K 1962 Gravitational waves in general relativity. VIII. Waves in asymptotically flat space-time *Proc. R. Soc. Lon. Ser. A. Math. Phys. Sci.* **270** 103–26
[4] Flanagan ÉÉ and Hughes S A 2005 The basics of gravitational wave theory *New J. Phys.* **7** 204
[5] Cai R-G, Yang X-Y and Zhao L 2022 On the energy of gravitational waves *Gen. Relativ. Gravitation* **54** 89
[6] Liang C and Zhou B 2000 *Introductory Differential Geometry and General Relativity I, II, III* (Beijing Normal University Press)
[7] Robert M 1984 *Wald. General Relativity* (Chicago, IL: The University of Chicago Press)
[8] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (Princeton University Press)
[9] Fan Z 2016 Accumulative coupling between magnetized tenuous plasma and gravitational waves *Phys. Rev.* D **94** 024048
[10] Lieu R, Lackeos K and Zhang B 2022 Damping of long wavelength gravitational waves by the intergalactic medium *Class. Quant. Grav.* **39** 075014
[11] Maggiore M 2008 *Gravitational Waves: Volume 1: Theory and experiments* (Oxford University Press)
[12] Li J, Liu F, Pan Y, Wang Z, Cao M, Wang M, Zhang F, Zhang J and Zhu Z-H 2024 Detecting gravitational wave with an interferometric seismometer array on lunar nearside *Sci. China Phys. Mech. Astron.* **67** 219551
[13] Yan H, Chen X, Zhang J, Zhang F, Shao L and Wang M 2024 Constraining the stochastic gravitational wave background using the future lunar seismometers *Phys. Rev.* D **110** 043009
[14] Racz I 2009 *Gravitational radiation and isotropic change of the spatial geometry* arXiv:0912.0128
[15] Chen X-S and Zhu B-C 2011 True radiation gauge for gravity *Phys. Rev.* D **83** 061501
[16] Zhu B-C and Chen X-S 2015 Tensor gauge condition and tensor field decomposition *Mod. Phys. Lett.* A **30** 1550192