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Some regularity properties of scattering data for the derivative nonlinear Schrödinger equation

Weifang Weng[1] **and Zhenya Yan**[2,3]

[1] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
[2] Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
[3] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

E-mail: zyyan@mmrc.iss.ac.cn

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Abstract

In this paper, we present some properties of scattering data for the derivative nonlinear Schrödinger equation in $H^s(\mathbb{R})(s \geqslant \frac{1}{2})$ starting from the Lax pair. We show that the reciprocal of the transmission coefficient can be expressed as the sum of some iterative integrals, and its logarithm can be written as the sum of some connected iterative integrals. We provide the asymptotic properties of the first few iterative integrals of the reciprocal of the transmission coefficient. Moreover, we provide some regularity properties of the reciprocal of the transmission coefficient related to scattering data in $H^s(\mathbb{R})$.

Keywords: derivative nonlinear Schrödinger equation, modified Zakharov–Shabat spectral problem, scattering data, inverse scattering transform, asymptotics

1. Introduction

The derivative nonlinear Schrödinger (DNLS) equation

$$\mathrm{i}q_t + q_{xx} \pm \mathrm{i}(|q|^2 q)_x = 0, \quad q = q(x, t), \quad x \in \mathbb{R}, \tag{1.1}$$

appears in many fields, such as the wave propagation of circular polarized nonlinear Alfvén waves in plasmas [1–5] and the weak nonlinear electromagnetic waves in ferromagnetic [6], antiferromagnetic [7], or dielectric [8] systems under external magnetic fields. Kaup and Newell [9] showed that equation (1.1) is completely integrable and has the following modified Zakharov–Shabat eigenvalue problem (Lax pair) [9]:

$$\begin{aligned} \psi_x &= U(x, t, \lambda)\psi, \\ \psi_t &= V(x, t, \lambda)\psi, \end{aligned} \tag{1.2}$$

with

$$\begin{gathered} U(x, t, \lambda) = -\mathrm{i}\sigma_3(\lambda^2 + \mathrm{i}\lambda Q), \\ V(x, t, \lambda) = \begin{pmatrix} -\mathrm{i}(2\lambda^4 \mp \lambda^2|q|^2) & 2\lambda^3 q \mp \lambda|q|^2 q + \mathrm{i}\lambda q_x \\ \mp 2\lambda^3 q^* + \lambda|q|^2 q^* \pm \mathrm{i}\lambda q_x^* & \mathrm{i}(2\lambda^4 \mp \lambda^2|q|^2) \end{pmatrix}, \\ Q = \begin{pmatrix} 0 & q(x, t) \\ \pm q^*(x, t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{gathered} \tag{1.3}$$

where $\psi(x, t; \lambda) = (\psi_1(x, t; \lambda), \psi_2(x, t; \lambda))^{\mathrm{T}}$ stands for the eigenvector, and $\lambda \in \mathbb{C}$ is the spectral parameter. Without loss of generality, in the following, we can take the $+$ sign (because the case of the $+$ sign can be transformed into the $-$ sign by means of $x \to -x$). Moreover, equation (1.1) also possesses an infinite number of conservation laws, for example,

$$\begin{cases} H_0 = \displaystyle\int_{\mathbb{R}} |q|^2 \mathrm{d}x, \\ H_1 = \mathrm{Im} \displaystyle\int_{\mathbb{R}} q^* q_x \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}} |q|^4 \mathrm{d}x, \\ H_2 = \displaystyle\int_{\mathbb{R}} |q_x|^2 - \frac{3}{2} \mathrm{Im}(|q|^2 q q_x^*) + \frac{1}{2}|q|^6 \mathrm{d}x, \end{cases} \tag{1.4}$$

where the star denotes the complex conjugate.

Note that the DNLS equation (1.1) is L^2-norm being invariant under the following scaling:

$$q(x, t) \to z^{\frac{1}{2}} q(zx, z^2 t), \quad z > 0. \tag{1.5}$$

Inverse scattering transform (IST) was investigated for the DNLS equation (1.1) with zero boundary conditions (ZBCs) to obtain its one-soliton solution [9] and N-soliton solutions [10]. IST was also considered for the DNLS equation (1.1) with nonzero boundary conditions (NZBCs) [11–14]. Explicit double-pole solutions were found for the DNLS equation (1.1) with ZBCs/NZBCs by IST with matrix Riemann–Hilbert problems [15]. The long-time leading-order asymptotic behavior was analyzed for the DNLS equation (1.1) [16, 17] via the Deift–Zhou method [18].

The local well-posedness for equation (1.1) was proved in the energy space $H^1(\mathbb{R})$ [19, 20]. By using mass and energy conservation laws, Hayashi and Ozawa [21, 22] proved that equation (1.1) was global well-posedness in the energy space $H^1(\mathbb{R})$ under the following condition:

$$\|q(x,0)\|_{L^2} < \sqrt{2\pi}. \tag{1.6}$$

Condition (1.6) was then improved by Wu [23, 24]. Moreover, Guo and Wu [25] proved that equation (1.1) is globally well-posed in the energy space $H^{\frac{1}{2}}(\mathbb{R})$. The global existence with solitons of the DNLS equation (1.1) was investigated [26, 27]. Recently, the global existence of the DNLS equation (1.1) was studied by IST [28–31]. Moreover, Bahouri and Perelman [32] showed that the DNLS equation (1.1) was globally well-posed for the general Cauchy condition in $H^{1/2}(\mathbb{R})$ and that the $H^{1/2}$-norm of the solutions still remained globally bounded in time.

Recently, Koch and Tataru [33] studied the (de)focusing cubic nonlinear Schrödinger (NLS) equation:

$$\mathrm{i}q_t + q_{xx} \pm 2|q|^2 q = 0, \quad q = q(x,t), \tag{1.7}$$

provided a modified conservation function for the NLS equation (1.7), and showed that there existed a conserved energy that is equivalent to the H^s-norm of the solution for each $s > 1/2$ with the aid of IST. Koch and Liao [34] studied the one-dimensional Gross–Pitaevskii (GP) equation:

$$\mathrm{i}q_t + q_{xx} = 2q(|q|^2 - 1), \quad q = q(x,t), \tag{1.8}$$

and proved the global-in-time well-posedness of the GP equation (1.8) in the energy space. Recently, they [35] further constructed a family of conserved energies for the one-dimensional GP equation (1.8) but in the low regularity case.

In this paper, motivated by the idea for the NLS equation [33], we present some properties of scattering data for the DNLS equation (1.1) with initial data $q(x) \in H^s(\mathbb{R}) (s \geqslant \frac{1}{2})$ in the energy space, which is a complete metric space equipped with a newly introduced metric and the energy norm describing the $H^s(\mathbb{R})$ regularities of the solutions. We provide some regularity properties of transmission coefficient related to scattering data in $H^s(\mathbb{R})$.

The main conclusion of this paper is the following theorem.

Theorem 1.1. *Let $q(x) \in L^2(\mathbb{R})$ and $s_{11}(\lambda)$ be the reciprocal of the transmission coefficient of the modified Zakharov–Shabat spectral problem (1.2) associated with the DNLS equation (1.1). Then one has the following properties:*

(1) $\ln s_{11}(\lambda) = \sum_{j=1}^{\infty} b_{2j}(\lambda)$ with

$$b_{2j}(\lambda) = (-1)^j \int_{\Sigma_j} \lambda^{2j} \prod_{k=1}^{j} \times q(y_k) q^*(x_k) \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} \mathrm{d}x_1 \mathrm{d}y_1 \cdots \mathrm{d}x_j \mathrm{d}y_j, \tag{1.9}$$

being formal linear combinations of connected integrals, where Σ_j is any domain that obeys the condition $x_k < y_k$ for all k ($k \leqslant j$), $\lambda \in \mathbb{C}$ is a spectral parameter, and the star denotes the complex conjugate.

(2) The following estimates hold:

$$\ln s_{11}(\lambda) \sim -\frac{\mathrm{i}}{2}\|q(x)\|^2_{L^2(\mathbb{R})}, \quad \lambda \to \infty, \tag{1.10}$$

and

$$s_{11}(\lambda) \sim \mathrm{e}^{-\frac{\mathrm{i}}{2}\|q(x)\|^2_{L^2(\mathbb{R})}}, \quad \lambda \to \infty. \tag{1.11}$$

The remainder of this paper is arranged as follows. In section 2, we introduce some basic properties about the IST of the DNLS equation (1.1) with $q(x) \in \mathcal{S}(\mathbb{R})$ (where $\mathcal{S}(\mathbb{R})$ represents Schwartz space). In section 3, we give the formal expansions of the reciprocal of the transmission coefficient, $s_{11}(\lambda)$, and its logarithmic function $\ln s_{11}(\lambda)$. In section 4, we construct iterative integrals $B_j(\lambda)$ arising from a formal expansion of $\ln s_{11}(\lambda)$ into a Hopf algebra, such that we can proof the first conclusion of theorem 1.1. In section 5, we give the boundary estimate for the leading term in both $s_{11}(\lambda) - 1$ and $\ln s_{11}(\lambda)$. In section 6, we recall the function spaces U^p, V^p, and DU^p, and give the boundary estimate for the iterative integrals s_{2j} of $s_{11}(\lambda) - 1$ with $q(x) \in H^s(\mathbb{R})$. In section 7, we have the asymptotic expressions for $b_4(\lambda)$ and $b_6(\lambda)$. In section 8, we give the expansions for the iterative integrals $b_{2j}(\lambda)$ with $q(x) \in H^s(\mathbb{R})$.

2. Preliminaries: Jost solutions and scattering data

In this section, we review some basic properties about the IST of the DNLS equation (1.1) with $q(x) \in \mathcal{S}(\mathbb{R})$ [26, 28–32, 36, 37]. For the Lax pair (1.2) of the DNLS equation (1.1), it is easy to see that the compatibility condition, $U_t - V_x + [U, V] = 0$ (i.e. zero-curvature equation), of the Lax pair (1.2) just generates the DNLS equation (1.1). The zero-curvature equation has the advantage that it is well defined even without decay assumptions on the initial data, because it is all formal calculations.

For the given $q(x) \in \mathcal{S}(\mathbb{R})$, i.e. the potential $q(x) \to 0$ as $x \to \pm\infty$, one has the asymptotics of Jost solutions (eigenfunctions) of Lax pair (1.2) as

$$\psi_x = \begin{pmatrix} -\mathrm{i}\lambda^2 & 0 \\ 0 & \mathrm{i}\lambda^2 \end{pmatrix} \psi, \quad x \to \pm\infty.$$

Therefore, it is natural to introduce the eigenfunction defined by the following boundary conditions:

$$\begin{aligned}
&\phi(x,\lambda)\sim \mathrm{e}^{-\mathrm{i}\lambda^2 x}\begin{pmatrix}1\\0\end{pmatrix}, \quad x\to-\infty,\\
&\overline{\phi}(x,\lambda)\sim \mathrm{e}^{\mathrm{i}\lambda^2 x}\begin{pmatrix}0\\1\end{pmatrix}, \quad x\to-\infty,\\
&\varphi(x,\lambda)\sim \mathrm{e}^{\mathrm{i}\lambda^2 x}\begin{pmatrix}0\\1\end{pmatrix}, \quad x\to+\infty,\\
&\overline{\varphi}(x,\lambda)\sim \mathrm{e}^{-\mathrm{i}\lambda^2 x}\begin{pmatrix}1\\0\end{pmatrix}, \quad x\to+\infty.
\end{aligned} \tag{2.1}$$

The functions $\phi(x,\lambda)$, $\overline{\phi}(x,\lambda)$, $\varphi(x,\lambda)$, and $\overline{\varphi}(x,\lambda)$ are called Jost solutions. The Jost solution $\phi(x,\lambda)$, $\varphi(x,\lambda)$ can be analytically extended to $L_+=\{\lambda\in\mathbb{C}|\,\mathrm{Im}\,\lambda^2>0\}$, C^∞ up to the boundary. The Jost solution $\overline{\phi}(x,\lambda)$, $\overline{\varphi}(x,\lambda)$ can be analytically extended to $L_-=\{\lambda\in\mathbb{C}|\,\mathrm{Im}\,\lambda^2<0\}$, C^∞ up to the boundary.

