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# The time-fractional (2+1)-dimensional Heisenberg ferromagnetic spin chain equation: its Lie symmetries, exact solutions and conservation laws

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**Abstract**

In this paper, the Lie symmetry analysis method is applied to the (2+1)-dimensional time-fractional Heisenberg ferromagnetic spin chain equation. We obtain all the Lie symmetries admitted by the governing equation and reduce the corresponding (2+1)-dimensional fractional partial differential equations with the Riemann–Liouville fractional derivative to (1+1)-dimensional counterparts with the Erdélyi–Kober fractional derivative. Then, we obtain the power series solutions of the reduced equations, prove their convergence and analyze their dynamic behavior graphically. In addition, the conservation laws for all the obtained Lie symmetries are constructed using the new conservation theorem and the generalization of Noether operators.

Keywords: Lie symmetries, fractional partial differential equation, Heisenberg ferromagnetic spin chain equation, power series solutions, conservation laws

## 1. Introduction

As is well known, nonlinear evolution equations play an important role in the field of mathematical physics. The (2+1)-dimensional Heisenberg ferromagnetic spin chain equation is a nonlinear integrable evolution equation, given by [1]

$$\mathrm{i}z_t + \gamma_1 z_{xx} + \gamma_2 z_{yy} + \gamma_3 z_{xy} - \gamma_4 |z|^2 z = 0, \tag{1}$$

where $z(t, x, y)$ signifies the appropriate continuum approximation of the coherent magnetism amplitude to the bosonic operators at spin–lattice sites, and $\gamma_i (i = 1, 2, 3, 4)$ are parameters regarding magnetic coupling coefficients. In recent years, this key nonlinear model has attracted the attention of many scholars and has become their research focus (see [2–11] and references therein). In particular, Osman *et al* [5], Sahoo and Tripathy [8] and Abdel-Aty [11] obtained new exact soliton solutions of the (2+1)-dimensional Heisenberg ferromagnetic spin chain equation using the new extended Fan sub-equation method, the modified Khater method and the $G'/(bG' + G + a)$-expansion method, respectively.

Fractional differential equations (FDEs), due to the nonlocal and memory effects of the fractional derivative [12–15], have been successfully applied in many aspects of science and technology recently. This paper extends the classical (2+1)-dimensional Heisenberg ferromagnetic spin chain equation to the following time-fractional version:

$$\mathrm{i}D_t^{\alpha} z + \gamma_1 z_{xx} + \gamma_2 z_{yy} + \gamma_3 z_{xy} - \gamma_4 |z|^2 z = 0,\quad 0 < \alpha < 1. \tag{2}$$

Many scholars have used different methods to obtain analytical or numerical solutions of this equation in

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different forms of fractional derivative definitions [16–23]. Among them, Bakcerler *et al* [18] used the modified extended tanh-function method to attain some analytic traveling wave solutions of the time-fractional (2+1)-dimensional nonlinear Heisenberg ferromagnetic spin chain equation with a conformable fractional derivative. Rani *et al* [20] applied the generalized Riccati equation mapping method to obtain some new optical soliton solutions for the space time conformable (2+1) dimensional Heisenberg ferromagnetic spin chain equation with Atangana's conformable derivative. Luo [23] investigated the soliton solutions and dynamical analysis of the (2+1)-dimensional Heisenberg ferromagnetic spin chains model with a beta fractional derivative by using the second-order complete discriminant system.

Assuming $z(t, x, y) = u(t, x, y) + iv(t, x, y)$, equation (2) can be rewritten as the following system:

$$\begin{cases} D_t^{\alpha} u = -\gamma_1 v_{xx} - \gamma_2 v_{yy} - \gamma_3 v_{xy} + \gamma_4 v(u^2 + v^2), \\ D_t^{\alpha} v = \gamma_1 u_{xx} + \gamma_2 u_{yy} + \gamma_3 u_{xy} - \gamma_4 u(u^2 + v^2). \end{cases} \tag{3}$$

Compared to numerical methods for FDEs, such as the finite difference method [24] and the finite element method [25], this paper adopts an analytical method: the Lie symmetry analysis method, which was initially advocated by Norwegian mathematician Sophus Lie at the end of the 19th century, and was further developed by Ovsianikov [26] and others [27–31]. The Lie symmetry analysis method can effectively handle various forms of integer-order differential equations [32–34] because it can treat differential equations uniformly, regardless of their forms, transforming some solutions of these equations into other forms of solutions [35]. It was extended to FDEs by Gazizov *et al* [36] in 2007, and then widely applied to various models of FDEs occurring in different areas of applied science (see [37–48]).

It should be noted that this paper adopts the most widely used Riemann–Liouville fractional derivative defined by [12]

$$ {}_aD_t^{\alpha} f(t, x) = D_t^n\, {}_aI_t^{n-\alpha} f(t, x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_a^t \frac{f(s, x)}{(t-s)^{\alpha-n+1}} \mathrm{d}s, & n-1 < \alpha < n, \\ D_t^n f(t, x), & \alpha = n \in \mathbb{N}, \end{cases}$$

for $t > a$. We denote the operator ${}_0D_t^{\alpha}$ as $D_t^{\alpha}$ for simplicity.

This paper is organized as follows. In section 2, we present the Lie symmetry analysis of equation (3). In section 3, we obtain the power series solutions of equation (3) via similarity reduction and prove their convergence. The conserved vectors for all Lie symmetries admitted by equation (3) are constructed in section 4. The conclusion is given in the last section.

## 2. Lie symmetry analysis of equation (3)

Let equation (3) remain invariant under the following one-parameter ($\epsilon$) continuous point transformation group:

$$\begin{aligned} t^* &= t + \epsilon\tau(t, x, y, u, v) + o(\epsilon), \\ x^* &= x + \epsilon\xi(t, x, y, u, v) + o(\epsilon), \\ y^* &= y + \epsilon\theta(t, x, y, u, v) + o(\epsilon), \\ u^* &= u + \epsilon\eta(t, x, y, u, v) + o(\epsilon), \\ v^* &= v + \epsilon\zeta(t, x, y, u, v) + o(\epsilon), \\ D_{t^*}^{\alpha} u^* &= D_t^{\alpha} u + \epsilon\eta^{\alpha,t} + o(\epsilon), \\ D_{t^*}^{\alpha} v^* &= D_t^{\alpha} v + \epsilon\zeta^{\alpha,t} + o(\epsilon), \\ D_{x^*} u^* &= D_x u + \epsilon\eta^x + o(\epsilon), \\ D_{x^*} v^* &= D_x v + \epsilon\zeta^x + o(\epsilon), \\ D_{y^*} u^* &= D_y u + \epsilon\eta^y + o(\epsilon), \\ D_{y^*} v^* &= D_y v + \epsilon\zeta^y + o(\epsilon), \\ D_{x^*}^2 u^* &= D_x^2 u + \epsilon\eta^{xx} + o(\epsilon), \\ D_{x^*}^2 v^* &= D_x^2 v + \epsilon\zeta^{xx} + o(\epsilon), \\ D_{y^*}^2 u^* &= D_y^2 u + \epsilon\eta^{yy} + o(\epsilon), \\ D_{y^*}^2 v^* &= D_y^2 v + \epsilon\zeta^{yy} + o(\epsilon), \\ D_{x^*} D_{y^*} u^* &= D_x D_y u + \epsilon\eta^{xy} + o(\epsilon), \\ D_{x^*} D_{y^*} v^* &= D_x D_y v + \epsilon\zeta^{xy} + o(\epsilon), \end{aligned} \tag{4}$$

