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# Axionic excitation and vorticity in a QCD string model with anomaly inflow

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**Abstract**

A new phenomenological model (axionic QCD string) is constructed to study the topological issues of the QCD vacuum and hadron structure. It provides an alternative way of tackling the Strong CP problem, which is different from the traditional Peccei–Quinn approach. Neither new particle nor extra symmetry is introduced, and the role of the Peccei–Quinn axion is played by a quasiparticle arising from the phase of the quark condensate, dubbed as axionic excitation. The derivative of this excitation field is decomposed into a regular part and a singular part, and the latter contains vorticity from the string configuration. A hidden gauge symmetry is revealed in this decomposition and vorticity is represented by an emergent gauge field associated with anomalies. These components, together with the anomaly-inflow mechanism, complete the effective Lagrangian description for the axionic QCD string.

Keywords: QCD vacuum, axion string, axial anomaly, quark condensate, quantized vorticity

## 1. Introduction

The quantum chromodynamics (QCD) axion is a hypothetical particle beyond the Standard Model, which was postulated to resolve the 'Strong CP problem' in 1977 [1–3]. It is also one of the leading particle candidates to provide dark matter in the cosmos [4–6]. The axion scenario, and the associated Peccei–Quinn mechanism, are certainly elegant solutions—to explain why the effective $\bar{\theta}$ parameter[3] is so small, i.e., $\bar{\theta} \leqslant 10^{-10} \sim 10^{-11}$ from the measurement of the neutron electric dipole moment [7, 8], while the Standard Model specifies no preferred value, Peccei and Quinn promote $\bar{\theta}$ to a new field, the axion field $a(x)$, with energetically favorable value $a = 0$ which follows from the Peccei–Quinn symmetry and its breaking at the low energy scale [1]. However, after having searched for nearly half a century, neither the axion nor axion-like particles (ALPs) are found in any laboratory experiments or celestial observations [2–6].

The missing axion particle motivates us to look for alternative solutions to the Strong CP problem, and probably even to the related $U_A(1)$ problem [9–11]. This relies on our understanding of the QCD vacuum and the hadron properties, especially the challenging questions such as chiral symmetry breaking, confinement and deconfinement. Take the instanton approach as an example: it can tackle the $U_A(1)$ problem of chiral symmetry breaking, but is not so efficient in explaining the confinement question. In our opinion, the correct and complete solution should cover all aspects of the QCD vacuum structure and hadron properties, not just one of them. Therefore our strategy is to start with models that can describe confinement at least phenomenologically, and then study the chiral symmetry breaking and anomalies, so finally it will be able to address the hadron spectrum issues like the mass of $\eta'$ (the $U_A(1)$ problem). Based on the chromoelectric flux-tube model [12–14], we previously incorporated the quark condensate as an order parameter for chiral symmetry breaking and obtained a new topological term that couples the derivative of the phase of the quark condensate to the Chern–Simons current [15–18]. This derivative coupling was originally obtained by Callan and Harvey [19] in studying

[3] The effective vacuum angle $\bar{\theta}$ is related to the QCD $\theta$-parameter as $\bar{\theta} = \theta + \mathrm{Arg}\det M$, where $M$ is the quark mass matrix.

anomaly inflow on topological defects such as strings and domain walls, which might appear in the QCD vacuum.[4] Although it can be connected to the usual topological charge, this coupling corresponds to a totally different configuration without the $\theta$ parameter. Our observation is that the phase of the quark condensate might play a role similar to the axion, and consequently, the axion and the new symmetry beyond the Standard Model, such as Peccei–Quinn symmetry [1], are not needed in our model.

In this paper, we focus on the phase of the quark condensate, identify an 'axionic excitation' and build an effective Lagrangian to describe an axionic QCD string model. An important step in this process is that we decompose the derivative of the phase into a regular part and a singular part [24], and treat the singular part as an emergent gauge field [24, 25], which measures vorticity and enstrophy if the singular part has features of vortex configurations. With the help of this decomposition, the new topological term is capable of addressing the axial anomaly, including the $U_A(1)$ problem without bringing in the Strong CP problem. In other words, what is needed to resolve the Strong CP problem is probably not to find a new fundamental particle (the axion), but an axionic excitation (quasiparticle) from the quark condensate.