For $\lambda\in\mathbb{R}\cup i\mathbb{R}$, because the Jost solutions solve both parts of the modified Zakharov–Shabat eigenvalue problem (1.2), there is a constant scattering matrix $S(\lambda)=(s_{ij})_{2\times 2}$ independent of x, t, which holds the following relation:

$$(\phi(x,\lambda),\overline{\phi}(x,\lambda))=(\overline{\varphi}(x,\lambda),\varphi(x,\lambda))\begin{pmatrix}s_{11}(\lambda)&s_{12}(\lambda)\\ s_{21}(\lambda)&s_{22}(\lambda)\end{pmatrix}. \tag{2.2}$$

The functions $s_{11}(\lambda)^{-1}$, $s_{22}(\lambda)^{-1}$ are called transmission coefficients, and $\frac{s_{21}(\lambda)}{s_{11}(\lambda)}$, $\frac{s_{12}(\lambda)}{s_{22}(\lambda)}$ are called reflection coefficients. The Jost solutions have the following symmetry:

$$\begin{aligned}
&\phi(x,\lambda)=\sigma_3\phi(x,-\lambda),\qquad \varphi(x,\lambda)=-\sigma_3\varphi(x,-\lambda),\\
&\phi(x,\lambda)=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\overline{\phi}^*(x,\lambda^*),\qquad \varphi(x,\lambda)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\overline{\varphi}^*(x,\lambda^*).
\end{aligned} \tag{2.3}$$

According to equations (2.2) and (2.3), the symmetry of the scattering data can be obtained. For $\lambda\in\mathbb{R}\cup\mathrm{i}\mathbb{R}$,

$$\begin{aligned}
&s_{11}(\lambda)=s_{11}(-\lambda),\qquad s_{11}(\lambda)=s_{22}^*(\lambda^*),\\
&s_{21}(\lambda)=-s_{21}(-\lambda),\qquad s_{12}(\lambda)=-s_{21}^*(\lambda^*).
\end{aligned}$$

Because $\det(S(\lambda))=1$, the following equation is obtained:

$$\begin{aligned}
&|s_{11}(\lambda)|^2+|s_{21}(\lambda)|^2=1,\qquad \lambda\in\mathbb{R},\\
&|s_{11}(\lambda)|^2-|s_{21}(\lambda)|^2=1,\qquad \lambda\in\mathrm{i}\mathbb{R}.
\end{aligned}$$

It follows from equation (2.2) that the scattering data have the Wronskian representations:

$$\begin{aligned}
&s_{11}(\lambda)=\det(\phi(x,\lambda),\varphi(x,\lambda)),\\
&s_{12}(\lambda)=\det(\overline{\phi}(x,\lambda),\varphi(x,\lambda)),\\
&s_{21}(\lambda)=-\det(\phi(x,\lambda),\overline{\varphi}(x,\lambda)),\\
&s_{22}(\lambda)=-\det(\overline{\phi}(x,\lambda),\overline{\varphi}(x,\lambda)).
\end{aligned} \tag{2.4}$$

Denoting $\overline{s_{11}}(\lambda)=\mathrm{e}^{\frac{\mathrm{i}}{2}\|q(x)\|^2_{L^2(\mathbb{R})}}s_{11}(\sqrt{\lambda})$, it has the following asymptotic expansion:

$$\ln\overline{s_{11}}(\lambda)=\sum_{j=1}^{\infty}D_k(q)\lambda^{-j},\qquad \lambda\to\infty, \tag{2.5}$$

where $D_k(q)$ is a polynomial with respect to $q(x)$ and its derivatives. For example,

$$D_1(q)=\frac{\mathrm{i}}{4}H_1,\qquad D_2(q)=-\frac{\mathrm{i}}{8}H_2. \tag{2.6}$$

Furthermore, one can show that $|\overline{s_{11}}(\lambda)|^2\in 1+\mathcal{S}(\mathbb{R})$, and

$$|\overline{s_{11}}(\lambda)|\geqslant 1,\ \ \lambda<0,\qquad |\overline{s_{11}}(\lambda)|\leqslant 1,\ \ \lambda>0.$$

The scattering data satisfy the following time evolution equation:

$$\frac{\partial s_{11}(\lambda)}{\partial t}=0,\qquad \frac{\partial s_{21}(\lambda)}{\partial t}=-4\mathrm{i}\lambda^4 s_{21}(\lambda). \tag{2.7}$$

Although the assumption $q(x)\in\mathcal{S}(\mathbb{R})$ can be weakened [26, 28–31], one needs at least $q(x)\in L^1(\mathbb{R})$ to define the scattering data. A way to overcome this difficulty and to keep a trace of the complete integrability for H^s solutions, for $\lambda\in L^+$, that remains well defined via equation (2.4) for $q(x)\in L^2(\mathbb{R})$ [32].

3. Formal expansion of the reciprocal of the transmission coefficient

Lemma 3.1. *[32] For any initial data $q_0(x)\in H^{\frac{1}{2}}(\mathbb{R})$, the Cauchy problem of the DNLS equation (1.1):*

$$\begin{cases} iq_t+q_{xx}\pm\mathrm{i}(|q|^2q)_x=0,\\ q(x,0)=q_0(x)\in H^s(\mathbb{R}),\end{cases} \tag{3.1}$$

is globally well-posed, and the corresponding solution q(t) *satisfies*

$$\sup\nolimits_{t\in\mathbb{R}}\|q(t)\|_{H^{\frac{1}{2}}(\mathbb{R})}<+\infty. \tag{3.2}$$

Moreover, if the initial datum is in $H^s(\mathbb{R})$ for some $s>\frac{1}{2}$, then the H^s*-norm of the solution of the Cauchy problem (3.1) remains globally bounded in time as well.*

The scattering transform associated with the DNLS equation (1.1) is defined via the first equation of (1.2), which can be written as a linear system:

$$\begin{cases} \dfrac{\mathrm{d}\psi_1}{\mathrm{d}x}=-\mathrm{i}\lambda^2\psi_1+\mathrm{i}\lambda q\psi_2,\\[2mm] \dfrac{\mathrm{d}\psi_2}{\mathrm{d}x}=\mathrm{i}\lambda^2\psi_2+\mathrm{i}\lambda q^*\psi_1,\end{cases} \tag{3.3}$$

Then, based on the asymptotic of $q_0(x)$, one can seek for the Jost solutions ψ_l with asymptotics:

$$\begin{cases} \psi_l \sim \begin{pmatrix} e^{-i\lambda^2 x} \\ 0 \end{pmatrix}, & x \to -\infty, \\ \psi_l \sim \begin{pmatrix} s_{11}(\lambda)e^{-i\lambda^2 x} \\ s_{21}(\lambda)e^{i\lambda^2 x} \end{pmatrix}, & x \to +\infty, \end{cases} \tag{3.4}$$

where $s_{11}^{-1}(\lambda)$ is the transmission coefficient, and $\frac{s_{21}}{s_{11}}(\lambda)$ is the reflection coefficient.

Theorem 3.1. *The reciprocal of the transmission coefficient,* $s_{11}(\lambda)$*, has a formal expansion as follows:*

$$s_{11}(\lambda) = 1 + \sum_{j=1}^{\infty} s_{2j}(\lambda), \tag{3.5}$$

where $s_{2j}(\lambda)$ *are multilinear integral forms with homogeneous of degree* $2j$ *in the potential* q *and its conjugate* q^**, that is,*

$$s_{2j}(\lambda) = (-1)^j \int_{x_1<y_1<\cdots<x_j<y_j} \lambda^{2j} \prod_{k=1}^{j} q(y_k)q^*(x_k)e^{2i\lambda^2(y_k-x_k)}dx_1dy_1\cdots dx_jdy_j. \tag{3.6}$$

Proof. We solve system (3.3) by using the iterative method to prove this theorem. First, we choose the initial value iteration function as

$$\psi_l^{(0)}(x) = \begin{pmatrix} e^{-i\lambda^2 x} \\ 0 \end{pmatrix}, \tag{3.7}$$

where the upper-right corner represents the number of iterations.

Substituting equation (3.7) into the second one of equation (3.3) yields

$$\psi^{(1)}_{l2,x} = i\lambda^2\psi^{(1)}_{l2} - \lambda q^* e^{-i\lambda^2 x}. \tag{3.8}$$

By solving ordinary differential equation (3.8), we have

$$\psi_l^{(1)}(x) = \begin{pmatrix} e^{-i\lambda^2 x} \\ -e^{i\lambda^2 x}\int_{-\infty}^{x} \lambda q^*(x_1)e^{-2i\lambda^2 x_1}dx_1 \end{pmatrix}. \tag{3.9}$$

Substituting the second component of equation (3.9) into the first one of system (3.3) yields

$$\psi^{(2)}_{l1,x} = -i\lambda^2\psi^{(2)}_{l1} + \lambda q\psi^{(1)}_{l2}, \tag{3.10}$$

and solving ordinary differential equation (3.10) results in

$$\psi^{(2)}_{l1,x} = e^{-i\lambda^2 x} - \int_{-\infty}^{x} \lambda q(y_1)e^{-i\lambda^2(x-y_1)}\int_{-\infty}^{y_1} \lambda q^*(x_1)e^{i\lambda^2(y_1-2x_1)}dx_1dy_1. \tag{3.11}$$

Simplifying equation (3.11) and using the second component of equation (3.9) yield

$$\psi_l^{(2)}(x) = \begin{pmatrix} e^{-i\lambda^2 x}\left(1 - \int_{x_1<y_1<x} \lambda^2 q(y_1)q^*(x_1)e^{2i\lambda^2(y_1-x_1)}dx_1dy_1\right) \\ -e^{i\lambda^2 x}\int_{-\infty}^{x} \lambda q^*(x_1)e^{-2i\lambda^2 x_1}dx_1 \end{pmatrix}. \tag{3.12}$$

By repeating the above process, we can obtain the results of the third and fourth iterations as follows:

$$\psi_l^{(3)}(x)=\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda^2 x}\left(1-\int_{x_1<y_1<x}\lambda^2 q(y_1)q^*(x_1)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1-x_1)}\mathrm{d}x_1\mathrm{d}y_1\right)\\ -\mathrm{e}^{\mathrm{i}\lambda^2 x}\int_{-\infty}^{x}\lambda q^*(x_2)\mathrm{e}^{-2\mathrm{i}\lambda^2 x_2}\left(1-\int_{x_1<y_1<x_2}\lambda^2 q(y_1)q^*(x_1)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1-x_1)}\mathrm{d}x_1\mathrm{d}y_1\right)\mathrm{d}x_2 \end{pmatrix}, \tag{3.13}$$

and

$$\psi_l^{(4)}(x)=\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda^2 x}\left(1+\int_{-\infty}^{x}\lambda q(y_2)\mathrm{e}^{\mathrm{i}\lambda^2 y_2}\psi_{l2}^{(3)}(y_2)\mathrm{d}y_2\right)\\ -\mathrm{e}^{\mathrm{i}\lambda^2 x}\int_{-\infty}^{x}\lambda q^*(x_2)\mathrm{e}^{-2\mathrm{i}\lambda^2 x_2}\left(1-\int_{x_1<y_1<x_2}\lambda^2 q(y_1)q^*(x_1)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1-x_1)}\mathrm{d}x_1\mathrm{d}y_1\right)\mathrm{d}x_2 \end{pmatrix}. \tag{3.14}$$

Simplifying equation (3.14) yields

$$\begin{aligned}\psi_{l1}^{(4)}(x)=\mathrm{e}^{-\mathrm{i}\lambda^2 x}\Bigg(1&-\int_{x_2<y_2<x}\lambda^2 q(y_2)q^*(x_2)\mathrm{e}^{2\mathrm{i}\lambda^2(y_2-x_2)}\\ &+\int_{x_1<y_1<x_2<y_2<x}\lambda^4 q(y_1)q^*(x_1)q(y_2)q^*(x_2)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2-x_1-x_2)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\Bigg).\end{aligned} \tag{3.15}$$

According to equations (3.3) and (3.4), we iterate the above procedure and obtain the following expression:

$$\begin{aligned} s_2(\lambda)&=-\int_{x_1<y_1}\lambda^2 q(y_1)q^*(x_1)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1-x_1)}\mathrm{d}x_1\mathrm{d}y_1,\\ s_4(\lambda)&=\int_{x_1<y_1<x_2<y_2}\lambda^4 q(y_1)q^*(x_1)q(y_2)q^*(x_2)\\ &\quad\times\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2-x_1-x_2)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2,\\ &\ \ \vdots\\ s_{2j}(\lambda)&=(-1)^j\int_{x_1<y_1<\cdots<x_j<y_j}\lambda^{2j}\prod_{k=1}^{j}q(y_k)q^*(x_k)\mathrm{e}^{2\mathrm{i}\lambda^2(y_k-x_k)}\mathrm{d}x_1\mathrm{d}y_1\cdots\mathrm{d}x_j\mathrm{d}y_j. \end{aligned} \tag{3.16}$$

Thus, the proof is completed. □

We remark that, at least as long as $q(x)\in L^2(\mathbb{R})$, each term $s_{2j}(\lambda)$ is pointwise defined for $\lambda\in L^+$. For convenience, we need the formal series of $\ln s_{11}(\lambda)$ even more. Therefore, we propose the following theorem.