where $\tau$, $\xi$, $\theta$, $\eta$ and $\zeta$ are infinitesimals and $\eta^{\alpha,t}$, $\zeta^{\alpha,t}$, $\eta^x$, $\zeta^x$, $\eta^y$, $\zeta^y$, $\eta^{xx}$, $\zeta^{xx}$, $\eta^{yy}$, $\zeta^{yy}$, $\eta^{xy}$, $\zeta^{xy}$ are the corresponding prolongations of orders $\alpha$, 1 and 2, respectively. The corresponding group generator is defined by

$$X = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \theta\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \zeta\frac{\partial}{\partial v}. \tag{5}$$

Then, the corresponding prolongation of $X$ has the form

$$\begin{aligned} \mathrm{Pr}^{(\alpha,2)} X = X &+ \eta^{\alpha,t}\frac{\partial}{\partial u_t^{\alpha}} + \zeta^{\alpha,t}\frac{\partial}{\partial v_t^{\alpha}} + \eta^{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \zeta^{xx}\frac{\partial}{\partial v_{xx}} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \zeta^{yy}\frac{\partial}{\partial v_{yy}} \\ &+ \eta^{xy}\frac{\partial}{\partial u_{xy}} + \zeta^{xy}\frac{\partial}{\partial v_{xy}}, \end{aligned} \tag{6}$$

where

$$\eta^x = D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi) - u_y D_x(\theta), \tag{7}$$

$$\zeta^x = D_x(\zeta) - v_t D_x(\tau) - v_x D_x(\xi) - v_y D_x(\theta), \tag{8}$$

$$\eta^y = D_y(\eta) - u_t D_y(\tau) - u_x D_y(\xi) - u_y D_y(\theta), \tag{9}$$

$$\zeta^y = D_y(\zeta) - v_t D_y(\tau) - v_x D_y(\xi) - v_y D_y(\theta), \tag{10}$$

$$\eta^{xx} = D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\theta), \tag{11}$$

$$\zeta^{xx} = D_x(\zeta^x) - v_{xt} D_x(\tau) - v_{xx} D_x(\xi) - v_{xy} D_x(\theta), \tag{12}$$

$$\eta^{yy} = D_y(\eta^y) - u_{yt} D_y(\tau) - u_{yx} D_y(\xi) - u_{yy} D_y(\theta), \tag{13}$$

$$\zeta^{yy} = D_y(\zeta^y) - v_{yt} D_y(\tau) - v_{yx} D_y(\xi) - v_{yy} D_y(\theta), \tag{14}$$

$$\eta^{xy} = D_x(\eta^y) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi) - u_{xy} D_x(\theta), \tag{15}$$

$$\zeta^{xy} = D_x(\zeta^y) - v_{xt} D_x(\tau) - v_{xx} D_x(\xi) - v_{xy} D_x(\theta), \tag{16}$$

$$\eta^{\alpha,t}=\frac{\partial^\alpha\eta}{\partial t^\alpha}+(\eta_u-\alpha D_t(\tau))\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\eta_u}{\partial t^\alpha}+(\eta_v\frac{\partial^\alpha v}{\partial t^\alpha}-v\frac{\partial^\alpha\eta_v}{\partial t^\alpha})+\sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(u)+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\eta_v}{\partial t^n}D_t^{\alpha-n}(v)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\theta)D_t^{\alpha-n}(u_y)+\mu_1+\mu_2, \tag{17}$$

with

$$\mu_1=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\times\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k},$$
$$\mu_2=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\times\frac{t^{n-\alpha}(-v)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m v^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial v^k},$$

and

$$\zeta^{\alpha,t}=\frac{\partial^\alpha\zeta}{\partial t^\alpha}+(\zeta_v-\alpha D_t(\tau))\frac{\partial^\alpha v}{\partial t^\alpha}-v\frac{\partial^\alpha\zeta_v}{\partial t^\alpha}+(\zeta_u\frac{\partial^\alpha u}{\partial t^\alpha}-u\frac{\partial^\alpha\zeta_u}{\partial t^\alpha})+\sum_{n=1}^{\infty}[\binom{\alpha}{n}\frac{\partial^n\zeta_v}{\partial t^n}-\binom{\alpha}{n+1}D_t^{n+1}(\tau)]D_t^{\alpha-n}(v)+\sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n\zeta_u}{\partial t^n}D_t^{\alpha-n}(u)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)D_t^{\alpha-n}(v_x)-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\theta)D_t^{\alpha-n}(v_y)+\mu_3+\mu_4, \tag{18}$$

with

$$\mu_3=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\times\frac{t^{n-\alpha}(-u)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m u^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial u^k},$$
$$\mu_4=\sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\times\frac{t^{n-\alpha}(-v)^r}{k!\Gamma(n+1-\alpha)}\frac{\partial^m v^{k-r}}{\partial t^m}\frac{\partial^{n-m+k}\zeta}{\partial t^{n-m}\partial v^k},$$

where $D_t$, $D_x$ and $D_y$ are the total derivatives with respect to $t$, $x$ and $y$, respectively.

**Remark 1.** The infinitesimal transformations, equation (4), should conserve the structure of the Riemann–Liouville fractional derivative operator, of which the lower limit in the integral is fixed. It means that $t=0$ should be invariant with respect to such transformations, i.e.

$$\tau(t,x,y,u,v)|_{t=0}=0. \tag{19}$$

**Remark 2.** From the expressions of $\mu_i(i=1,2,3,4)$, if the infinitesimals $\eta$ and $\zeta$ are linear with respect to the variables $u$ and $v$, then $\mu_i=0$, that is,

$$\frac{\partial^2\eta}{\partial u^2}=\frac{\partial^2\eta}{\partial v^2}=\frac{\partial^2\zeta}{\partial u^2}=\frac{\partial^2\zeta}{\partial v^2}=0. \tag{20}$$

The invariance criterion of equation (3) under equation (4) is

$$\begin{cases}\mathrm{Pr}^{(\alpha,2)}X(D_t^\alpha u+\gamma_1 v_{xx}+\gamma_2 v_{yy}+\gamma_3 v_{xy}-\gamma_4 v(u^2+v^2))|_0=0,\\ \mathrm{Pr}^{(\alpha,2)}X(D_t^\alpha v-\gamma_1 u_{xx}-\gamma_2 u_{yy}-\gamma_3 u_{xy}+\gamma_4 u(u^2+v^2))|_0=0,\end{cases} \tag{21}$$

which can be rewritten as

$$\begin{cases}(\eta^{\alpha,t}+\gamma_1\zeta^{xx}+\gamma_2\zeta^{yy}+\gamma_3\zeta^{xy}-2\gamma_4 uv\eta-\gamma_4(u^2+3v^2)\zeta)|_0=0,\\ (\zeta^{\alpha,t}-\gamma_1\eta^{xx}-\gamma_2\eta^{yy}-\gamma_3\eta^{xy}+2\gamma_4 uv\zeta+\gamma_4(3u^2+v^2)\eta)|_0=0.\end{cases} \tag{22}$$