## 2. Quark condensate, emergent gauge field and vorticity

As one of the characterizing features of the QCD vacuum, the quark condensate is usually treated as a constant [26]. This is quite different from how we usually consider the condensates in condensed matter physics [27, 28]. For instance, in quantum fluids such as the Bose–Einstein condensate (BEC) of cold atoms, superfluid helium and the Cooper pairs of electrons in superconductors, condensates are described by macroscopic wave functions or order parameters, which satisfy certain nonlinear differential equations such as the Gross–Pitaevskii equation and the Ginzburg–Landau equation. Therefore, it is natural to consider inhomogeneous quark condensates in general, and the physical laws governing their spatial and temporal variations. There are also some similar examples in QCD—the chiral condensate may be spatially modulated at high densities (see [29] for a review); In some flux-tube models of hadrons [30], the chiral condensate is considered strongly distorted near the hadron and vanishes inside the hadron, which suggests a restoration of chiral symmetry in there, provided that the chromoelectric field is strong enough to pull the quark–antiquark pairs apart. Therefore it is legitimate to introduce a complex scalar field,

$$\hat{\Phi}(x) = \Phi(x) + \hat{\varphi}(x), \tag{1}$$

to describe the quark condensate $\Phi(x)$ and the fluctuation $\hat{\varphi}(x)$. The macroscopic wave function $\Phi(x)$, also called the order parameter of the system,

$$\Phi(x) \equiv F(x)\,\mathrm{e}^{\mathrm{i}\sigma(x)}, \tag{2}$$

is a complex function with $F(x)$ and $\sigma(x)$ being its magnitude and phase, respectively. They are related to the quark condensate as

$$\langle \bar{q}_R^i q_L^j \rangle = -\Phi(x)\,\delta^{ij} = -F(x)\,\mathrm{e}^{\mathrm{i}\sigma(x)}\,\delta^{ij}. \tag{3}$$

Note that here we *do not assume that the QCD vacuum has a definite parity*, hence $\Phi$ is complex in general [31], similar to the aforementioned cases of BEC, superfluidity and superconductivity in which the order parameters are all complex functions [27, 28]. This is because the standard QCD Lagrangian with the $\theta$-term reads,

$$\mathcal{L}_{\mathrm{QCD}} = \mathcal{L}_{\mathrm{QCD}}^{(\theta=0)} + \frac{g_3^2}{64\pi^2}\theta\, G_{\mu\nu}^a \tilde{G}^{a\mu\nu}, \tag{4}$$

where $G_{\mu\nu}^a$ is the gluon field strength, and it is well known that the first term above is $P$- and $CP$-even while the second term (topological charge) is $P$- and $CP$-odd, so the QCD vacuum as an eigenstate of the QCD Hamiltonian is not a $P$ and $CP$ eigenstate, hence does not have a definite parity in general if we start with the above QCD Lagrangian.

The chiral symmetry group is $SU_V(N_f) \otimes SU_A(N_f) \otimes U_V(1) \otimes U_A(1)$, where $N_f$ is the number of light quark flavors. Under the $U_A(1)$ transformation the quark fields transform as,

$$q_L \to \mathrm{e}^{-\mathrm{i}\xi/2} q_L, \qquad q_R \to \mathrm{e}^{\mathrm{i}\xi/2} q_R, \tag{5}$$

hence the scalar fields $\Phi$ and $\Phi^*$ transform as, respectively,

$$\Phi \to \mathrm{e}^{-\mathrm{i}\xi}\Phi, \qquad \Phi^* \to \mathrm{e}^{\mathrm{i}\xi}\Phi^*. \tag{6}$$