Theorem 3.2. *The function* $\ln s_{11}(\lambda)$ *has a formal expansion as follows:*

$$\ln s_{11}(\lambda)=\sum_{j=1}^{\infty}b_{2j}(\lambda), \tag{3.17}$$

where each component $b_{2j}(\lambda)$ *is a linear combination of the following expression:*

$$\begin{aligned} b_{2j}(\lambda)&=(-1)^j\int_{\Sigma_j}\lambda^{2j}\\ &\times\prod_{k=1}^{j}q(y_k)q^*(x_k)\mathrm{e}^{2\mathrm{i}\lambda^2(y_k-x_k)}\mathrm{d}x_1\mathrm{d}y_1\cdots\mathrm{d}x_j\mathrm{d}y_j, \end{aligned} \tag{3.18}$$

where Σ_j *is any possible domain that obeys the condition* $x_k<y_k$ *for all k.*

Proof. Expanding the Taylor expansion of $\ln s_{11}(\lambda)$, we have

$$\ln s_{11}(\lambda)=\ln\left(1+\sum_{j=1}^{\infty}s_{2j}(\lambda)\right)=\sum_{k=1}^{\infty}(-1)^{k-1}k^{-1}\left(\sum_{j=1}^{\infty}s_{2j}(\lambda)\right)^k. \tag{3.19}$$

Based on the sum of subscripts, we have

$$\begin{cases} b_2(\lambda) = s_2(\lambda), \\ b_4(\lambda) = s_4(\lambda) - \dfrac{s_2(\lambda)^2}{2}, \\ b_6(\lambda) = s_6(\lambda) - s_2(\lambda)s_4(\lambda) + \dfrac{s_2(\lambda)^3}{3}, \\ b_8(\lambda) = s_8(\lambda) - s_2(\lambda)s_6(\lambda) - \dfrac{s_4(\lambda)^2}{2} + s_2(\lambda)^2 s_4(\lambda) - \dfrac{s_2(\lambda)^4}{4}, \\ b_{10}(\lambda) = s_{10}(\lambda) - s_2(\lambda)s_8(\lambda) - s_4(\lambda)s_6(\lambda) + s_2(\lambda)^2 s_6(\lambda) + s_2(\lambda)s_4(\lambda)^2 - s_2(\lambda)^3 s_4(\lambda) + \dfrac{s_2(\lambda)^5}{5}, \\ \cdots \end{cases} \tag{3.20}$$

Thus, the proof is completed. □

4. Iterative integrals

The goal of this section is to construct iterative integrals given by equation (3.18) into a Hopf algebra. Our iterative integrals are in the following form:

$$\int_{\Sigma_j} \prod_{k=1}^{j} q(y_k)q^*(x_k)\mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)}\mathrm{d}x_1\mathrm{d}y_1 \cdots \mathrm{d}x_j\mathrm{d}y_j, \tag{4.1}$$

where Σ_j is an appropriate domain that obeys $x_k < y_k$ for all k.

Omitting the indices, we use X to represent x_j and Y to represent y_j. For example,

$$\begin{aligned} &x_1 < y_1 \longrightarrow XY, \\ &x_1 < y_1 < x_2 < y_2 \longrightarrow XYXY, \\ &x_1 < x_2 < x_3 < y_1 < y_2 < y_3 \longrightarrow XXXYYY. \end{aligned}$$

In what follows, we only consider the situation where the quantities of X and Y are the same, and x_l, y_l satisfy the following constraint conditions: $x_l < y_l$. Then, we can use letters X, Y to simply represent iterative integrals. For instance,

$$\begin{aligned} XY &:= \int_{x_1<y_1} q(y_1)q^*(x_1)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1-x_1)}\mathrm{d}x_1\mathrm{d}y_1, \\ XYXY &:= \int_{x_1<y_1<x_2<y_2} q(y_1)q^*(x_1)q(y_2)q^*(x_2)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2-x_1-x_2)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2, \\ (XY)^{(j)} &:= \int_{x_1<y_1<\cdots<x_j<y_j} \prod_{k=1}^{j} q(y_k)q^*(x_k)\mathrm{e}^{2\mathrm{i}\lambda^2(y_k-x_k)}\mathrm{d}x_1\mathrm{d}y_1 \cdots \mathrm{d}x_j\mathrm{d}y_j, \\ XXYXYY &:= \int_{x_1<x_2<y_1<x_3<y_2<y_3} \prod_{j=1}^{3} q(y_j)q^*(x_j)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2+y_3-x_1-x_2-x_3)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3, \\ XXXYYY &:= \int_{x_1<x_2<x_3<y_1<y_2<y_3} \prod_{j=1}^{3} q(y_j)q^*(x_j)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2+y_3-x_1-x_2-x_3)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3. \end{aligned} \tag{4.2}$$

According to equation (4.2), $s_{11}(\lambda)$ can be expressed in the following form:

$$s_{11}(\lambda) = 1 + \sum_{j=1}^{\infty}(-1)^j\lambda^{2j}(XY)^{(j)}. \tag{4.3}$$

In addition, $b_{2j}(\lambda)$ in $\ln s_{11}(\lambda)$ can be expressed in the following form:

$$\begin{cases} b_2(\lambda) = -\lambda^2 XY, \\ b_4(\lambda) = \lambda^4 (XY)^{(2)} - \dfrac{\lambda^4 (XY)^2}{2}, \\ b_6(\lambda) = -\lambda^6 (XY)^{(3)} + \lambda^6 XY \times (XY)^{(2)} - \lambda^6 \dfrac{(XY)^3}{3}, \\ b_8(\lambda) = \lambda^8 (XY)^{(4)} - \lambda^8 XY \times (XY)^{(3)} - \dfrac{\lambda^8 ((XY)^{(2)})^2}{2} + \lambda^8 (XY)^2 \times (XY)^{(2)} - \lambda^8 \dfrac{(XY)^4}{4}. \\ \cdots \end{cases} \tag{4.4}$$

Next, we provide the pairing principle for X and Y. Starting from left to right, each X pairs with its nearest Y (see figure 1).

We call an integral (4.1) connected if its first X is paired with its last Y. Now, we want to prove that $\ln s_{11}(\lambda)$ is composed of connected integrals. According to Fubini's theorem, we have

$$\begin{aligned} &\left(\int_{x_1<y_1} f(x_1) g(y_1) \mathrm{d}x_1 \mathrm{d}y_1\right)^2 \\ &= 2\int_{x_1<y_1<x_2<y_2} f(x_1) g(y_1) f(x_2) g(y_2) \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \\ &+4\int_{x_1<x_2<y_1<y_2} f(x_1) g(y_1) f(x_2) g(y_2) \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2. \end{aligned}$$

Represented by letters X, Y, the above equation is written as

$$(XY)^2 = 2XYXY + 4XXYY.$$

Theorem 4.1. $\ln s_{11}(\lambda)$ *has a formal expansion as follows:*

$$\ln s_{11}(\lambda) = -\lambda^2 XY - 2\lambda^4 XXYY - 4\lambda^6 \left(XXYXYY + 3XXXYYY\right) + \cdots. \tag{4.5}$$

Proof. According to equation (4.4), we calculate each item separately. First, we have

$$\begin{aligned} b_4(\lambda) &= \lambda^4 (XY)^{(2)} - \frac{\lambda^4 (XY)^2}{2} \\ &= \lambda^4 \left((XY)^{(2)} - \frac{2XYXY + 4XXYY}{2}\right) \\ &= -2\lambda^4 XXYY. \end{aligned} \tag{4.6}$$

Expanding $b_6(\lambda)$ yields

$$\begin{aligned} b_6(\lambda) &= -\lambda^6 (XY)^{(3)} + \lambda^6 XY \times (XY)^{(2)} - \lambda^6 \frac{(XY)^3}{3} \\ &= -\lambda^6 \left((XY)^{(3)} - XY \times (XY)^{(2)} + \frac{(2XYXY + 4XXYY) \times XY}{3}\right) \\ &= -\lambda^6 \left((XY)^{(3)} - \frac{XY \times (XY)^{(2)}}{3} + \frac{4XXYY \times XY}{3}\right). \end{aligned} \tag{4.7}$$

Since

$$\begin{aligned}&XY\times(XY)^{(2)}=X\times(\dot{Y}XYXY+X\dot{Y}YXY\\&\quad+XY\dot{Y}XY+XYX\dot{Y}Y+XYXY\dot{Y})\\&\quad=XYXYXY+2XXYYXY+2XXYYXY+XYXYXY+2XXYXYY\\&\quad+2XYXXYY+2XXYXYY+2XYXXYY+XYXYXY\\&\quad=3(XY)^{(3)}+4XXYYXY+4XXYXYY+4XYXXYY,\end{aligned}\tag{4.8}$$

and

$$\begin{aligned}&XXYY\times XY=(XXYY\dot{X}+XXY\dot{X}Y+XX\dot{X}YY\\&\quad+X\dot{X}XYY+\dot{X}XXYY)\times Y\\&\quad=XXYYXY+2XXYXYY+3XXXYYY+3XXXYYY+XXYXYY\\&\quad+3XXXYYY+XXYXYY+XYXXYY\\&\quad=XXYYXY+4XXYXYY+9XXXYYY+XYXXYY.\end{aligned}\tag{4.9}$$

According to equations (4.7)–(4.9), we have

$$b_6(\lambda)=-4\lambda^6(XXYXYY+3XXXYYY).$$

Thus, the proof is completed. □

We need to construct a Hopf algebra. Let H be a graded algebra, and these are the following operations on H:

$$\times\text{:natural multiplication},\quad\otimes\text{:tensor product},$$
$$\Delta\colon H\to H\times H,$$

where

$$\Delta a=\sum_{a_1a_2=a}a_1\otimes a_2.$$

We call a word $a\in H$ group-like if

$$\Delta a=a\otimes a.$$

Then, we know that the set G of all group-like words with natural multiplication is a group. The primitive words are

$$P=\{p\in H|\Delta p=1\otimes p+p\otimes 1\}.$$

The primitive words are linear combinations of connected integrals. There is a relationship between groups G and P as follows:

$$G=\mathrm{e}^P.\tag{4.10}$$

We obtain the following lemma:

Lemma 4.1. *The expression*

$$s_{11}(\lambda)=1+\sum_{j=1}^{\infty}(-1)^j\lambda^{2j}(XY)^{(j)}\tag{4.11}$$

belongs to G.

Proof. For the sake of simplicity, let $(XY)^{(0)}=1$. Then

$$\begin{aligned}\Delta s_{11}(\lambda)&=\Delta\sum_{j=0}^{\infty}(-1)^j\lambda^{2j}(XY)^{(j)}\\&=\sum_{j=0}^{\infty}\Delta(-1)^j\lambda^{2j}(XY)^{(j)}\\&=\sum_{j=0}^{\infty}\sum_{k=0}^{n}(-1)^k\lambda^{2k}(XY)^{(k)}\\&\quad\otimes(-1)^{j-k}\lambda^{2(j-k)}(XY)^{(j-k)}\\&=s_{11}(\lambda)\otimes s_{11}(\lambda).\end{aligned}$$

Thus, the proof is completed. □

Therefore, $b_{2j}(\lambda)'s$ are formal linear combinations of connected integrals. Then, the proof of the first conclusion of theorem 1.1 is completed.

5. Bounding the integral term $s_2(\lambda)$

The leading term in both $s_{11}(\lambda) - 1$ and $\ln s_{11}(\lambda)$ away from $\Sigma := \{\lambda | \lambda^2 \in \mathbb{R}\}$ is $s_2(\lambda)$. Thus, here, we analyze the term $s_2(\lambda)$.