Putting $\eta^{\alpha,t}$, $\zeta^{\alpha,t}$, $\eta^{xx}$, $\zeta^{xx}$, $\eta^{yy}$, $\zeta^{yy}$, $\eta^{xy}$, $\zeta^{xy}$ into equation (22) and solving the obtained determining equations, we can get the infinitesimals as follows:

$$\tau=c_1 t,\qquad \xi=\frac{\alpha}{2}c_1 x+c_2,\qquad \theta=\frac{\alpha}{2}c_1 y+c_3,$$
$$\eta=-\frac{\alpha}{2}c_1 u+c_4 v,\qquad \zeta=-\frac{\alpha}{2}c_1 v-c_4 u, \tag{23}$$

where $c_1$, $c_2$, $c_3$ and $c_4$ are arbitrary constants. Therefore, equation (3) admitted the four-dimension Lie algebra spanned by

$$X_1=t\frac{\partial}{\partial t}+\frac{\alpha}{2}x\frac{\partial}{\partial x}+\frac{\alpha}{2}y\frac{\partial}{\partial y}-\frac{\alpha}{2}u\frac{\partial}{\partial u}-\frac{\alpha}{2}v\frac{\partial}{\partial v},$$
$$X_2=v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},\qquad X_3=\frac{\partial}{\partial x},\qquad X_4=\frac{\partial}{\partial y}. \tag{24}$$

## 3. Similarity reductions and exact solutions of equation (3)

In this section, the aimed equations (3) can be reduced to solvable (1+1)-dimensional fractional partial differential equations with the left-hand Erdélyi–Kober fractional derivative.

**Case 1**: $X_1=t\frac{\partial}{\partial t}+\frac{\alpha}{2}x\frac{\partial}{\partial x}+\frac{\alpha}{2}y\frac{\partial}{\partial y}-\frac{\alpha}{2}u\frac{\partial}{\partial u}-\frac{\alpha}{2}v\frac{\partial}{\partial v}$.

From the characteristic equation corresponding to the group generator $X_1$, i.e.

$$\frac{\mathrm{d}t}{t}=\frac{\mathrm{d}x}{\frac{\alpha}{2}x}=\frac{\mathrm{d}y}{\frac{\alpha}{2}y}=\frac{\mathrm{d}u}{-\frac{\alpha}{2}u}=\frac{\mathrm{d}v}{-\frac{\alpha}{2}v}, \tag{25}$$

we obtain the similarity variables $xt^{-\frac{\alpha}{2}}$, $yt^{-\frac{\alpha}{2}}$, $ut^{\frac{\alpha}{2}}$ and $vt^{\frac{\alpha}{2}}$. Therefore, we get the invariant solutions of equation (3) as follows:

$$u(t,x,y)=t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2),\qquad v(t,x,y)=t^{-\frac{\alpha}{2}}g(\omega_1,\omega_2),\tag{26}$$

with $\omega_1=xt^{-\frac{\alpha}{2}}$, $\omega_2=yt^{-\frac{\alpha}{2}}$.

**Theorem 3.1.** *The similarity transformations* $u(t,x,y)=t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2)$, $v(t,x,y)=t^{-\frac{\alpha}{2}}g(\omega_1,\omega_2)$ *with the similarity variable* $\omega_1=xt^{-\frac{\alpha}{2}}$, $\omega_2=yt^{-\frac{\alpha}{2}}$ *reduce equation (3) to the (1+1)-dimensional fractional partial differential equations given by*

$$\begin{cases}(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2)=-\gamma_1 g_{\omega_1\omega_1}-\gamma_2 g_{\omega_2\omega_2}-\gamma_3 g_{\omega_1\omega_2}+\gamma_4 g(f^2+g^2),\\ (\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}g)(\omega_1,\omega_2)=\gamma_1 f_{\omega_1\omega_1}+\gamma_2 f_{\omega_2\omega_2}+\gamma_3 f_{\omega_1\omega_2}-\gamma_4 f(f^2+g^2),\end{cases}\tag{27}$$

*where* $(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2})$ *is the left-hand Erdélyi–Kober fractional differential operator defined by*

$$\begin{aligned}&(\mathcal{P}^{\iota,\kappa}_{\delta_1,\delta_2}\psi)(\omega_1,\omega_2):=\prod_{j=0}^{m-1}\left(\iota+j-\frac{1}{\delta_1}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1}-\frac{1}{\delta_2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2}\right)\\&\quad\times(\mathcal{K}^{\iota+\kappa,m-\kappa}_{\delta_1,\delta_2}\psi)(\omega_1,\omega_2),\quad \kappa>0,\\&m=\begin{cases}[\kappa]+1, & \kappa\notin\mathbb{N},\\ \kappa, & \kappa\in\mathbb{N},\end{cases}\end{aligned}\tag{28}$$

*where*

$$(\mathcal{K}^{\iota,\kappa}_{\delta_1,\delta_2}\psi)(\omega_1,\omega_2):=\begin{cases}\frac{1}{\Gamma(\kappa)}\int_1^\infty(s-1)^{\kappa-1}s^{-(\iota+\kappa)}\psi(\omega_1 s^{\frac{1}{\delta_1}},\omega_2 s^{\frac{1}{\delta_2}})\mathrm{d}s, & \kappa>0,\\ \psi(\omega_1,\omega_2), & \kappa=0.\end{cases}\tag{29}$$

**Proof.** For $0<\alpha<1$, the Riemann–Liouville time-fractional derivative of $u(t,x,y)$ can be obtained as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial^\alpha}{\partial t^\alpha}(t^{-\frac{\alpha}{2}}f(\omega_1,\omega_2))=\frac{\partial}{\partial t}\left[\frac{1}{\Gamma(1-\alpha)}\right.\\&\quad\left.\times\int_0^t(t-s)^{-\alpha}s^{\frac{-\alpha}{2}}f(xs^{-\frac{\alpha}{2}},ys^{-\frac{\alpha}{2}})\mathrm{d}s\right].\end{aligned}$$

Assuming $r=\frac{t}{s}$, we have

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=\frac{\partial}{\partial t}\left[\frac{t^{1-\frac{3\alpha}{2}}}{\Gamma(1-\alpha)}\int_1^\infty(r-1)^{-\alpha}r^{\frac{3\alpha}{2}-2}f(\omega_1 r^{\frac{\alpha}{2}},\omega_2 r^{\frac{\alpha}{2}})\mathrm{d}r\right]\\&=\frac{\partial}{\partial t}[t^{1-\frac{3\alpha}{2}}(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2)].\end{aligned}$$