Thus, $\Phi$ does transform under the $U_A(1)$ symmetry while stays invariant under the $U_V(1)$ symmetry. For simplicity, we also take the same value for different quark condensates (in general we should put multiple $\Phi$ fields in a matrix form, like the chiral field in the chiral perturbation theory). Suppose that as an order parameter the function $\Phi$ is described by the Lagrangian density

$$\mathcal{L}_0(\Phi, \Phi^*) = -(\partial^\mu + \mathrm{i}S^\mu)\Phi^*(\partial_\mu - \mathrm{i}S_\mu)\Phi - V(\Phi^*, \Phi), \tag{7}$$

where $S_\mu$ is some external gauge field or source (e.g. those external currents used in the chiral Lagrangian approach),[5] and $V(\Phi^*, \Phi)$ a nonlinear potential of $\Phi\Phi^*$ (e.g. a Higgs-type potential [32]). Now we decompose the derivative of the phase of $\Phi$ into two parts,

$$\partial_\mu \sigma = \partial_\mu \vartheta + X_\mu, \tag{8}$$

where the scalar $\vartheta(x)$ is the smooth part of the phase function $\sigma$ and the vector $X_\mu$ is the singular part.[6] For topological defects like vortices, a rigorous definition for the

[4] For example, the 'spaghetti vacuum' [20, 21] and the center vortices [22, 23] are such vortex configurations for studying the confinement problem.

[5] Note that if $S_\mu$ is introduced as a gauge field, then it should be the gauge field of some other symmetry (e.g. the electromagnetic potential $A_\mu$), instead of the global $U_A(1)$ symmetry. One may set $S_\mu = 0$ for simplicity and all the following steps can proceed without the presence of $S_\mu$.

[6] As it will be shown later, with a 'gauge fixing' condition, the decomposition (8) is simply a relativistic generalization of the usual Helmholtz decomposition for velocities. In [24] it has been applied to the relativistic superfluid case to separate vorticity from regular flow.

'smoothness' and 'singularity' is

$$[\partial_\mu, \partial_\nu]\,\vartheta = 0, \qquad \text{while} \quad [\partial_\mu, \partial_\nu]\,\sigma = \partial_\mu X_\nu - \partial_\nu X_\mu \equiv X_{\mu\nu} \neq 0. \tag{9}$$

Therefore, the tensor $X_{\mu\nu}$ measures the vorticity associated with the singularity of the topological defect in the condensate. Note that there is a 'symmetry' in the decomposition of $\partial_\mu \sigma$ in equation (8)—the phase derivative $\partial_\mu \sigma$ is invariant under the following transformation,

$$\vartheta \to \vartheta + \alpha, \qquad X_\mu \to X_\mu - \partial_\mu \alpha, \tag{10}$$

where $\alpha$ is an arbitrary smooth function. Such a type of 'hidden' symmetry also occurs in the slave–boson method in condensed matter physics [25, 33]. Interestingly, this 'gauge' symmetry in decomposing $\partial_\mu \sigma$ leads to interpretation of the vector field $X_\mu$ as an *emergent gauge field*, based on the following observation: If we define a 'smooth' order parameter, $\phi$, with $\vartheta$ being its phase and $F$ its amplitude, respectively,

$$\phi \equiv F(x)\,\mathrm{e}^{\mathrm{i}\vartheta(x)}, \tag{11}$$

then the original Lagrangian (7) can be rewritten in terms of $\phi$ and $X_\mu$,

$$\mathcal{L}_0 = -(\partial^\mu + \mathrm{i}S^\mu - \mathrm{i}X^\mu)\phi^* (\partial_\mu - \mathrm{i}S_\mu + \mathrm{i}X_\mu)\phi - V(\phi^*, \phi), \tag{12}$$

with an emergent gauge symmetry

$$\phi \to \phi' = \mathrm{e}^{\mathrm{i}\alpha}\phi$$
$$X_\mu \to X'_\mu = X_\mu - \partial_\mu \alpha. \tag{13}$$