First, we know that

$$s_2(\lambda) = -\int_{x<y} \lambda^2 q(y) q^*(x) \mathrm{e}^{2\mathrm{i}\lambda^2(y-x)} \mathrm{d}x\mathrm{d}y. \tag{5.1}$$

For convenience, we choose the unitary Fourier transform

$$\hat{q}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} q(x) \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{d}x, \tag{5.2}$$

and the corresponding Fourier inversion formula as follows:

$$\check{q}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} q(\xi) \mathrm{e}^{\mathrm{i}x\xi} \mathrm{d}x. \tag{5.3}$$

Note that $\delta(x)$ is a Dirac delta function and has properties:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{i\omega(\xi-\eta)} d\omega = 2\pi\delta(\xi - \eta). \tag{5.4}$$

According to equations (5.2), (5.3), and (5.4), we have

$$\begin{aligned}
s_2(\lambda) = &-\int_{x<y} \lambda^2 q(y) q^*(x) \mathrm{e}^{2\mathrm{i}\lambda^2(y-x)} \mathrm{d}x\mathrm{d}y \\
=&-\frac{1}{2\pi}\int_{x<y}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda^2 \mathrm{e}^{2\mathrm{i}\lambda^2(y-x)} \hat{q}(\xi) \mathrm{e}^{iy\xi} \hat{q}^*(\eta) \mathrm{e}^{-ix\eta} \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}x\mathrm{d}y \\
=&-\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda^2 \left\{\int_{-\infty}^{y} \mathrm{e}^{\mathrm{i}y(2\lambda^2+\xi)-ix(2\lambda^2+\eta)}\mathrm{d}x\right\} \hat{q}(\xi)\hat{q}^*(\eta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}y \\
=&\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\lambda^2}{2\mathrm{i}\lambda^2 + i\eta} \mathrm{e}^{\mathrm{i}y(\xi-\eta)} \hat{q}(\xi)\hat{q}^*(\eta)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}y \\
=&-\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\lambda^2}{2\lambda^2 + \eta} \left\{\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}y(\xi-\eta)}\mathrm{d}y\right\} \hat{q}(\xi)\hat{q}^*(\eta)\mathrm{d}\xi\mathrm{d}\eta \\
=&-\mathrm{i}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\lambda^2}{2\lambda^2 + \eta} \delta(\xi - \eta) \hat{q}(\xi)\hat{q}^*(\eta)\mathrm{d}\xi\mathrm{d}\eta \\
=&-\mathrm{i}\int_{-\infty}^{+\infty} \frac{\lambda^2}{2\lambda^2 + \xi} |\hat{q}(\xi)|^2 \mathrm{d}\xi.
\end{aligned} \tag{5.5}$$

Simplifying the last equation of equation (5.5), we obtain

$$s_2(\lambda) = -\frac{\mathrm{i}}{2}\|q(x)\|_{L^2}^2 + \mathrm{i}\int_{-\infty}^{+\infty} \frac{\xi}{4\lambda^2 + 2\xi} |\hat{q}(\xi)|^2 \mathrm{d}\xi. \tag{5.6}$$

For convenience, we introduce a new variable as follows:

$$\begin{cases} c_2(\lambda) := s_2(\lambda) + \dfrac{\mathrm{i}}{2}\|q(x)\|_{L^2}^2 = \mathrm{i}\displaystyle\int_{-\infty}^{+\infty} \frac{\xi}{4\lambda^2 + 2\xi} |\hat{q}(\xi)|^2 \mathrm{d}\xi, \\ c_{2j}(\lambda) := b_{2j}(\lambda), \quad j \geqslant 2. \end{cases} \tag{5.7}$$

Obviously, we have

$$\ln s_{11}(\lambda) + \frac{\mathrm{i}}{2}\|q(x)\|_{L^2}^2 = \sum_{j=1}^{\infty} c_{2j}(\lambda).$$

Through the above analysis, we obtain the following theorem.

Theorem 5.1.

(a) For $\lambda \in \Sigma_+ := \{\lambda | \operatorname{Im}(\lambda^2) > 0\}$, we have

$$|\operatorname{Re} c_2(\lambda)| \leqslant \int_{-\infty}^{+\infty} \frac{|\xi| \operatorname{Im}(\lambda^2)}{(2\operatorname{Re}\lambda^2 + \xi)^2 + (2\operatorname{Im}(\lambda^2))^2} |\hat{q}(\xi)|^2 \mathrm{d}\xi. \tag{5.8}$$

(b) For all $N \in \mathbb{N}$, we have

$$\begin{aligned} |\operatorname{Re}\left(c_2\left(\frac{\lambda}{\sqrt{2}}\right) - \frac{\mathrm{i}}{2}\sum_{k=0}^{N-1} M_{k,2}\lambda^{-2k-2}\right)| \leqslant \frac{|\lambda|^{-2N}}{2} \\ \times \int_{-\infty}^{+\infty} |\xi|^{N+1} \frac{\operatorname{Im}\lambda^2 + |\operatorname{Re}\lambda^2 + \xi|}{(\operatorname{Re}\lambda^2 + \xi)^2 + (\operatorname{Im}\lambda^2)^2} |\hat{q}(\xi)|^2 \mathrm{d}\xi, \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} M_{k,2} = &\int_{-\infty}^{+\infty} (-1)^k \xi^{k+1} |\hat{q}(\xi)|^2 \mathrm{d}\xi \\ = &\begin{cases} -\displaystyle\int_{-\infty}^{+\infty} |q^{(k)}|^2 \mathrm{d}x, & j = 2k - 1, \\ -\operatorname{Im}\displaystyle\int_{-\infty}^{+\infty} q^{(k+1)} q^{*(k)} \mathrm{d}x, & j = 2k. \end{cases} \end{aligned}$$

Figure 1. Pairing principle for X and Y.

Proof. First, we prove the property (a). We have

$$\begin{aligned}|\operatorname{Re} c_2(\lambda)| &= |\operatorname{Re}\, \mathrm{i}\int_{-\infty}^{+\infty} \frac{\xi}{4\lambda^2 + 2\xi}|\hat{q}(\xi)|^2 \mathrm{d}\xi|,\\ &= |\operatorname{Im} \int_{-\infty}^{+\infty} \frac{\xi}{4\lambda^2 + 2\xi}|\hat{q}(\xi)|^2 \mathrm{d}\xi|,\\ &= |\int_{-\infty}^{+\infty} \frac{4\xi \operatorname{Im}\lambda^2}{(4\operatorname{Re}\lambda^2 + 2\xi)^2 + (4\operatorname{Im}\lambda^2)^2}|\hat{q}(\xi)|^2 \mathrm{d}\xi|,\\ &\leqslant \int_{-\infty}^{+\infty} \frac{|\xi| \operatorname{Im}\lambda^2}{(2\operatorname{Re}\lambda^2 + \xi)^2 + (2\operatorname{Im}\lambda^2)^2}|\hat{q}(\xi)|^2 \mathrm{d}\xi.\end{aligned}$$

Then, we prove property (b). For $c_2(\lambda)$, we can rewrite it as

$$\begin{aligned}c_2(\lambda) &= \mathrm{i}\int_{-\infty}^{+\infty} \frac{\xi}{4\lambda^2 + 2\xi}|\hat{q}(\xi)|^2 \mathrm{d}\xi\\ &= \frac{\mathrm{i}}{4\lambda^2}\int_{-\infty}^{+\infty} \frac{\xi}{1 - (-\frac{\xi}{2\lambda^2})}|\hat{q}(\xi)|^2 \mathrm{d}\xi\\ &= \frac{\mathrm{i}}{4\lambda^2}\sum_{j=0}^{\infty}\int_{-\infty}^{+\infty} \xi\left(-\frac{\xi}{2\lambda^2}\right)^j |\hat{q}(\xi)|^2 \mathrm{d}\xi\\ &= \frac{\mathrm{i}}{2}\left(\sum_{j=0}^{N-1}\int_{-\infty}^{+\infty} (-1)^j \xi^{j+1}(2\lambda^2)^{-j-1}|\hat{q}(\xi)|^2 \mathrm{d}\xi\right.\\ &\quad \left. + \sum_{j=N}^{\infty}\int_{-\infty}^{+\infty} (-1)^j \xi^{j+1}(2\lambda^2)^{-j-1}|\hat{q}(\xi)|^2 \mathrm{d}\xi\right)\\ &= \frac{\mathrm{i}}{2}\left(\sum_{j=0}^{N-1}\int_{-\infty}^{+\infty} (-1)^j \xi^{j+1}(2\lambda^2)^{-j-1}|\hat{q}(\xi)|^2 \mathrm{d}\xi\right.\\ &\quad \left. + \int_{-\infty}^{+\infty} \frac{(-2\lambda^2)^{-N}\xi^{N+1}}{2\lambda^2 + \xi}|\hat{q}(\xi)|^2 \mathrm{d}\xi\right).\end{aligned}$$

Then, we have

$$\begin{aligned}&|\operatorname{Re}\left(c_2\left(\frac{\lambda}{\sqrt{2}}\right) - \frac{\mathrm{i}}{2}\sum_{k=0}^{N-1} M_{k,2}\lambda^{-2k-2}\right)|\\ &= |\operatorname{Im}\frac{1}{2}\int_{-\infty}^{+\infty} \frac{(-\lambda^2)^{-N}\xi^{N+1}}{\lambda^2 + \xi}|\hat{q}(\xi)|^2 \mathrm{d}\xi|\\ &\leqslant \frac{|\lambda|^{-2N}}{2}\int_{-\infty}^{+\infty} |\xi|^{N+1}\frac{\operatorname{Im}\lambda^2 + |\operatorname{Re}\lambda^2 + \xi|}{(\operatorname{Re}\lambda^2 + \xi)^2 + (\operatorname{Im}\lambda^2)^2}|\hat{q}(\xi)|^2 \mathrm{d}\xi.\end{aligned}$$

Thus, the proof is completed. □

6. Bounding the iterative integrals s_{2j}

Here, we first recall the function spaces U^p, V^p, and DU^p [33].

Definition 6.1.

(a) We define the space V^p as the space of the function that the following norm is finite:

$$\|v\|_{V^p} = \sup_{-\infty < t_1 < t_2 \cdots < t_N = +\infty}\left(\sum_{j=1}^{N-1}|v(t_{j+1}) - v(t_j)|^p\right)^{\frac{1}{p}},$$

$1 < p < \infty$,

where $v(t_N) = 0$.

(b) A U^p atom is defined as

$$u(x) = \sum_{j=1}^{N-1} c_j\chi_{[t_j, t_{j+1})}(x), \quad \mathit{if} \sum_{j=1}^{N-1}|c_j|^p \leqslant 1,$$

where χ is the characteristic function as follows:

$$\chi_{[t_j, t_{j+1})}(t) = \begin{cases}1, & t_j \leqslant t < t_{j+1}\\ 0, & \text{otherwise.}\end{cases} \tag{6.1}$$

We define the space U^p as

$$U^p = \left\{\sum_{j=1}^{\infty} c_j a_j | (c_j)_j \in l^1,\ a_j\ \mathit{is}\ U^p\ \text{atom}\right\},$$

with the following norm:

$$\|u\|_{U^p} = \inf\left\{\sum_{j=1}^{\infty}|c_j|\ \middle|\ u = \sum_{j=1}^{\infty} c_j a_j,\ (c_j)_j \in l^1,\ a_j\ \mathit{is}\ U^p\ \text{atom}\right\}.$$

(c) We define the space DU^p as

$$DU^p = \{u' | u \in U^p\},$$

with the following norm:

$$\|f\|_{DU^p} = \sup\left\{\int_{-\infty}^{+\infty} f\phi dt|\ \|\phi\|_{V^q} \leqslant 1,\ \phi \in C_c^{\infty}\right\}.$$

(d) We define the space DV^p as

$$DV^p = \{v' | v \in V^p,\ v \text{ is left} - \text{continuous functions with limit 0 at the right endpoint }\},$$

with the following norm:

$$\|f\|_{DV^p} = \sup\left\{\int_{-\infty}^{+\infty} f\phi \mathrm{d}t|\ \|\phi\|_{U^q} \leqslant 1,\ \phi \in C_c^{\infty}\right\}.$$

(e) Let $\sigma > 0$, we define

$$\|u\|_{l_\sigma^p U^2} = \|\|\chi_{[\frac{k}{\sigma}, \frac{k+1}{\sigma}]}u\|_{U^2}\|_{l_k^p},$$

and

$$\|u\|_{l_\sigma^p DU^2} = \|\|\chi_{[\frac{k}{\sigma}, \frac{k+1}{\sigma}]}u\|_{DU^2}\|_{l_k^p},$$

where $\chi_{[\frac{k}{\sigma}, \frac{k+1}{\sigma}]}$ is a smooth cutoff function in interval $[\frac{k}{\sigma}, \frac{k+1}{\sigma}]$.

Let us recall some basic properties of the spaces U^p, V^p, and DU^p.

Lemma 6.1.