Because of $\omega_1=xt^{-\frac{\alpha}{2}}$ and $\omega_2=yt^{-\frac{\alpha}{2}}$, the following relation holds:

$$\begin{aligned}t\frac{\partial}{\partial t}\psi(\omega_1,\omega_2)&=t(-\frac{\alpha}{2}xt^{-\frac{\alpha}{2}-1}\psi_{\omega_1}-\frac{\alpha}{2}xt^{-\frac{\alpha}{2}-1}\psi_{\omega_2})\\&=-\frac{\alpha}{2}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1}\psi-\frac{\alpha}{2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2}\psi.\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha}&=t^{-\frac{3\alpha}{2}}\left[\left(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega_1\frac{\mathrm{d}}{\mathrm{d}\omega_1}-\frac{\alpha}{2}\omega_2\frac{\mathrm{d}}{\mathrm{d}\omega_2}\right)\right.\\&\quad\left.\times(\mathcal{K}^{1-\frac{\alpha}{2},1-\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega)\right]=t^{-\frac{3\alpha}{2}}(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}f)(\omega_1,\omega_2).\end{aligned}$$

Similarly,

$$\frac{\partial^\alpha v}{\partial t^\alpha}=t^{-\frac{3\alpha}{2}}(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}g)(\omega_1,\omega_2).$$

Meanwhile,

$$\begin{aligned}&-\gamma_1 v_{xx}-\gamma_2 v_{yy}-\gamma_3 v_{xy}+\gamma_4 v(u^2+v^2)=t^{-\frac{3\alpha}{2}}\\&\quad\times(-\gamma_1 g_{\omega_1\omega_1}-\gamma_2 g_{\omega_2\omega_2}-\gamma_3 g_{\omega_1\omega_2}+\gamma_4 g(f^2+g^2)),\\&\gamma_1 u_{xx}+\gamma_2 u_{yy}+\gamma_3 u_{xy}-\gamma_4 u(u^2+v^2)=t^{-\frac{3\alpha}{2}}\\&\quad\times(\gamma_1 f_{\omega_1\omega_1}+\gamma_2 f_{\omega_2\omega_2}+\gamma_3 f_{\omega_1\omega_2}-\gamma_4 f(f^2+g^2)).\end{aligned}$$

This completes the proof. □

Next, we use the power series method to derive the power series solutions of equation (27). Assuming

$$f(\omega_1,\omega_2)=\sum_{k=0}^\infty a_k\omega^k,\qquad g(\omega_1,\omega_2)=\sum_{k=0}^\infty b_k\omega^k,\qquad \omega=C_1\omega_1+C_2\omega_2,\tag{30}$$

with constants $a_k$ and $b_k$ to be known later, we can get

$$\begin{aligned}f'(\omega)&=\sum_{k=0}^\infty(k+1)a_{k+1}\omega^k,\\ f''(\omega)&=\sum_{k=0}^\infty(k+2)(k+1)a_{k+2}\omega^k,\\ g'(\omega)&=\sum_{k=0}^\infty(k+1)b_{k+1}\omega^k,\\ g''(\omega)&=\sum_{k=0}^\infty(k+2)(k+1)b_{k+2}\omega^k,\end{aligned}\tag{31}$$

and

$$\begin{aligned}&(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha}}f)(\omega)=(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\frac{1}{\Gamma(1-\alpha)}\\&\times\int_1^\infty(s-1)^{-\alpha}s^{\frac{3\alpha}{2}-2}f(\omega s^{\frac{\alpha}{2}})\mathrm{d}s)\\&=(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\frac{1}{\Gamma(1-\alpha)}\int_1^\infty(s-1)^{-\alpha}s^{\frac{3\alpha}{2}-2}\\&\times\sum_{k=0}^\infty(a_k\omega^k s^{\frac{k\alpha}{2}})\mathrm{d}s)\\&=(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\sum_{k=0}^\infty a_k\omega^k\frac{1}{\Gamma(1-\alpha)}\\&\times\int_1^\infty(s-1)^{-\alpha}s^{\frac{k+3}{2}\alpha-2}\mathrm{d}s)\\&=(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})\left(\sum_{k=0}^\infty a_k\omega^k\frac{B\left(1-\frac{(k+1)\alpha}{2},1-\alpha\right)}{\Gamma(1-\alpha)}\right)\\&=(1-\frac{3\alpha}{2}-\frac{\alpha}{2}\omega\frac{\mathrm{d}}{\mathrm{d}\omega})(\sum_{k=0}^\infty\frac{\Gamma\left(1-\frac{(k+1)\alpha}{2}\right)}{\Gamma\left(2-\frac{(k+3)\alpha}{2}\right)}a_k\omega^k)\\&=\sum_{k=0}^\infty\frac{\Gamma\left(1-\frac{(k+1)\alpha}{2}\right)}{\Gamma\left(1-\frac{(k+3)\alpha}{2}\right)}a_k\omega^k.\end{aligned}\tag{32}$$

Similarly,

$$(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha},\frac{2}{\alpha}}g)(\omega_1,\omega_2)=(\mathcal{P}^{1-\frac{3\alpha}{2},\alpha}_{\frac{2}{\alpha}}g)(\omega) =\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}b_k\omega^k. \tag{33}$$

Substituting equations (30)–(33) into equation (27) arrives at the following system:

$$\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k\omega^k=-(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)\times\sum_{k=0}^{\infty}(k+2)(k+1)b_{k+2}\omega^k+\gamma_4\Big(\sum_{l+m+n=k}a_la_mb_n+\sum_{l+m+n=k}b_lb_mb_n\Big)\omega^k,$$

$$\sum_{k=0}^{\infty}\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}b_k\omega^k=(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)\times\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\omega^k-\gamma_4\Big(\sum_{l+m+n=k}a_la_ma_n+\sum_{l+m+n=k}a_lb_mb_n\Big)\omega^k.$$

Next, we equate the coefficients of different powers of $\omega$ to obtain the explicit expressions of $a_k$ and $b_k$. For $k\geqslant 0$, we have

$$\begin{cases} a_{k+2}=\dfrac{1}{(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)(k+2)(k+1)}\left[\dfrac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}b_k+\gamma_4\Big(\sum\limits_{l+m+n=k}a_la_ma_n+\sum\limits_{l+m+n=k}a_lb_mb_n\Big)\right],\\ b_{k+2}=\dfrac{-1}{(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)(k+2)(k+1)}\left[\dfrac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k-\gamma_4\Big(\sum\limits_{l+m+n=k}a_la_mb_n+\sum\limits_{l+m+n=k}b_lb_mb_n\Big)\right], \end{cases} \tag{34}$$

with the initial conditions $a_0=f(0)$, $b_0=g(0)$, $a_1=f'(0)$ and $b_1=g'(0)$. Therefore, the power series solutions of equation (3) are

$$\begin{aligned} u(t,x,y)&=a_0t^{-\frac{\alpha}{2}}+a_1(C_1x+C_2y)t^{-\alpha}+\sum_{k=0}^{\infty}\frac{(C_1x+C_2y)^{k+2}t^{-\frac{(k+3)\alpha}{2}}}{(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)(k+2)(k+1)}\\ &\times\left[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}b_k+\gamma_4\Big(\sum_{l+m+n=k}a_la_ma_n+\sum_{l+m+n=k}a_lb_mb_n\Big)\right],\\ v(t,x,y)&=b_0t^{-\frac{\alpha}{2}}+b_1(C_1x+C_2y)t^{-\alpha}-\sum_{k=0}^{\infty}\frac{(C_1x+C_2y)^{k+2}t^{-\frac{(k+3)\alpha}{2}}}{(\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2)(k+2)(k+1)}\\ &\times\left[\frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}a_k-\gamma_4\Big(\sum_{l+m+n=k}a_la_mb_n+\sum_{l+m+n=k}b_lb_mb_n\Big)\right]. \end{aligned} \tag{35}$$