Now the hidden symmetry in the decomposition of $\partial_\mu \sigma$ (8) has become explicit with the help of the new order parameter $\phi$. One can impose extra constraint(s) on the decomposition (8) such that it resembles the Helmholtz decomposition of a vector field. In that case, the extra constrain(s) will play the role of 'gauge fixing' condition for the transformation (13). For example, a possible choice of gauge fixing reads

$$\partial^\mu X_\mu = 0, \tag{14}$$

which is similar to the Lorentz gauge in quantum electrodynamics. This gauge is particularly suitable for the dual string picture in which $X_\mu$ is expressed in terms of the Kalb–Ramond potential $B^{\mu\nu}$ [24, 34–37],

$$X_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial^\nu B^{\lambda\rho}. \tag{15}$$

In this gauge, it is easy to see that equation (8) generalizes the usual Helmholtz decomposition for a three-dimensional vector $\boldsymbol{v}$, i.e.,

$$\boldsymbol{v} = \boldsymbol{v}_\parallel + \boldsymbol{v}_\perp, \qquad \nabla \times \boldsymbol{v}_\parallel = \nabla \cdot \boldsymbol{v}_\perp = 0, \tag{16}$$

since the 'longitudinal' component $\partial_\mu \vartheta$ is irrotational due to $[\partial_\mu, \partial_\nu]\,\vartheta = 0$, while the 'transverse' component $X_\mu$ is divergenceless by definition (for a regular Kalb–Ramond potential $B^{\mu\nu}$) as it can be seen from the dual representation

$$\partial^\mu X_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial^\mu\partial^\nu B^{\lambda\rho} = 0. \tag{17}$$

Here some remarks are in order: we have decomposed the derivative of the phase of quark condensate for a vortex configuration, $\partial_\mu \sigma = \partial_\mu \vartheta + X_\mu$, satisfying $[\partial_\mu, \partial_\nu]\,\vartheta = 0$, $\partial_\mu X_\nu - \partial_\nu X_\mu \neq 0$; Because of this decomposition, a symmetry emerges (equation (13)) and $X_\mu$ is interpreted as an emergent gauge field, which describes the vortex configuration; Interestingly, choosing the Lorentz gauge $\partial^\mu X_\mu = 0$ naturally leads to an identification $X_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial^\nu B^{\lambda\rho}$, which connects the emergent gauge field to the Kalb–Ramond potential $B^{\mu\nu}$—that is precisely what we expect from the field-string duality point of view[7].

The emergent gauge field $X_\mu$ suggests that a theory with global symmetry of field $\Phi$ in (7) is equivalent to another theory with local gauge symmetry of fields $\phi$ and $X_\mu$ in (12). Of course, this equivalence relies on certain conditions which make the decomposition possible, e.g., vortex-like topological defects existing in the condensate as shown from equation (8) to equation (13). The above procedure effectively separates the singularities of the theory from the regular part of the phase derivative. Such singularities occur in real physical systems, e.g. superfluid $^4$He and BEC of cold atoms [27, 28], where quantized vortices appear in these quantum fluids so we have to distinguish the rotational flow from the potential flow, or the turbulent flow from the laminar flow in more complicated circumstances. Therefore this emergent gauge field provides a new method to study excitations with singularities, which can be applied to the case of quark condensates.

For quantum vortices, $X_\mu$ is further restricted by the quantized circulation condition,

$$\oint_C X_\mu \mathrm{d}x^\mu = 2\pi n \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{18}$$

where $C$ refers to a closed loop around the vortex configuration. This condition follows from the fact that both $\sigma$ and $\vartheta$ are the phase angle of (macroscopic) wave functions. Noticing that the LHS of equation (18) has the same form as the Wilson loop, $\oint_C A_\mu \mathrm{d}x^\mu$, we can consider this constraint as a *quantized Wilson loop* condition for the emergent gauge field $X_\mu$. Furthermore, $X_\mu$ can be used to capture the vorticity feature in other physical quantities such as currents and energy-momentum tensors. As mentioned earlier, the $U_A(1)$ symmetry of the Lagrangian (7) leads to an axial current