(a) For all $1 < p < \infty$, we have

$$U^p \subset V^p, \text{ and } \|u\|_{V^p} \leqslant \|u\|_{U^p}. \tag{6.2}$$

If $g \in L^1$, *we have*

$$\|g * v\|_{V^p} \leqslant \|g\|_{L^1}\|v\|_{V^p}, \quad \|g * u\|_{U^p} \leqslant \|g\|_{L^1}\|u\|_{U^p}. \tag{6.3}$$

(b) If $u \in U^2$, $v \in V^2$, and v is left-continuous functions with limit 0 at the right endpoint, then

$$\|u\|_{U^2} = \|u'\|_{DU^2}, \quad \|v\|_{V^2} = \|v'\|_{DV^2}. \tag{6.4}$$

(c) The bilinear estimates

$$\|vu\|_{DU^2} \leqslant 2\|v\|_{V^2}\|u\|_{DU^2}. \tag{6.5}$$

For convenience, we define the one-step operator as follows:

$$L(f)(t) = -\int_{x<y<t} q(y)q^*(x)\mathrm{e}^{2\mathrm{i}\lambda^2(y-x)}f(x)\mathrm{d}x\mathrm{d}y.$$

Lemma 6.2. *For* $\mathrm{Im}\,\lambda^2 > 0$, *we have*

$$\|L\|_{V^2\to U^2} \leqslant 4\sqrt{2}\,\|\mathrm{e}^{-\mathrm{i}\,\mathrm{Re}\,\lambda^2 x}q\|^2_{DU^2}. \tag{6.6}$$

Proof. It suffices to consider $\lambda^2 = \mathrm{i}$. Then, according to lemma 6.1, we have

$$\begin{aligned}\|Lf\|_{U^2} &= \|\left(\int_{-\infty}^{t}\int_{-\infty}^{y} q(y)q^*(x)\mathrm{e}^{2(x-y)}f(x)\mathrm{d}x\mathrm{d}y\right)'\|_{DU^2}\\ &= \|\int_{-\infty}^{t} q(t)q^*(x)\mathrm{e}^{2(x-t)}f(x)\mathrm{d}x\|_{DU^2}\\ &\leqslant 2\|q\|_{DU^2}\,\|\chi_{t<0}\mathrm{e}^{2t}*(q^*f)\|_{V^2}\\ &\leqslant 2\sqrt{2}\,\|q\|_{DU^2}\,\|\chi_{t<0}\mathrm{e}^{2t}*(q^*f)\|_{U^2}\\ &\leqslant 2\sqrt{2}\,\|q\|_{DU^2}\,\|(\chi_{t<0}\mathrm{e}^{2t}*(q^*f))'\|_{DU^2}\\ &\leqslant 4\sqrt{2}\,\|q\|_{DU^2}\,\|\chi_{t<0}\mathrm{e}^{2t}*(q^*f)\|_{DU^2}\\ &\leqslant 2\sqrt{2}\,\|q\|_{DU^2}\,\|q^*f\|_{DU^2}\\ &\leqslant 4\sqrt{2}\,\|q\|^2_{DU^2}\,\|f\|_{V^2}.\end{aligned}$$

Thus, the proof is completed. □

This bound is very sharp on the region Σ, but we want to move λ into the region Ω_+. Therefore, we need the following lemma:

Lemma 6.3.

(a) We have

$$\|q\|_{l^p_\sigma U^2} \lesssim \|\partial q\|_{l^p_\sigma DU^2} + \sigma\|q\|_{l^p_\sigma DU^2}. \tag{6.7}$$

(b) The space $l^2_\sigma U^2$ can be seen as

$$l^2_\sigma U^2 = DU^2 + \sqrt{\sigma}L^2. \tag{6.8}$$

(c) The following relationship hold:

$$B^{-\frac{1}{2}}_{2,1} \subset l^2_1 U^2 \subset B^{-\frac{1}{2}}_{2,\infty}. \tag{6.9}$$

(d) For all $p > 2$, we have

$$\|q\|_{l^p_\tau DU^2} \lesssim \tau^{\frac{1}{p}-1}\|q\|_{\dot{H}^{\frac{1}{2}-\frac{1}{p}}}. \tag{6.10}$$

If $0 \leqslant \tau_1 \leqslant \tau_2$, *then*

$$\|q\|_{l^p_{\tau_2} DU^2} \lesssim \|q\|_{l^p_{\tau_1} DU^2} \lesssim \left(\frac{\tau_2}{\tau_1}\right)^{1-\frac{1}{p}}\|q\|_{l^p_{\tau_2} DU^2}. \tag{6.11}$$

Lemma 6.4. *For* $\mathrm{Im}\,\lambda^2 > 0$, *we have*

$$\|L\|_{U^2\to U^2} \lesssim \|\mathrm{e}^{-\mathrm{iRe}\lambda^2 x}q\|^2_{l^2_{\mathrm{Im}\,\lambda^2}DU^2}. \tag{6.12}$$

Proof. It suffices to consider $\lambda^2 = \mathrm{i}$. Then, we have

$$\begin{aligned}\|Lf\|_{U^2} &= \|\int_{-\infty}^{t} q(t)q^*(x)\mathrm{e}^{2(x-t)}f(x)\mathrm{d}x\|_{DU^2}\\ &\lesssim \|q\|_{l^2DU^2}\,\|\chi_{t<0}\mathrm{e}^{2t}*(q^*f)\|_{l^2U^2}\\ &\lesssim \|q\|_{l^2DU^2}\,\|(\chi_{t<0}\mathrm{e}^{2t}*(q^*f))'\|_{l^2DU^2}\\ &\lesssim \|q\|_{l^2DU^2}\,\|q^*f\|_{l^2DU^2}\\ &\lesssim \|q\|^2_{l^2DU^2}\,\|f\|_{U^2}.\end{aligned}$$

Thus, the proof is completed. □

Based on the above analysis, we provide an estimate of $s_{2j}(\lambda)$ and $b_{2j}(\lambda)$.

Theorem 6.1. *The iterated integrals* $s_{2j}(\lambda)$ *and* $b_{2j}(\lambda)$ *have the following estimate:*

$$|\lambda^{-2j}s_{2j}(\lambda)| + |\lambda^{-2j}b_{2j}(\lambda)| \leqslant C\|\mathrm{e}^{-\mathrm{iRe}\lambda^2 x}q\|^{2j}_{l^2_{\mathrm{Im}\,\lambda^2}DU^2}. \tag{6.13}$$

Proof. According to theorem 3.1, the first component of the Jost solution can be rewritten as

$$\psi_1(x) = \mathrm{e}^{-\mathrm{i}\lambda^2 x}\sum_{j=0}^{\infty}\lambda^{2j}L^j 1(x).$$

Then, the transmission coefficient $s_{11}(\lambda)$ can be expressed as

$$s_{11}(\lambda) = \lim_{x\to+\infty}\sum_{j=0}^{\infty}\lambda^{2j}L^j 1(x).$$

First, we introduce a partial order $\preceq$; $f_1 \preceq f_2$ means that each coefficient of Taylor expansion at zero for f_1 is not greater than the coefficient of Taylor expansion at zero for f_2. Because both the iterated integrals s_{2j} and b_{2j} are homogeneous forms, we have

$$\sum_{j=1}^{\infty}(z\lambda^{-2})^j b_{2j} = \ln\left(1 + \sum_{j=1}^{\infty}(z\lambda^{-2})^j s_{2j}\right).$$

Let f: $z \to \frac{z}{1-z}$, and we note that $\ln(1+z) \preceq f(z)$. Then, we have

$$\sum_{j=1}^{\infty}(z\lambda^{-2})^{j}b_{2j}\preceq f(f(C_1 z)), \tag{6.14}$$

where

$$C_1=\left(\frac{C}{2^{j-1}}\right)^{\frac{1}{j}}\|\mathrm{e}^{-\mathrm{i}\mathrm{Re}\lambda^2 x}q\|^2_{l^2_{\mathrm{Im}\,\lambda^2}DU^2}.$$

Simplifying equation (6.14) yields

$$\sum_{j=1}^{\infty}(z\lambda^{-2})^{j}b_{2j}\preceq\sum_{j=0}^{\infty}2^{j}(C_1 z)^{j+1}. \tag{6.15}$$

Comparing the coefficients of each power of z, we get

$$|\lambda^{-2j}b_{2j}(\lambda)|\leqslant 2^{j-1}C_1^{j}.$$

Thus, the proof is completed. □

Theorem 6.2. *Suppose that* $q\in H^s$. *If* $-\frac{1}{2}<s\leqslant\frac{j-1}{2}$, *then we have*

$$\begin{aligned}&|s_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})|+|b_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})|\\&\leqslant C(1+\frac{1}{2s+1})\zeta^{j-2s-1}\|q\|^2_{H^s}\|q\|^{2j-2}_{l_1^2DU^2},\end{aligned} \tag{6.16}$$

and

$$\begin{aligned}&\int_1^{\infty}\zeta^{2s-j}(|s_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})|+|b_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})|)\mathrm{d}\zeta\\&\lesssim\left(1+\frac{1}{j-1-2s}+\frac{1}{(2s+1)^2}\right)\|q\|^2_{H^s}\|q\|^{2j-2}_{l_1^2DU^2}.\end{aligned} \tag{6.17}$$

Proof. For convenience, we define the following symbols:

$$\begin{cases}\sum\limits_{k}\cdot:=\sum\limits_{j=0,k=2^j}^{\infty}\cdot, & q=\sum\limits_{k}q_k,\\ \hat{q}_1=\chi_{|\xi|<1}\hat{q}, \quad \hat{q}_{<k}=\chi_{|\xi|<k}\hat{q}, & \hat{q}_k=\chi_{k\leqslant|\xi|<2k}\hat{q}.\end{cases}$$

According to theorem 6.1, we have

$$\left|s_{2j}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}}\right)\right|\lesssim\zeta^{j}\sum_{k_1\geqslant k_2}\|q_{k_1}\|_{l_\zeta^2DU^2}\|q_{k_2}\|_{l_\zeta^2DU^2}\|q_{\leqslant k_2}\|^{2j-2}_{l_\zeta^2DU^2}. \tag{6.18}$$

Below, we investigate the classification for the above inequality. According to lemma 6.3, we have

$$\begin{aligned}&\|q_k\|_{l_\zeta^2DU^2}\lesssim k^{-s}\zeta^{-\frac{1}{2}}\|q\|_{H^s}, && \text{if } 1<k\leqslant\zeta\\&\|q_k\|_{l_\zeta^2DU^2}\lesssim k^{-s-\frac{1}{2}}\|q\|_{H^s}, && \text{if } k\geqslant\zeta\\&\|q_{<k}\|_{l_\zeta^2DU^2}\lesssim\|q_{<k}\|_{l_1^2DU^2}, && \text{if } k\geqslant\zeta\\&\|q_{<k}\|_{l_\zeta^2DU^2}\lesssim k^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\|q_{<k}\|_{l_1^2DU^2}, && \text{if } k<\zeta.\end{aligned} \tag{6.19}$$

Then, we obtain

$$\begin{aligned}&|s_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})|\lesssim\zeta^{j-2s-1}\\&\times\sum_{k_1\geqslant k_2}C(\zeta,k_1,k_2)\|q_{k_1}\|_{H^s}\|q_{k_2}\|_{H^s}\|q\|^{2j-2}_{l_1^2DU^2},\end{aligned} \tag{6.20}$$

where

$$C(\zeta,k_1,k_2)=\begin{cases}\left(\frac{k_1}{\zeta}\right)^{j-2s-1}\left(\frac{k_2}{k_1}\right)^{j-s-1}, & k_2\leqslant k_1\leqslant\zeta,\\ \left(\frac{\zeta}{k_1}\right)^{s+\frac{1}{2}}\left(\frac{k_2}{\zeta}\right)^{j-s-1}, & k_2\leqslant\zeta\leqslant k_1,\\ \left(\frac{\zeta}{k_1}\right)^{s+\frac{1}{2}}\left(\frac{\zeta}{k_2}\right)^{s+\frac{1}{2}}, & \zeta\leqslant k_2\leqslant k_1.\end{cases} \tag{6.21}$$

Then, the critical coefficients are $\frac{1}{2s+1}$; thus, we get equation (6.16). Moreover, by the Cauchy–Schwarz inequality and Schur's lemma, we get

$$\int_1^{\infty}\zeta^{2s-j}\left|s_{2j}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}}\right)\right|\mathrm{d}\zeta\lesssim c\|q\|^2_{H^s}l_1^2DU^2,$$

where

$$c=\max\left\{\sup_{k_1}\sum_{\zeta,k_2}C(\zeta,k_1,k_2),\ \sup_{k_2}\sum_{\zeta,k_1}C(\zeta,k_1,k_2)\right\}.$$

Then, the critical coefficients are $\frac{1}{j-1-2s}$, $\frac{1}{(2s+1)^2}$; therefore, we get equation (6.17). Thus, the proof is completed. □

Remark 6.1. For a single integral element, such as $s_2(\lambda)$ and $b_4(\lambda)$, we do not need to transform their independent variables to analyze their properties. However, for expression (6.16) in theorem 6.2, if we want to obtain the regularity of $|s_{2j}|+|b_{2j}|$, we must transform their independent variables. This is also the reason why we cannot obtain conserved energies similar to those in the NLS equation [33]. However, similar to the NLS equation, we can draw the following conclusion for the DNLS equation. Let

$$\begin{aligned}E_s:=&-\frac{2\sin(\pi s)}{\pi}\int_1^{\infty}(\zeta^2-1)^s[\mathrm{Re}\ln I(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})\\&-\sum_{j=0}^{M}(-1)^jH_{2j}\zeta^{-2j-1}]\mathrm{d}\zeta+\sum_{j=0}^{M}C_s^jH_{2j},\end{aligned} \tag{6.22}$$

where

$$H_{2j}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\zeta^k\mathrm{Re}\ln I\left(\sqrt{\frac{\zeta}{2}}\right)\mathrm{d}\zeta. \tag{6.23}$$

Then, E_s is conserved along the DNLS flow if $s\geqslant\frac{1}{2}$.