**Theorem 3.2.** *The power series solutions, equation (35), are convergent in a neighborhood of the point (0, $|a_0|$, $|b_0|$).*

**Proof.** From equation (34), according to the generalized absolute value triangle inequality, i.e. $|\sum_{i=1}^{n}d_i|\leqslant\sum_{i=1}^{n}|d_i|$, we can obtain

$$\begin{cases} |a_{k+2}|\leqslant\dfrac{1}{|\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2|(k+2)(k+1)}\left[\left|\dfrac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}\right||b_k|+|\gamma_4|\Big(\sum\limits_{l+m+n=k}|a_l||a_m||a_n|+\sum\limits_{l+m+n=k}|a_l||b_m||b_n|\Big)\right],\\ |b_{k+2}|\leqslant\dfrac{1}{|\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2|(k+2)(k+1)}\left[\left|\dfrac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})}\right||a_k|+|\gamma_4|\Big(\sum\limits_{l+m+n=k}|a_l||a_m||b_n|+\sum\limits_{l+m+n=k}|b_l||b_m||b_n|\Big)\right]. \end{cases} \tag{36}$$

From the properties of the gamma function, it can be found that $\frac{|\Gamma(1-\frac{(k+1)\alpha}{2})|}{|\Gamma(1-\frac{(k+3)\alpha}{2})|}\leqslant 1$ for arbitrary $k$. Thus, equation (36) can be written as

$$|a_{k+2}|\leqslant M\left(\left|b_k\right|+\sum_{l+m+n=k}|a_l||a_m||a_n|+\sum_{l+m+n=k}|a_l||b_m||b_n|\right),\quad k\geqslant 0, \tag{37}$$

$$|b_{k+2}|\leqslant M\left(\left|a_k\right|+\sum_{l+m+n=k}|a_l||a_m||b_n|+\sum_{l+m+n=k}|b_l||b_m||b_n|\right),\quad k\geqslant 0, \tag{38}$$

where $M=\max\{\frac{1}{|\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2|(k+2)(k+1)},\frac{|\gamma_4|}{|\gamma_1C_1^2+\gamma_2C_2^2+\gamma_3C_1C_2|(k+2)(k+1)}\}$.

Consider another power series

$$P(\omega)=\sum_{k=0}^{\infty}p_k\omega^k,\qquad Q(\omega)=\sum_{k=0}^{\infty}q_k\omega^k, \tag{39}$$

where $p_0=|a_0|$, $p_1=|a_1|$, $q_0=|b_0|$, $q_1=|b_1|$ and

$$p_{k+2}=M\left(q_k+\sum_{l+m+n=k}p_lp_mp_n+\sum_{l+m+n=k}p_lq_mq_n\right),\quad k\geqslant 0, \tag{40}$$

$$q_{k+2}=M\left(p_k+\sum_{l+m+n=k}p_lp_mq_n+\sum_{l+m+n=k}q_lq_mq_n\right),\quad k\geqslant 0. \tag{41}$$

Therefore, it is easily seen that $|a_k|\leqslant p_k$ and $|b_k|\leqslant q_k$ for $k=0, 1, 2, \ldots$, that is, the power series, equation (39), are the majorant series of equation (30). We next show that the power series, equation (39), are convergent. By simple calculation,

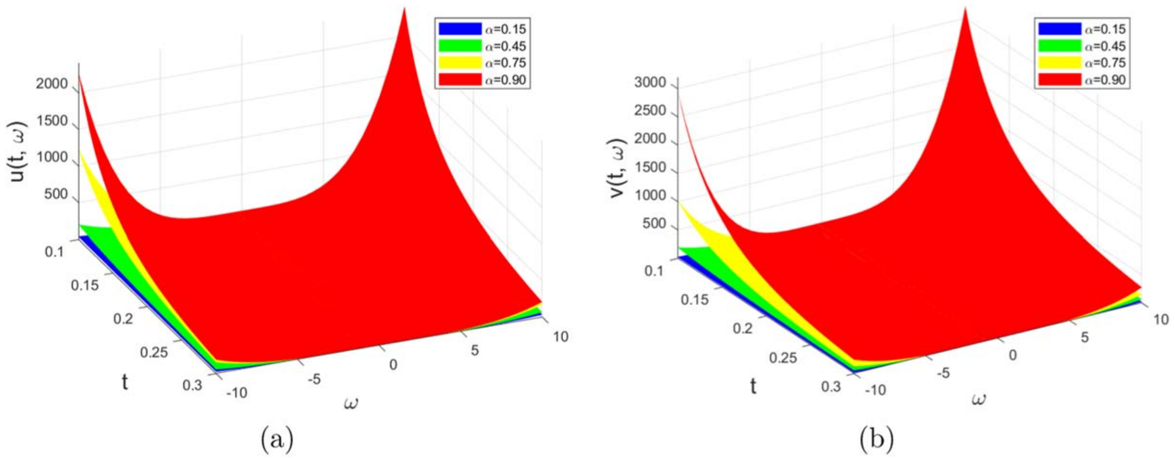


**Figure 1.** Dynamical profiles of the truncated power series solution, equation (35).

we can get

$$P(\omega) = p_0 + p_1\omega + M(Q(\omega) + P^3(\omega) + P(\omega)Q^2(\omega))\omega^2, \tag{42}$$

$$Q(\omega) = q_0 + q_1\omega + M(P(\omega) + Q^3(\omega) + Q(\omega)P^2(\omega))\omega^2. \tag{43}$$

Consider the following implicit function with respect to the independent variable $\omega$:

$$\begin{aligned} F(\omega, P, Q) = P - P_0 - P_1\omega \\ -M(Q(\omega) + P^3(\omega) + P(\omega)Q^2(\omega))\omega^2, \end{aligned} \tag{44}$$

$$\begin{aligned} G(\omega, P, Q) = Q - q_0 - q_1\omega \\ -M(P(\omega) + Q^3(\omega) + Q(\omega)P^2(\omega))\omega^2, \end{aligned} \tag{45}$$

which are analytic in a neighborhood of $(0, p_0, q_0)$, and $F(0, p_0, q_0) = 0$, $G(0, p_0, q_0) = 0$, $\frac{\partial(F,G)}{\partial(P,Q)}|_{(0,p_0,q_0)} = 1 \neq 0$. Therefore, via the implicit function theorem, the power series, equation (39), are analytic in the neighborhood of the point $(0, p_0, q_0)$. It implies that the power series solutions, equation (30), are convergent in a neighborhood of the point $(0, |a_0|, |b_0|)$. □

In figure 1, we illustrate the physical features of the power series solutions, equation (35), for different chosen parameters. Assuming $a_0 = b_0 = a_1 = b_1 = 1, C_1 = C_2 = \frac{1}{2}$ and $\gamma_i = 1 (i = 1, 2, 3, 4)$, these graphs show that as the fractional order changes slightly, the solutions, equation (35), exhibit subtle differences for the initial values and constants $C_1$, $C_2$ and $\gamma_i$. Therefore, the fractional form is more suitable for adjusting the evolution of the model in practical situations based on changes in order $\alpha$.