$$J^A_\mu \equiv -\frac{\mathrm{i}}{2}(\Phi^*\partial_\mu\Phi - \Phi\partial_\mu\Phi^*) = F^2\partial_\mu\sigma, \tag{19}$$

which can be decomposed into a smooth part and a singular

[7] The so-called 'QCD string' usually refers to a reformulation of Yang-Mills gauge theory as a string theory, assumed to exist at least at the large-$N_c$ limit, where $N_c$ is the number of colors. In this paper, we do not distinguish in terminology 'string' from 'flux tube', since, obviously, we are not considering quantized strings in higher-dimensional spacetimes as in its modern version. The duality between $X_\mu$ and the Kalb–Ramond potential $B^{\mu\nu}$ also holds in string field theory [24, 35–37].

part as well,

$$J^A_\mu = J^A_{\vartheta\,\mu} + J^A_{X\,\mu},$$
$$J^A_{\vartheta\,\mu} \equiv F^2 \partial_\mu \vartheta,$$
$$J^A_{X\,\mu} \equiv F^2 X_\mu. \tag{20}$$

At the classical level, it is easy to see that $J^A_\mu$ is conserved, i.e., $\partial^\mu J^A_\mu = 0$; at the quantum level, it is well known that the axial anomaly occurs and plays a key role in breaking the $U_A(1)$ symmetry. In our formulation, this axial anomaly is obtained via a new mechanism in which we will see how the above axial current is coupled to a Chern–Simons current of gluons. Besides the current $J^A_\mu$, we also need the energy-momentum tensor to describe the condensate as a relativistic quantum fluid [27, 28]. The chemical potential is defined as $\mu \equiv \sqrt{-\partial^\tau \sigma \partial_\tau \sigma}$ and a 4-velocity is introduced [27, 28, 32],

$$U_\tau = \frac{1}{\mu}\partial_\tau \sigma, \qquad U^\tau U_\tau = -1. \tag{21}$$

Then the stress-energy tensor can be put in the usual hydrodynamical form

$$T_{\mu\nu} = T^F_{\mu\nu} + \epsilon\, U_\mu U_\nu + p(g_{\mu\nu} + U_\mu U_\nu), \tag{22}$$

where $U_\tau = \frac{1}{\mu}(\partial_\tau \vartheta + X_\tau)$, $\epsilon \equiv \mu^2 F^2 + V(F^2)$ and $p \equiv \mu^2 F^2 - V(F^2)$ are the total energy density and the pressure of the system, respectively, and the tensor $T^F_{\mu\nu} \equiv 2\partial_\mu F \partial_\nu F - g_{\mu\nu}\partial^\tau F \partial_\tau F$ vanishes for constant $F$. However, for vortex configurations which will be discussed in the next section, the magnitude of the field $F$ changes drastically in the core of the vortices. Thus the $T^F_{\mu\nu}$ part contributes significantly in that case. One can compute other hydrodynamical quantities like a twist tensor, shear tensor and expansion scalar. Note that with the vorticity tensor $X_{\mu\nu} = \partial_\mu X_\nu - \partial_\nu X_\mu$ one can build a scalar

$$\omega^2 \equiv X^{\mu\nu} X_{\mu\nu}, \tag{23}$$

which works as the kinetic term for $X_\mu$ if we consider $X_{\mu\nu}$ as the 'field strength' of $X_\mu$. The quantity $\omega^2$ can also be thought of as the relativistic generalization of the enstrophy density in the hydrodynamics of vortices and hence has a distinct physics meaning.