7. Asymptotic analysis of $b_4(\lambda)$ and $b_6(\lambda)$

In this section, we provide asymptotic expressions for $b_4(\lambda)$ and $b_6(\lambda)$ and some related conclusions. For the analysis of

$s_2(\lambda)$, we use the Fourier transform method. Here, we use the same technique to analyze $b_4(\lambda)$ and $b_6(\lambda)$. We recall that $b_4(\lambda)$ is given by

$$\begin{aligned}
b_4(\lambda) &= -2\lambda^4 XXYY\\
&= -2\lambda^4 \int_{x_1<x_2<y_1<y_2} q(y_1)q^*(x_1)\\
&\quad\times q(y_2)q^*(x_2)\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2-x_1-x_2)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\\
&= -\frac{\lambda^4}{2\pi^2}\int_{\mathbb{R}^4}\int_{x_1<x_2<y_1<y_2}\\
&\quad\times \mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2-x_1-x_2)+\mathrm{i}(y_1\eta_1+y_2\eta_2-x_1\xi_1-x_2\xi_2)}\\
&\quad\times \hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\\
&\quad\hat{q}^*(\xi_2)\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\\
&= -\frac{\lambda^4}{2\pi^2}\int_{\mathbb{R}^4}\Bigg(\int_{x_1<x_2<y_1<y_2}\\
&\quad\mathrm{e}^{\mathrm{i}(2\lambda^2+\eta_1)y_1+\mathrm{i}(2\lambda^2+\eta_2)y_2-\mathrm{i}(2\lambda^2+\xi_1)x_1-\mathrm{i}(2\lambda^2+\xi_2)x_2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\Bigg)\\
&\quad\times \hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2.
\end{aligned}\tag{7.1}$$

Let

$$\begin{aligned}
K(\xi_1,\ \xi_2,\ \eta_1,\ \eta_2) &:= \int_{x_1<x_2<y_1<y_2}\\
&\quad\times \mathrm{e}^{\mathrm{i}(2\lambda^2+\eta_1)y_1+\mathrm{i}(2\lambda^2+\eta_2)y_2-\mathrm{i}(2\lambda^2+\xi_1)x_1-\mathrm{i}(2\lambda^2+\xi_2)x_2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{y_2}\int_{-\infty}^{y_1}\int_{-\infty}^{x_2}\mathrm{e}^{\mathrm{i}(2\lambda^2+\eta_1)y_1+\mathrm{i}(2\lambda^2+\eta_2)y_2-\mathrm{i}(2\lambda^2+\xi_1)x_1-\mathrm{i}(2\lambda^2+\xi_2)x_2}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\\
&= -\frac{2i\pi}{(2\lambda^2+\xi_1)(4\lambda^2+\xi_1+\xi_2)(2\lambda^2-\eta_1+\xi_1+\xi_2)}\delta(\eta_1+\eta_2-\xi_1-\xi_2).
\end{aligned}\tag{7.2}$$

Lemma 7.1. *We have the following identity:*

$$\begin{aligned}
b_4(\lambda) = \frac{\mathrm{i}}{2\pi}\int_{\xi_1+\xi_2=\eta_1+\eta_2}\frac{\lambda^4}{(2\lambda^2+\xi_1)(2\lambda^2+\eta_1)(2\lambda^2+\eta_2)}\\
\times \mathrm{Re}\,(\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2))\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2.
\end{aligned}\tag{7.3}$$

Suppose that q *is a Schwartz function. Then, we have the following asymptotic series:*

$$b_4(\lambda) \sim \mathrm{i}\sum_{j=2}^{\infty} H_{j4}\frac{\lambda^{2-2j}}{2^{j+1}},\tag{7.4}$$

where

$$H_{j4} = -\mathrm{Re}\left(\mathrm{i}^j\sum_{\alpha_1+\alpha_2+\alpha_3=j-2}(-1)^{\alpha_1}\int q^{(\alpha_2)}q^{(\alpha_3)}\overline{q^{(\alpha_3)}q}\mathrm{d}x\right).$$

Proof. Substituting equation (7.2) into equation (7.1) yields

$$\begin{aligned}
b_4(\lambda) &= -\frac{\lambda^4}{2\pi^2}\\
&\quad\times\int_{\mathbb{R}^4}K(\xi_1,\ \xi_2,\ \eta_1,\ \eta_2)\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\\
&= \frac{\mathrm{i}}{\pi}\int_{\mathbb{R}^4}\frac{\lambda^4\delta(\eta_1+\eta_2-\xi_1-\xi_2)}{(2\lambda^2+\xi_1)(4\lambda^2+\xi_1+\xi_2)(2\lambda^2-\eta_1+\xi_1+\xi_2)}\\
&\quad\times \hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\\
&= \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}^3}\frac{\lambda^4}{\left(2\lambda^2+\xi_1)(2\lambda^2+\frac{\eta_1}{2}+\frac{\eta_2}{2})(2\lambda^2+\eta_2\right)}\\
&\quad\times \hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\eta_1+\eta_2-\xi_1)\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2.
\end{aligned}$$

We note that

$$
\begin{aligned}
&\frac{1}{2}\Bigg[\frac{1}{\left(2\lambda^2+\frac{\eta_1}{2}+\frac{\eta_2}{2}\right)\left(2\lambda^2+\eta_1\right)}\\
&+\frac{1}{\left(2\lambda^2+\frac{\eta_1}{2}+\frac{\eta_2}{2}\right)\left(2\lambda^2+\eta_2\right)}\Bigg]\\
&=\frac{1}{(2\lambda^2+\eta_1)(2\lambda^2+\eta_2)}.
\end{aligned}
$$

We can take advantage of the symmetry between ξ_1, ξ_2 and η_1, η_2, then

$$
b_4(\lambda)=\frac{\mathrm{i}}{2\pi}\int_{\xi_1+\xi_2=\eta_1+\eta_2}\frac{\lambda^4}{(2\lambda^2+\xi_1)(2\lambda^2+\eta_1)(2\lambda^2+\eta_2)}\mathrm{Re}\,(\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2))\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2.
$$

Expanding equation (7.3) to the negative power, we have

$$
\begin{aligned}
b_4(\lambda)\sim{}&\frac{\mathrm{i}}{2\pi}\int_{\xi_1+\xi_2=\eta_1+\eta_2}\frac{1}{8\lambda^2}\\
&\times\sum_{j_1=0}^{\infty}\left(-\frac{\xi_1}{2\lambda^2}\right)^{j_1}\sum_{j_2=0}^{\infty}\left(-\frac{\eta_1}{2\lambda^2}\right)^{j_2}\sum_{j_3=0}^{\infty}\left(-\frac{\eta_2}{2\lambda^2}\right)^{j_3}\\
&\times\mathrm{Re}\,(\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2))\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2.
\end{aligned}
$$

Then, the corresponding coefficient of $i\frac{\lambda^{2-2j}}{2^{j+1}}$ as

$$
\begin{aligned}
H_{j4}:={}&\frac{1}{2\pi}\mathrm{Re}\sum_{\alpha_1+\alpha_2+\alpha_3=j-2}(-1)^j\\
&\times\int_{\xi_1+\xi_2=\eta_1+\eta_2}\xi_1^{\alpha_1}\eta_1^{\alpha_2}\eta_2^{\alpha_3}\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}(\eta_1)\hat{q}(\eta_2)\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2\\
={}&\frac{1}{2\pi}\mathrm{Re}\sum_{\alpha_1+\alpha_2+\alpha_3=j-2}(-1)^j\mathrm{i}^{2-j}\\
&\times\int_{\xi_1+\xi_2=\eta_1+\eta_2}(\mathrm{i}\xi_1)^{\alpha_1}(\mathrm{i}\eta_1)^{\alpha_2}(\mathrm{i}\eta_2)^{\alpha_3}\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}(\eta_1)\hat{q}(\eta_2)\mathrm{d}\xi_1\mathrm{d}\eta_1\mathrm{d}\eta_2\\
={}&\frac{1}{2\pi}\mathrm{Re}\sum_{\alpha_1+\alpha_2+\alpha_3=j-2}\mathrm{i}^{j-2}(-1)^{\alpha_1}\widehat{\overline{q^{(\alpha_1)}}}*\widehat{\overline{q}}*\widehat{q^{(\alpha_2)}}*\widehat{q^{(\alpha_3)}}(0)\\
={}&-\mathrm{Re}\left(\mathrm{i}^j\sum_{\alpha_1+\alpha_2+\alpha_3=j-2}(-1)^{\alpha_1}\int q^{(\alpha_2)}q^{(\alpha_3)}\overline{q^{(\alpha_3)}}q\mathrm{d}x\right).
\end{aligned}
$$

Thus, the proof is completed. □

Similarly, we provide an asymptotic expression for $b_6(\lambda)$.

Lemma 7.2. *We have the following identity:*

$$
\begin{aligned}
b_6(\lambda)={}&-\frac{\mathrm{i}}{4\pi^2}\int_{\xi_1+\xi_2+\xi_3=\eta_1+\eta_2+\eta_3}\frac{\lambda^6}{(2\lambda^2+\xi_1)(2\lambda^2+\xi_2)(2\lambda^2+\eta_2)(2\lambda^2+\eta_3)}\\
&\times\left(\frac{1}{2\lambda^2+\eta_1}+\frac{1}{2\lambda^2+\xi_1+\xi_2-\eta_1}\right)\\
&\times\hat{q}(\xi_1)\hat{q}(\xi_2)\hat{q}(\xi_3)\hat{q}^*(\eta_1)\hat{q}^*(\eta_2)\hat{q}^*(\eta_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3.
\end{aligned}
\tag{7.5}
$$

Suppose that q *is a Schwartz function. Then, we have the following asymptotic series:*

$$b_6(\lambda) \sim -\mathrm{i}\sum_{j=4}^{\infty} H_{j6}\frac{\lambda^{4-2j}}{2^{j+1}}, \tag{7.6}$$

where

$$\begin{aligned} H_{j6} = \mathrm{Re}\Bigg(&\mathrm{i}^j \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5=j-4} (-1)^{\alpha_1+\alpha_2} \\ &\times \int q^{(\alpha_1)}q^{(\alpha_2)}q\overline{q^{(\alpha_3)}q^{(\alpha_4)}q^{(\alpha_5)}} + q^{(\alpha_1)}q^{(\alpha_2)}q^*(q\overline{q^{(\alpha_4)}q^{(\alpha_5)}})^{\alpha_3}\mathrm{d}x\Bigg). \end{aligned}$$

Proof. According to theorem 4.1, we have

$$b_6(\lambda) = -4\lambda^6(XXYXYY + 3XXXYYY). \tag{7.7}$$

We calculate the two terms on the right side of equation (7.7).

$$\begin{aligned} -4\lambda^6 XXYXYY &= -4\lambda^6\int_{x_1<x_2<y_1<x_3<y_2<y_3} q(y_1)q^*(x_1)q(y_2)q^*(x_2)q(y_3)q^*(x_3) \\ &\times \mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2+y_3-x_1-x_2-x_3)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3 \\ &= -\frac{\lambda^6}{2\pi^3}\int_{\mathbb{R}^6}\left(\int_{x_1<x_2<y_1<x_3<y_2<y_3} \mathrm{e}^{\sum_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\right) \\ &\times\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3. \end{aligned} \tag{7.8}$$

Let

$$\begin{aligned} K(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3) &:= \int_{x_1<x_2<y_1<x_3<y_2<y_3} \mathrm{e}^{\sum_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{y_3}\int_{-\infty}^{y_2}\int_{-\infty}^{x_3}\int_{-\infty}^{y_1}\int_{-\infty}^{x_2} \mathrm{e}^{\sum_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3 \\ &= 2\pi i\frac{1}{2\lambda^2+\xi_1}\frac{1}{4\lambda^2+\xi_1+\xi_2} \\ &\times\frac{1}{2\lambda^2+\xi_1+\xi_2-\eta_1}\frac{1}{4\lambda^2+\xi_1+\xi_2+\xi_3-\eta_1} \\ &\times\frac{1}{2\lambda^2+\xi_1+\xi_2+\xi_3-\eta_1-\eta_2}\delta(\eta_1+\eta_2+\eta_3-\xi_1-\xi_2-\xi_3). \end{aligned} \tag{7.9}$$