**Case 2**: $X_3 = \frac{\partial}{\partial x}$.

From the characteristic equation corresponding to the group generator $X_3$, i.e.

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0} = \frac{dv}{0}, \tag{46}$$

we obtain the similarity variables $t$, $y$, $u$ and $v$. Therefore, we get the invariant solutions of equation (3) as follows:

$$u = f(t, y), \qquad v = g(t, y). \tag{47}$$

Substituting equation (47) into equation (3), we have the following reduced equations:

$$\begin{cases} D_t^\alpha f = -\gamma_2 g_{yy} + \gamma_4 g(f^2 + g^2), \\ D_t^\alpha g = \gamma_2 f_{yy} - \gamma_4 f(f^2 + g^2), \end{cases} \tag{48}$$

which is the time-fractional cubic Schrödinger equation [40] with $\gamma_2 = 1$ and $\gamma_4 = -\gamma$. Similar to [40], we can further reduce equation (48) to fractional ordinary differential equations using the Lie symmetry analysis method again, and obtain the following convergent power series solutions:

$$\begin{aligned} u(t, y) &= a_0 t^{-\frac{\alpha}{2}} + a_1 y t^{-\alpha} + \sum_{k=0}^{\infty} \frac{y^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{\gamma_2 (k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1 - \frac{(k+1)\alpha}{2})}{\Gamma(1 - \frac{(k+3)\alpha}{2})} b_k + \gamma_4 \Big( \sum_{l+m+n=k} a_l a_m a_n + \sum_{l+m+n=k} a_l b_m b_n \Big) \right], \\ v(t, y) &= b_0 t^{-\frac{\alpha}{2}} + b_1 y t^{-\alpha} - \sum_{k=0}^{\infty} \frac{y^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{\gamma_2 (k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1 - \frac{(k+1)\alpha}{2})}{\Gamma(1 - \frac{(k+3)\alpha}{2})} a_k - \gamma_4 \Big( \sum_{l+m+n=k} a_l a_m b_n + \sum_{l+m+n=k} b_l b_m b_n \Big) \right], \end{aligned} \tag{49}$$

which is consistent with equation (35) when $C_1 = 0$ and $C_2 = 1$.

**Case 3**: $X_4 = \frac{\partial}{\partial y}$.

From the characteristic equation corresponding to the group generator $X_4$, i.e.

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0} = \frac{dv}{0}, \tag{50}$$

we obtain the similarity variables $t$, $x$, $u$ and $v$. Therefore, we get the invariant solutions of equation (3) as follows:

$$u = f(t, x), \qquad v = g(t, x). \tag{51}$$

Substituting equation (51) into equation (3), we have the following reduced equations:

$$\begin{cases} D_t^\alpha f = -\gamma_1 g_{xx} + \gamma_4 g(f^2 + g^2), \\ D_t^\alpha g = \gamma_1 f_{xx} - \gamma_4 f(f^2 + g^2), \end{cases} \tag{52}$$

which is the time-fractional cubic Schrödinger equation [40] with $\gamma_1 = 1$ and $\gamma_4 = -\gamma$. Similar to case 2, we can obtain the following convergent power series solutions:

$$\begin{aligned} u(t,x) &= a_0 t^{-\frac{\alpha}{2}} + a_1 x t^{-\alpha} + \sum_{k=0}^{\infty} \frac{x^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{\gamma_1 (k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})} b_k + \gamma_4 (\sum_{l+m+n=k} a_l a_m a_n + \sum_{l+m+n=k} a_l b_m b_n) \right], \\ v(t,x) &= b_0 t^{-\frac{\alpha}{2}} + b_1 x t^{-\alpha} - \sum_{k=0}^{\infty} \frac{x^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{\gamma_1 (k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})} a_k - \gamma_4 (\sum_{l+m+n=k} a_l a_m b_n + \sum_{l+m+n=k} b_l b_m b_n) \right], \end{aligned} \tag{53}$$

which is consistent with equation (35) when $C_1 = 1$ and $C_2 = 0$.

**Case 4**: $X_3 - X_4 = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$.

The characteristic equation corresponding to the group generator $X_3 - X_4$ is

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}y}{-1} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}v}{0}, \tag{54}$$

from which, we obtain the similarity variables $t$, $x + y$, $u$ and $v$. Therefore, we get the invariant solutions of equation (3)

$$\begin{aligned} u(t,x,y) &= f(\omega_1, \omega_2), \quad v(t,x,y) = g(\omega_1, \omega_2), \\ \omega_1 &= t, \quad \omega_2 = x + y, \end{aligned} \tag{55}$$

and the reduced equations

$$\begin{cases} D_{\omega_1}^{\alpha} f = -(\gamma_1 + \gamma_2 + \gamma_3) g_{\omega_2\omega_2} + \gamma_4 g (f^2 + g^2), \\ D_{\omega_1}^{\alpha} g = (\gamma_1 + \gamma_2 + \gamma_3) f_{\omega_2\omega_2} - \gamma_4 f (f^2 + g^2). \end{cases} \tag{56}$$

Similar to cases 2 and 3, we can apply the Lie symmetry analysis method again to obtain the following convergent power series solutions:

$$\begin{aligned} u(t,x,y) &= a_0 t^{-\frac{\alpha}{2}} + a_1 (x+y) t^{-\alpha} + \sum_{k=0}^{\infty} \frac{(x+y)^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{(\gamma_1+\gamma_2+\gamma_3)(k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})} b_k + \gamma_4 (\sum_{l+m+n=k} a_l a_m a_n + \sum_{l+m+n=k} a_l b_m b_n) \right], \\ v(t,x,y) &= b_0 t^{-\frac{\alpha}{2}} + b_1 (x+y) t^{-\alpha} - \sum_{k=0}^{\infty} \frac{(x+y)^{k+2} t^{-\frac{(k+3)\alpha}{2}}}{(\gamma_1+\gamma_2+\gamma_3)(k+2)(k+1)} \\ &\times \left[ \frac{\Gamma(1-\frac{(k+1)\alpha}{2})}{\Gamma(1-\frac{(k+3)\alpha}{2})} a_k - \gamma_4 (\sum_{l+m+n=k} a_l a_m b_n + \sum_{l+m+n=k} b_l b_m b_n) \right], \end{aligned} \tag{57}$$

which is consistent with equation (35) when $C_1 = 1$ and $C_2 = 1$.