Substituting equation (7.9) into equation (7.8) yields

$$\begin{aligned}
&-4\lambda^6 XXYXYY = -\frac{\lambda^6}{2\pi^3}\int_{\mathbb{R}^6} K(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)\\
&\quad\times\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\\
&\quad=-\frac{\mathrm{i}\lambda^6}{4\pi^2}\int_{\xi_1+\xi_2+\xi_3=\eta_1+\eta_2+\eta_3}\frac{1}{2\lambda^2+\xi_1}\frac{1}{2\lambda^2+\frac{1}{2}\xi_1+\frac{1}{2}\xi_2}\frac{1}{2\lambda^2+\xi_1+\xi_2-\eta_1}\\
&\quad\times\frac{1}{2\lambda^2+\frac{1}{2}\eta_2+\frac{1}{2}\eta_3}\frac{1}{2\lambda^2+\eta_3}\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\\
&=-\frac{\mathrm{i}\lambda^6}{4\pi^2}\int_{\xi_1+\xi_2+\xi_3=\eta_1+\eta_2+\eta_3}\frac{1}{2\lambda^2+\xi_1}\frac{1}{2\lambda^2+\xi_2}\frac{1}{2\lambda^2+\xi_1+\xi_2-\eta_1}\frac{1}{2\lambda^2+\eta_2}\\
&\quad\times\frac{1}{2\lambda^2+\eta_3}\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3.
\end{aligned}\tag{7.10}$$

On the other hand,

$$\begin{aligned}
&-12\lambda^6 XXXYYY = -12\lambda^6\int_{x_1<x_2<x_3<y_1<y_2<y_3} q(y_1)q^*(x_1)q(y_2)q^*(x_2)q(y_3)q^*(x_3)\\
&\quad\times\mathrm{e}^{2\mathrm{i}\lambda^2(y_1+y_2+y_3-x_1-x_2-x_3)}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3\\
&\quad=-\frac{3\lambda^6}{2\pi^3}\int_{\mathbb{R}^6}\left(\int_{x_1<x_2<x_3<y_1<y_2<y_3}\mathrm{e}^{\sum\limits_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum\limits_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\right)\\
&\quad\times\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3.
\end{aligned}\tag{7.11}$$

Let

$$\begin{aligned}
&H(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)\\
&\quad:=\int_{x_1<x_2<x_3<y_1<y_2<y_3}\mathrm{e}^{\sum\limits_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum\limits_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\\
&\quad=\int_{-\infty}^{+\infty}\int_{-\infty}^{y_3}\int_{-\infty}^{y_2}\int_{-\infty}^{y_1}\int_{-\infty}^{x_3}\int_{-\infty}^{x_2}\mathrm{e}^{\sum\limits_{j=1}^{3}\mathrm{i}(2\lambda^2+\eta_j)y_j-\sum\limits_{k=1}^{3}\mathrm{i}(2\lambda^2+\xi_k)x_k}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}y_3\\
&\quad=2\pi i\frac{1}{2\lambda^2+\xi_1}\frac{1}{4\lambda^2+\xi_1+\xi_2}\frac{1}{6\lambda^2+\xi_1+\xi_2+\xi_3}\frac{1}{4\lambda^2+\xi_1+\xi_2+\xi_3-\eta_1}\\
&\quad\times\frac{1}{2\lambda^2+\xi_1+\xi_2+\xi_3-\eta_1-\eta_2}\delta(\eta_1+\eta_2+\eta_3-\xi_1-\xi_2-\xi_3).
\end{aligned}\tag{7.12}$$

Substituting equation (7.12) into equation (7.11) yields

$$\begin{aligned}
&-12\lambda^6 XXXYYY = -\frac{3\lambda^6}{2\pi^3}\int_{\mathbb{R}^6} H(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3)\\
&\quad\times\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\\
&\quad=-\frac{\mathrm{i}\lambda^6}{4\pi^2}\int_{\xi_1+\xi_2+\xi_3=\eta_1+\eta_2+\eta_3}\frac{1}{2\lambda^2+\xi_1}\frac{1}{2\lambda^2+\frac{1}{2}\xi_1+\frac{1}{2}\xi_2}\frac{1}{2\lambda^2+\frac{1}{3}(\xi_1+\xi_2+\xi_3)}\\
&\quad\times\frac{1}{2\lambda^2+\frac{1}{2}\eta_2+\frac{1}{2}\eta_3}\frac{1}{2\lambda^2+\eta_3}\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3\\
&\quad=-\frac{\mathrm{i}\lambda^6}{4\pi^2}\int_{\xi_1+\xi_2+\xi_3=\eta_1+\eta_2+\eta_3}\frac{1}{2\lambda^2+\xi_1}\frac{1}{2\lambda^2+\xi_2}\frac{1}{2\lambda^2+\eta_1}\frac{1}{2\lambda^2+\eta_2}\\
&\quad\times\frac{1}{2\lambda^2+\eta_3}\hat{q}(\eta_1)\hat{q}(\eta_2)\hat{q}(\eta_3)\hat{q}^*(\xi_1)\hat{q}^*(\xi_2)\hat{q}^*(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3,
\end{aligned}\tag{7.13}$$

since

$$\begin{aligned}&\frac{1}{3}\left(\frac{1}{(2\lambda^2+\eta_1)(2\lambda^2+\eta_2)}+\frac{1}{(2\lambda^2+\eta_1)(2\lambda^2+\eta_3)}\right.\\&\left.+\frac{1}{(2\lambda^2+\eta_2)(2\lambda^2+\eta_3)}\right)\frac{1}{2\lambda^2+\frac{1}{3}(\eta_1+\eta_2+\eta_3)}\\&=\frac{1}{(2\lambda^2+\eta_1)(2\lambda^2+\eta_2)(2\lambda^2+\eta_3)}.\end{aligned}$$

Expanding equation (7.5) to the negative power, we can obtain equation (7.6). Thus, the proof is completed. ☐

Based on the above analysis, we provide an asymptotic estimate of b_{2j}.

Lemma 7.3. *The following estimate holds:*

$$b_{2j}(\lambda)\sim\mathcal{O}(\lambda^{-2j+2}),\quad j\geqslant 2.$$

Proof. Based on the properties of the Hopf algebra that we constructed earlier, we know that $b_{2j}(\lambda)'s$ are formal linear combinations of connected integrals. We then obtain this lemma from the properties of connected integrals. ☐

We recall that

$$s_2(\lambda)=-\frac{\mathrm{i}}{2}\|q(x)\|_{L^2}^2+\mathrm{i}\int_{-\infty}^{+\infty}\frac{\xi}{4\lambda^2+2\xi}|\hat{q}(\xi)|^2\mathrm{d}\xi. \tag{7.14}$$

According to lemma 7.3, we can obtain equations (1.10) and (1.11). Then, theorem 1.1 has been proven.

8. Expansions for the iterative integrals $b_{2j}(\lambda)$

The overall properties of $s_{2j}(\lambda)$ and $b_{2j}(\lambda)$ were given in the previous section, and the properties of $b_{2j}(\lambda)$ need to be considered separately in this section.

Lemma 8.1. $\mathrm{b}_{2\mathrm{j}}$ *has the following estimation:*

$$|\lambda^{-2j}b_{2j}(\lambda)|\lesssim\|\mathrm{e}^{\mathrm{iRe}\lambda^2x}q\|_{l_{\mathrm{Im}\,\lambda^2}^{2j}DU^2}^{2j}. \tag{8.1}$$

Proof. The proof is a direct consequence of theorem 6.1. ☐

Theorem 8.1. *Suppose that $q\in H^s$. Then, we have*

$$|b_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\zeta})|\lesssim\zeta^{j-2s-1}\|q\|_{H^s}^2\|q\|_{l_1^2DU^2}^{2j-2},\quad s\leqslant j-1$$

and

$$\begin{aligned}&\int_1^{\infty}\zeta^{2s-j}\left|b_{2j}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}}\right)\right|\mathrm{d}\zeta\lesssim 2^{-j}\left(1+\frac{1}{j-1-s}\right)\\&\times\|q\|_{H^s}^2\|q\|_{l_1^2DU^2}^{2j-2},\quad 0\leqslant s<j-1.\end{aligned}$$

Proof. First, we have

$$|b_{2j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\zeta})| \lesssim \zeta^{j} \sum_{k_1 \geqslant k_2} \|q_{k_1}\|_{l_{\zeta}^{2j}DU^2} \|q_{k_2}\|_{l_{\zeta}^{2j}DU^2} \|q_{\leqslant k_2}\|_{l_{\zeta}^{2j}DU^2}^{2j-2}. \tag{8.2}$$

As a proof method similar to theorem 6.2, first, if $1 \leqslant k < \zeta$, then we have

$$\|q_k\|_{l_{\zeta}^{2j}DU^2} \lesssim k^{\frac{1}{2}-s-\frac{1}{2j}} \zeta^{\frac{1}{j}-2} \|q\|_{H^s}, \tag{8.3}$$

and

$$\|q_{\leqslant k}\|_{l_{\zeta}^{2j}DU^2} \lesssim k^{1-\frac{1}{2j}} \zeta^{\frac{1}{2j}-1} \|q\|_{l_1^2 DU^2}. \tag{8.4}$$

If $k \geqslant \zeta$, then we have

$$\|q_k\|_{l_{\zeta}^{2j}DU^2} \lesssim k^{-s-\frac{1}{2}} \|q\|_{H^s} \tag{8.5}$$

and

$$\|q_{\leqslant k}\|_{l_{\zeta}^{2j}DU^2} \lesssim \|q\|_{l_1^2 DU^2}. \tag{8.6}$$

Then, we obtain

$$\begin{aligned} |s_{2j}\left(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}}\right)| &\lesssim \zeta^{j-2s-1} \\ &\times \sum_{k_1 \geqslant k_2} C(\zeta, k_1, k_2) \|q_{k_1}\|_{H^s} \|q_{k_2}\|_{H^s} \|q\|_{l_1^2 DU^2}^{2j-2}, \end{aligned} \tag{8.7}$$

where

$$C(\zeta, k_1, k_2) = \begin{cases} \left(\frac{k_1}{\zeta}\right)^{2(j-s-1)} \left(\frac{k_2}{k_1}\right)^{2j-s-\frac{5}{2}+\frac{1}{2j}}, & k_2 \leqslant k_1 \leqslant \zeta, \\ \left(\frac{\zeta}{k_1}\right)^{s+\frac{1}{2}} \left(\frac{k_2}{\zeta}\right)^{2j-s-\frac{5}{2}+\frac{1}{2j}}, & k_2 \leqslant \zeta \leqslant k_1, \\ \left(\frac{\zeta}{k_1}\right)^{s+\frac{1}{2}} \left(\frac{\zeta}{k_2}\right)^{s+\frac{1}{2}}, & \zeta \leqslant k_2 \leqslant k_1. \end{cases} \tag{8.8}$$

Then, by the Cauchy–Schwarz inequality and Schur's lemma, the critical coefficient obtained is $\frac{1}{j-1-s}$. Thus, the proof is completed. □

Given Σ_j a connected symbol of length $2j$, we study the asymptotic expressions of the following iterated integral:

$$T_{\Sigma_j}(\lambda) = \lambda^{2j} \int_{\Sigma_j} \prod_{k=1}^{j} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} q(y_k) q^*(x_k) \mathrm{d}x_1 \mathrm{d}y_1 \cdot \cdot\cdot \mathrm{d}x_j \mathrm{d}y_j.$$

Theorem 8.2. *The connected integrals* $T_{\Sigma_j}(\lambda)$ *have the following asymptotic expressions:*

$$T_{\Sigma_j}(\lambda) \sim \sum_{l=0}^{\infty} T_{\Sigma_j}^{l} 2^{1-2j-l} \lambda^{-(2j-2+2l)},$$

where

$$T_{\Sigma_j}^{l} = \sum_{|\alpha|+|\beta|=l} c_{\alpha\beta} \int \prod_{k=1}^{j} \partial^{\alpha_k} q_k^* \partial^{\beta_k} q_k \mathrm{d}x,$$

with

$$c_{\alpha\beta} = \frac{1}{\alpha!\beta!} \int_{\Sigma_j, x_1=0} \prod \mathrm{e}^{y_j - x_j} x_j^{\alpha} y_j^{\beta} \mathrm{d}x_j \mathrm{d}y_j,$$

and the errors in the above expansion have the following bounds:

$$\begin{aligned} &|T_{\Sigma_j}\left(\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\zeta}{\sqrt{2}}\right) - \sum_{l=0}^{k} T_{\Sigma_j}^{l} 2^{-j} i^{-(j-1+l)} \zeta^{-(2j-2+2l)}| \\ &\lesssim \sum_{k+1 \leqslant |\alpha|+|\beta| \leqslant 2j-1+k} 2^{-j} |\zeta^2|^{j-|\alpha|-|\beta|} \\ &\quad \times \prod_{k} \|\partial^{\alpha_k} q_k^*\|_{l_{\zeta^2}^{2j}DU^2} \|\partial^{\beta_k} q_k\|_{l_{\zeta^2}^{2j}DU^2}, \end{aligned}$$

where $\max\{\alpha_k, \beta_k\} \leqslant [\frac{k}{2}] + 1$ *and* $\zeta \geqslant 1$.

Proof. First, we have

$$\begin{aligned} T_{\Sigma_j}(\lambda) &= \lambda^{2j} \int_{\Sigma_j} \prod_{k=1}^{j} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} q(y_k) q^*(x_k) \mathrm{d}x_1 \mathrm{d}y_1 \cdot \cdot\cdot \mathrm{d}x_j \mathrm{d}y_j \\ &= \lambda^{2j} \int_{\Sigma_j} \prod_{k=1}^{j} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} \sum_{\beta_k=0}^{\infty} \frac{1}{\beta_k!} \partial^{\beta_k} q(x_1)(y_k - x_1)^{\beta_k} \\ &\quad \times \sum_{\alpha_k=0}^{\infty} \frac{1}{\alpha_k!} \partial^{\alpha_k} q^*(x_1)(x_k - x_1)^{\alpha_k} \mathrm{d}x_1 \mathrm{d}y_1 \cdot \cdot\cdot \mathrm{d}x_j \mathrm{d}y_j \\ &= \sum_{l=0}^{\infty} \lambda^{2j} \int_{\Sigma_j} \sum_{|\alpha|+|\beta|=l} \prod_{k=1}^{j} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} \frac{1}{\beta_k!} \partial^{\beta_k} q(x_1)(y_k - x_1)^{\beta_k} \\ &\quad \times \frac{1}{\alpha_k!} \partial^{\alpha_k} q^*(x_1)(x_k - x_1)^{\alpha_k} \mathrm{d}x_1 \mathrm{d}y_1 \cdot \cdot\cdot \mathrm{d}x_j \mathrm{d}y_j \\ &=: \sum_{l=0}^{\infty} \sum_{|\alpha|+|\beta|=l} T_{\Sigma_j}^{\alpha\beta}, \end{aligned} \tag{8.9}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_j)$, $\beta = (\beta_1, \beta_2, \cdots, \beta_j)$, $|\alpha| = \sum_{k=1}^{j} \alpha_k$, $|\beta| = \sum_{k=1}^{j} \beta_k$. For convenience, we redefine the following notations:

$$\{x_1, y_1, x_2, y_2, \cdots, x_j, y_j\}_{\Sigma_j} = \{t_1, t_2, t_3, t_4, \cdots, t_{2j-1}, t_{2j}\},$$
$$\{q, q^*, \cdots, q, q^*\}_{\Sigma_j} = \{v_1, v_2, \cdots, v_{2j-1}, v_{2j}\}.$$

Then, we note that

$$\begin{aligned} T_{\Sigma_j}^{\alpha\beta} &= \int_{\Sigma_j} \lambda^{2j} \prod_{k=1}^{j} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} \frac{1}{\beta_k!} \partial^{\beta_k} q(x_1)(y_k - x_1)^{\beta_k} \\ &\quad \times \frac{1}{\alpha_k!} \partial^{\alpha_k} q^*(x_1)(x_k - x_1)^{\alpha_k} \mathrm{d}x_1 \mathrm{d}y_1 \cdot \cdot\cdot \mathrm{d}x_j \mathrm{d}y_j \\ &= \int_{-\infty}^{+\infty} \int_{t_1}^{+\infty} \int_{t_2}^{+\infty} \cdots \int_{t_{2j-1}}^{+\infty} \lambda^{2j} \prod_{k=1}^{j-1} \mathrm{e}^{2\mathrm{i}\lambda^2(y_k - x_k)} \\ &\quad \times \frac{1}{\beta_k!} \partial^{\beta_k} q(t_1)(y_k - t_1)^{\beta_k} \\ &\quad \times \frac{1}{\alpha_k!} \partial^{\alpha_k} q^*(t_1)(x_k - t_1)^{\alpha_k} \mathrm{e}^{-2\mathrm{i}\lambda^2 x_j} \\ &\quad \times \frac{1}{\alpha_j!} \partial^{\alpha_j} q^*(t_1)(x_j - t_1)^{\alpha_j} \\ &\quad \times \mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j}} \frac{1}{\beta_j!} \partial^{\beta_j} q(t_1)(t_{2j} - t_1)^{\beta_j} \mathrm{d}t_1 \mathrm{d}t_2 \cdot \cdot\cdot \mathrm{d}t_{2j-1} \mathrm{d}t_{2j}. \end{aligned} \tag{8.10}$$

We first calculate the integral $T_{\Sigma_j}^{\alpha\beta}$ about t_{2j}, and we have

$$\begin{aligned}&\int_{t_{2j-1}}^{+\infty} \mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j}}\frac{1}{\beta_j!}\partial^{\beta_j}q(t_1)(t_{2j}-t_1)^{\beta_j}\mathrm{d}t_{2j}\\
&=-\frac{1}{2\mathrm{i}\lambda^2}\mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j-1}}\frac{1}{\beta_j!}\partial^{\beta_j}q(t_1)(t_{2j-1}-t_1)^{\beta_j}\\
&-\int_{t_{2j-1}}^{+\infty}\frac{1}{2\mathrm{i}\lambda^2}\mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j}}\frac{1}{(\beta_j-1)!}\partial^{\beta_j}q(t_1)(t_{2j}-t_1)^{\beta_j-1}\mathrm{d}t_{2j}\\
&=\sum_{k=1}^{\beta_j}(-1)^k\frac{1}{(2\mathrm{i}\lambda^2)^k}\mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j-1}}\\
&\times\frac{1}{(\beta_j-k+1)!}\partial^{\beta_j}q(t_1)(t_{2j-1}-t_1)^{\beta_j-k+1}\\
&+(-1)^{\beta_j}\int_{t_{2j-1}}^{+\infty}\frac{1}{(2\mathrm{i}\lambda^2)^{\beta_j}}\mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j}}\partial^{\beta_j}q(t_1)\mathrm{d}t_{2j}\\
&=\sum_{k=1}^{\beta_j+1}(-1)^k\frac{1}{(2\mathrm{i}\lambda^2)^k}\mathrm{e}^{2\mathrm{i}\lambda^2 t_{2j-1}}\\
&\times\frac{1}{(\beta_j-k+1)!}\partial^{\beta_j}q(t_1)(t_{2j-1}-t_1)^{\beta_j-k+1}.\end{aligned}\tag{8.11}$$

Thus, the corresponding terms in the errors are linear combinations of the following integrals:

$$\begin{aligned}R=&\frac{1}{2^j\zeta^{2j-2+2l}}\int_{t_1=\cdots=t_{j_-+1}<\cdots<t_{2j-j_+}=\cdots=t_{2j}}\mathrm{e}^{\zeta^2\sum_{i=1}^{j}x_i-y_i}\\
&\times\prod_{i=1}^{2j}\partial^{\alpha_i}v_i(t_i)\mathrm{d}t_{j_-+1}\cdot\cdot\cdot\mathrm{d}t_{2j-j_+},\end{aligned}$$

where

$$\begin{aligned}&0\leqslant j_-\leqslant\alpha_-=[\frac{k+2}{2}],\quad 0\leqslant j_+\leqslant\alpha_+=k+1\\
&-\left[\frac{k+2}{2}\right],\quad j_-+j_+\leqslant 2j-2,\\
&\sum_{i=1}^{1+j_-}\alpha_i=[\frac{k+1}{2}],\qquad\sum_{i=2j-j_+}^{2j}\alpha_i=[\frac{k+2}{2}].\end{aligned}$$

Similar to the proof method of theorem 6.1, we have

$$|R|\lesssim\frac{1}{2^j\zeta^{2j-2+2l}}\|v_-\|_{l_{\zeta^2}^{\frac{2j}{j_-+1}}DU^2}\prod_{i=2+j_-}^{2j-j_+-1}\|v_i\|_{l_{\zeta^2}^{2j}DU^2}\|v_+\|_{l_{\zeta^2}^{\frac{2j}{j_++1}}DU^2},$$

where

$$v_-=\prod_{i=1}^{1+j_-}\partial^{\alpha_i}v_i,\quad v_-=\prod_{i=2j-j_+}^{2j}\partial^{\alpha_i}v_i.$$

We note that

$$\begin{aligned}&\|q_1q_2\|_{l_{\zeta^2}^{p}DU^2}\lesssim\left\|\|\chi_{[\frac{k}{\zeta^2},\frac{k+1}{\zeta^2})}q_1\|_{V^2}\right\|_{l^q}\left\|\|\chi_{[\frac{k}{\zeta^2},\frac{k+1}{\zeta^2})}q_2\|_{DU^2}\right\|_{l^r},\\
&\frac{1}{p}=\frac{1}{q}+\frac{1}{r},\\
&\|q\|_{l_{\zeta^2}^{q}V^2}\lesssim\|q'\|_{l^qDU^2}+\zeta^2\|q\|_{l^qDU^2},\end{aligned}$$

where $p\geqslant 2$. Then, we have the following estimate:

$$\begin{aligned}\|v_-\|_{l_{\zeta^2}^{\frac{2j}{j_-+1}}DU^2}&\lesssim\|\partial^{\alpha_1}v_1\|_{l^{2j}DU^2}\\
&\times\prod_{i=2}^{j_--1}(\|\partial^{\alpha_i+1}v_i\|_{l^{2j}DU^2}+\frac{1}{\zeta^2}\|\partial^{\alpha_i}v_i\|_{l^{2j}DU^2}).\end{aligned}$$

We argue similarly for v_+. Thus, the proof is completed. □

Based on the above analysis, we obtain the following corollary.

Corollary 8.1. *The following estimate holds:*

$$\begin{aligned}&|T_{\Sigma_j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})-\sum_{l=0}^{k}T_{\Sigma_j}^{l}2^{-j}(i\zeta)^{-(j-1+l)}|\\
&\lesssim\sum_{k+1\leqslant|\alpha|+|\beta|\leqslant 2j-1+k}2^{-j}\zeta^{j-|\alpha|-|\beta|}\\
&\times\prod_k\|\partial^{\alpha_k}q_k^*\|_{l_{\zeta^2}^{2j}DU^2}\|\partial^{\beta_k}q_k\|_{l_{\zeta^2}^{2j}DU^2}.\end{aligned}$$

where $\max\{\alpha_k,\beta_k\}\leqslant[\frac{k}{2}]+1$ *and* $\zeta\geqslant 1$.

Theorem 8.3. *Let* $q(x)\in H^s(\mathbb{R})$ *and* $j-1+\frac{k_1}{2}\leqslant s\leqslant j-1+\frac{k_1+1}{2}$ $(j, k_1\in\mathbb{Z}^+)$. *Define the following iterated integral:*

$$T_{\Sigma_j}(\lambda)=\lambda^{2j}\int_{\Sigma_j}\prod_{k=1}^{j}\mathrm{e}^{2\mathrm{i}\lambda^2(y_k-x_k)}q(y_k)q^*(x_k)\mathrm{d}x_1\mathrm{d}y_1\cdot\cdot\cdot\mathrm{d}x_j\mathrm{d}y_j,$$

and

$$T_{\Sigma_j}^{l}=\sum_{|\alpha|+|\beta|=l}c_{\alpha\beta}\int\prod_{k=1}^{j}\partial^{\alpha_k}q_k^*\partial^{\beta_k}q_k\mathrm{d}x,$$

with

$$c_{\alpha\beta}=\frac{1}{\alpha!\beta!}\int_{\Sigma_j,x_1=0}\prod\mathrm{e}^{y_j-x_j}x_j^{\alpha}y_j^{\beta}\mathrm{d}x_j\mathrm{d}y_j.$$

Then, the following error estimates hold:

$$\begin{aligned}&|T_{\Sigma_j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})-\sum_{l=0}^{k_1}T_{\Sigma_j}^{l}2^{-j}(i\zeta)^{-(j-1+l)}|\\
&\lesssim 2^{-j}\zeta^{j-2s-1}\|q\|_{H^s}^2\|q\|_{l_1^2DU^2}^{2j-2},\end{aligned}$$

and

$$\begin{aligned}&\int_1^{+\infty}2^j\zeta^{2s-j}|T_{\Sigma_j}(\mathrm{e}^{\frac{\mathrm{i}\pi}{4}}\sqrt{\frac{\zeta}{2}})-\sum_{l=0}^{k_1}T_{\Sigma_j}^{l}2^{-j}(i\zeta)^{-(j-1+l)}|d\zeta\\
&\lesssim\frac{1}{|\sin(2\pi s)|}\|q\|_{H^s}^2\|q\|_{l_1^2DU^2}^{2j-2},\end{aligned}$$

where Σ_j *is an appropriate domain that obeys* $x_k<y_k$ *for all* k $(k\leqslant j)$.

Proof. According to corollary 8.1, we can show this theorem. □

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