## 4. Conservation laws of equation (3)

We denote equation (3) as

$$\begin{cases} F_1 = D_t^{\alpha} u + \gamma_1 v_{xx} + \gamma_2 v_{yy} + \gamma_3 v_{xy} - \gamma_4 v (u^2 + v^2) = 0, \\ F_2 = D_t^{\alpha} v - \gamma_1 u_{xx} - \gamma_2 u_{yy} - \gamma_3 u_{xy} + \gamma_4 u (u^2 + v^2) = 0, \end{cases} \tag{58}$$

of which the formal Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= p(t,x,y) F_1 + q(t,x,y) F_2 \\ &= p(t,x,y)(D_t^{\alpha} u + \gamma_1 v_{xx} + \gamma_2 v_{yy} + \gamma_3 v_{xy} \\ &\quad - \gamma_4 v (u^2 + v^2)) \\ &\quad + q(t,x,y)(D_t^{\alpha} v - \gamma_1 u_{xx} - \gamma_2 u_{yy} \\ &\quad - \gamma_3 u_{xy} + \gamma_4 u (u^2 + v^2)), \end{aligned} \tag{59}$$

where $p(t, x, y)$ and $q(t, x, y)$ are new dependent variables. The Euler–Lagrange operators are

$$\begin{aligned} \frac{\delta}{\delta u} &= \frac{\partial}{\partial u} + (D_t^{\alpha})^* \frac{\partial}{\partial (D_t^{\alpha} u)} \\ &\quad + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u_{i_1 \cdots i_s}}, \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{\delta}{\delta v} &= \frac{\partial}{\partial v} + (D_t^{\alpha})^* \frac{\partial}{\partial (D_t^{\alpha} v)} \\ &\quad + \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial v_{i_1 \cdots i_s}}, \end{aligned} \tag{61}$$

where $(D_t^{\alpha})^*$ is the adjoint operator of $D_t^{\alpha}$. It is defined by the right-side of the Caputo fractional derivative, i.e.

$$(D_t^{\alpha})^* f(t,x) \equiv {}_t^c D_T^{\alpha} f(t,x) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_t^T \frac{1}{(t-s)^{\alpha-n+1}} \frac{\partial^n}{\partial s^n} f(s,x) \mathrm{d}s, & n-1 < \alpha < n, \\ D_t^n f(t,x), & \alpha = n \in \mathbb{N}. \end{cases}$$

The adjoint equations to equation (58) are given by

$$\begin{cases} F_1^* = \frac{\delta \mathcal{L}}{\delta u} = (D_t^{\alpha})^* p + \gamma_1 q_{xx} + \gamma_2 q_{yy} + \gamma_3 q_{xy} - \gamma_4 (2puv - 3qu^2 - qv^2) = 0, \\ F_2^* = \frac{\delta \mathcal{L}}{\delta v} = (D_t^{\alpha})^* q - \gamma_1 p_{xx} - \gamma_2 p_{yy} - \gamma_3 p_{xy} + \gamma_4 (2quv - 3pv^2 - pu^2) = 0. \end{cases} \tag{62}$$

Next, we will use the above adjoint equations and the new conservation theorem [49] to construct conservation laws of equation (3). From the classical definition of the conservation laws, a vector $C = (C^t, C^x, C^y)$ is called the conserved vector for the governing equation if it satisfies the conservation equation $[D_t C^t + D_x C^x + D_y C^y]_{F_1, F_2 = 0} = 0$. We can obtain the components of the conserved vector using the generalization of the Noether operators [50]. From the fundamental operator identity, i.e.

$$\begin{aligned} &\mathrm{Pr}^{(\alpha,2)} X + D_t \tau \cdot \mathcal{I} + D_x \xi \cdot \mathcal{I} + D_y \theta \cdot \mathcal{I} \\ &= W^u \cdot \frac{\delta}{\delta u} + W^v \cdot \frac{\delta}{\delta v} + D_t \mathcal{N}^t + D_x \mathcal{N}^x + D_y \mathcal{N}^y, \end{aligned} \tag{63}$$

where $\mathrm{Pr}^{(\alpha,2)} X$ is mentioned in equation (6), $\mathcal{I}$ is the identity operator, and $W^u = \eta - \tau u_t - \xi u_x - \theta u_y$, $W^v = \zeta - \tau v_t - \xi v_x - \theta u_y$ are the characteristics for the group generator $X$. We can get the Noether operators as follows:

$$\mathcal{N}^t = \tau\mathcal{I} + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W^u)D_t^k\frac{\partial}{\partial(D_t^\alpha u)} - (-1)^n J\left(W^u, D_t^n\frac{\partial}{\partial(D_t^\alpha u)}\right) + \sum_{k=0}^{n-1}(-1)^k D_t^{\alpha-1-k}(W^v)D_t^k\frac{\partial}{\partial(D_t^\alpha v)} - (-1)^n J\left(W^v, D_t^n\frac{\partial}{\partial(D_t^\alpha v)}\right), \quad (64)$$

$$\mathcal{N}^x = \xi\mathcal{I} - W^u\left(D_x\frac{\partial}{\partial u_{xx}} + D_y\frac{\partial}{\partial u_{xy}}\right) - W^v\left(D_x\frac{\partial}{\partial v_{xx}} + D_y\frac{\partial}{\partial v_{xy}}\right) + D_x(W^u)\frac{\partial}{\partial u_{xx}} + D_y(W^u)\frac{\partial}{\partial u_{xy}} + D_x(W^v)\frac{\partial}{\partial v_{xx}} + D_y(W^v)\frac{\partial}{\partial v_{xy}}, \quad (65)$$

$$\mathcal{N}^y = \theta\mathcal{I} - W^u\left(D_y\frac{\partial}{\partial u_{yy}} + D_x\frac{\partial}{\partial u_{yx}}\right) - W^v\left(D_y\frac{\partial}{\partial v_{yy}} + D_x\frac{\partial}{\partial v_{yx}}\right) + D_y(W^u)\frac{\partial}{\partial u_{yy}} + D_x(W^u)\frac{\partial}{\partial u_{yx}} + D_y(W^v)\frac{\partial}{\partial v_{yy}} + D_x(W^v)\frac{\partial}{\partial v_{yx}}, \quad (66)$$

where $n = [\alpha] + 1$, and $J$ is given by

$$J(f, g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau, x)g(\theta, x)}{(\theta-\tau)^{\alpha+1-n}}\mathrm{d}\theta\mathrm{d}\tau. \quad (67)$$

The components of the conserved vector are defined by

$$C^t = \mathcal{N}^t\mathcal{L}, \qquad C^x = \mathcal{N}^x\mathcal{L}, \qquad C^y = \mathcal{N}^y\mathcal{L}. \quad (68)$$

**Case 1**: $X_1 = t\frac{\partial}{\partial t} + \frac{\alpha}{2}x\frac{\partial}{\partial x} + \frac{\alpha}{2}y\frac{\partial}{\partial y} - \frac{\alpha}{2}u\frac{\partial}{\partial u} - \frac{\alpha}{2}v\frac{\partial}{\partial v}$.
The characteristics of $X_1$ are

$$W^u = -\frac{\alpha}{2}u - tu_t - \frac{\alpha}{2}xu_x - \frac{\alpha}{2}yu_y, \quad W^v = -\frac{\alpha}{2}v - tv_t - \frac{\alpha}{2}xv_x - \frac{\alpha}{2}yv_y. \quad (69)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -pD_t^{\alpha-1}\left(\frac{\alpha}{2}u + tu_t + \frac{\alpha}{2}xu_x + \frac{\alpha}{2}yu_y\right) - J\left(\frac{\alpha}{2}u + tu_t + \frac{\alpha}{2}xu_x + \frac{\alpha}{2}yu_y, p_t\right) - qD_t^{\alpha-1}\left(\frac{\alpha}{2}v + tv_t + \frac{\alpha}{2}xv_x + \frac{\alpha}{2}yv_y\right) - J\left(\frac{\alpha}{2}v + tv_t + \frac{\alpha}{2}xv_x + \frac{\alpha}{2}yv_y, q_t\right), \quad (70)$$

$$C^x = -(\gamma_1 q_x + \gamma_3 q_y)\left(\frac{\alpha}{2}u + tu_t + \frac{\alpha}{2}xu_x + \frac{\alpha}{2}yu_y\right) + (\gamma_1 p_x + \gamma_3 p_y)\left(\frac{\alpha}{2}v + tv_t + \frac{\alpha}{2}xv_x + \frac{\alpha}{2}yv_y\right) + \gamma_1 q\left(\alpha u_x + tu_{tx} + \frac{\alpha}{2}u_{xx} + \frac{\alpha}{2}yu_{xy}\right) + \gamma_3 q\left(\alpha u_y + tu_{ty} + \frac{\alpha}{2}xu_{xy} + \frac{\alpha}{2}yu_{yy}\right) - \gamma_1 p\left(\alpha v_x + tv_{tx} + \frac{\alpha}{2}v_{xx} + \frac{\alpha}{2}yv_{xy}\right) - \gamma_3 p\left(\alpha v_y + tv_{ty} + \frac{\alpha}{2}xv_{xy} + \frac{\alpha}{2}yv_{yy}\right), \quad (71)$$

$$C^y = -(\gamma_1 q_y + \gamma_3 q_x)\left(\frac{\alpha}{2}u + tu_t + \frac{\alpha}{2}xu_x + \frac{\alpha}{2}yu_y\right) + (\gamma_1 p_y + \gamma_3 p_x)\left(\frac{\alpha}{2}v + tv_t + \frac{\alpha}{2}xv_x + \frac{\alpha}{2}yv_y\right) + \gamma_1 q\left(\alpha u_y + tu_{ty} + \frac{\alpha}{2}xu_{xy} + \frac{\alpha}{2}yu_{yy}\right) + \gamma_3 q\left(\alpha u_x + tu_{tx} + \frac{\alpha}{2}u_{xx} + \frac{\alpha}{2}yu_{xy}\right) - \gamma_1 p\left(\alpha v_y + tv_{ty} + \frac{\alpha}{2}xv_{xy} + \frac{\alpha}{2}yv_{yy}\right) - \gamma_3 p\left(\alpha v_x + tv_{tx} + \frac{\alpha}{2}v_{xx} + \frac{\alpha}{2}yv_{xy}\right). \quad (72)$$

**Case 2**: $X_2 = v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}$.
The characteristics of $X_2$ are

$$W^u = v, \qquad W^v = -u. \quad (73)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = pD_t^{\alpha-1}v + J(v, p_t) - qD_t^{\alpha-1}u - J(u, q_t), \quad (74)$$

$$C^x = v(\gamma_1 q_x + \gamma_3 q_y) + u(\gamma_1 p_x + \gamma_3 p_y) - \gamma_1 qv_x - \gamma_3 qv_y - \gamma_1 pu_x - \gamma_3 pu_y, \quad (75)$$

$$C^y = v(\gamma_1 q_y + \gamma_3 q_x) + u(\gamma_1 p_x + \gamma_3 p_y) - \gamma_1 qv_y - \gamma_3 qv_x - \gamma_1 pu_y - \gamma_3 pu_x. \quad (76)$$

**Case 3**: $X_3 = \frac{\partial}{\partial x}$.
The characteristics of $X_3$ are

$$W^u = -u_x, \qquad W^v = -v_x. \quad (77)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -pD_t^{\alpha-1}u_x - J(u_x, p_t) - qD_t^{\alpha-1}v_x - J(v_x, q_t), \quad (78)$$

$$C^x = -u_x(\gamma_1 q_x + \gamma_3 q_y) + v_x(\gamma_1 p_x + \gamma_3 p_y) + \gamma_1 qu_{xx} + \gamma_3 qu_{xy} - \gamma_1 pv_{xx} - \gamma_3 pv_{xy}, \quad (79)$$

$$C^y = -u_x(\gamma_1 q_y + \gamma_3 q_x) + v_x(\gamma_1 p_x + \gamma_3 p_y) + \gamma_1 qu_{xy} + \gamma_3 qu_{xx} - \gamma_1 pv_{xy} - \gamma_3 pv_{xx}. \quad (80)$$

**Case 4**: $X_4 = \frac{\partial}{\partial y}$.
The characteristics of $X_4$ are

$$W^u = -u_y, \qquad W^v = -v_y. \quad (81)$$

Therefore, for $0 < \alpha < 1$,

$$C^t = -pD_t^{\alpha-1}u_y - J(u_y, p_t) - qD_t^{\alpha-1}v_y - J(v_y, q_t), \quad (82)$$

$$C^x = -u_y(\gamma_1 q_x + \gamma_3 q_y) + v_y(\gamma_1 p_x + \gamma_3 p_y) + \gamma_1 qu_{xy} + \gamma_3 qu_{yy} - \gamma_1 pv_{xy} - \gamma_3 pv_{yy}, \quad (83)$$

$$C^y = -u_y(\gamma_1 q_y + \gamma_3 q_x) + v_y(\gamma_1 p_x + \gamma_3 p_y) + \gamma_1 qu_{yy} + \gamma_3 qu_{xy} - \gamma_1 pv_{yy} - \gamma_3 pv_{xy}. \quad (84)$$

## 5. Conclusion

This paper shows that the Lie symmetry analysis method is effective in solving nonlinear fractional evolution equations. We obtained all the Lie symmetries of the (2+1)-dimensional time-fractional Heisenberg ferromagnetic spin chain equation and used them to reduce the equation, thereby getting some convergent power series solutions. In addition, the new conservation theorem and the generalization of the Noether operators are developed to construct the conservation laws for the equation studied. Inspired by this, our next step is to apply the Lie symmetry analysis method to more nonlinear fractional evolution equations that have significant value in mathematics and physics.

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## Confict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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