## 3. Axionic QCD string and anomaly-inflow mechanism

To construct our axionic QCD string model, we propose an effective Lagrangian as follows,

$$\mathcal{L}^0_{\rm eff} = -\tfrac{1}{4}\kappa\, G^a_{\mu\nu} G^{a\mu\nu} - \partial^\mu \Phi \partial_\mu \Phi^* - V(\Phi, \Phi^*)$$
$$+\bar{q}[\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}g G^a_\mu T^a) - g_Y(\Phi_1 + \mathrm{i}\gamma^5 \Phi_2)]q, \tag{24}$$

where $\Phi_1$, $\Phi_2$ are the real and imaginary parts of $\Phi$, respectively; $G^a_\mu$ are the gluon fields and $T^a$ are the generators of the $SU(N_c)$ group. The function $\kappa = \kappa(\Phi, \Phi^*)$ describes the color electric and magnetic polarization properties of the QCD vacuum as a physical medium, similar to the bag model and the flux-tube models (see e.g. [12–16, 30]). For instance, one may choose $\kappa = 1 - \sqrt{\Phi\Phi^*}/F_0$ for the bag model [12]. For a vortex or string configuration, we first need to check the boundary condition of $\Phi$. For a single straight vortex-line configuration with winding number $m \in \mathbb{Z}$, we take the amplitude and the phase of $\Phi$ to be, respectively,

$$F = F(\rho), \qquad \sigma = m\theta, \qquad m = \pm 1, \pm 2, \cdots, \tag{25}$$

with boundary condition

$$F(0) = 0, \qquad \text{and} \qquad F(\infty) \longrightarrow F_0 = \text{constant}, \tag{26}$$

where $\rho$ and $\theta$ are the polar coordinates describing the cross section of the straight vortex line. The constant $F(\infty)$ comes naturally from the vacuum expectation value of the quark condensate, and then the question is whether $F(0) = 0$ can be satisfied inside the flux tube. This has been studied before in the context of chiral symmetry restoration [15, 16, 30]. The result is that when the chromoelectric field exceeds a critical strength, (see e.g. a numerical simulation by Suganuma and Tatsumi [30]: $E_{\rm crit} \approx 4.0$ GeV/fm, $E \approx 5.3$ GeV/fm $> E_{\rm crit}$), the chiral symmetry is restored inside the flux tube. The chromoelectric field, if strong enough, can pull quark and antiquark pairs apart, hence destroying the quark condensate. Therefore, the quark condensate vanishes inside the flux tube, $\langle \bar{q}q \rangle = 0$, while outside the flux tube, the QCD vacuum has non-vanishing $\langle \bar{q}q \rangle \neq 0$. This shows that it is possible for the quark condensate to satisfy the boundary condition of a vortex configuration. It is actually not a surprising result from the original dual (color) superconductor picture of the QCD vacuum. One may consider it as a *type-II color superconducting vacuum*, which allows chromoelectric field lines to penetrate and squeeze them into a flux tube, then it can be used to study the confinement problem. We further assume that the vortex configuration is energetically favored, in analogy to the appearance of the Abrikosov vortices in the type-II superconductors in an external magnetic field [27, 28].

Another ingredient for characterizing the QCD string in our model is to apply an abelian projection of the gluon potential, which aims to tackle the confinement problem. In the present work, we will not repeat this projection technique (see [15, 16] for details and [38] for a review), since our focus here is not the confinement, but the axionic excitation and vorticity in the quark condensate which are independent of particular gluon projections, as it should be.

With this vortex configuration of the quark condensate $\Phi$ and the previous string and flux tube models for the hadrons [15, 16], the effective Lagrangian (24) is capable of describing an axionic QCD string. It contains the gluon potential $G^a_\mu$, the quarks $q$, and the order parameter of the quark condensate $\Phi$. In this model, the *axionic excitation is not introduced as a new particle, but as an excitation of the quark condensate, or a quasiparticle* from the condensed matter physics point of view. We then study the Dirac equation for the quarks, treating $\Phi$ and $G^a_\mu$ as background fields, and what follows is similar to the anomaly-inflow scenario proposed by Callan and Harvey [19]. It is easy to see that chiral zero modes of quarks are localized in the vortex since they have an exponential profile

$\psi_L = \chi_L \exp[-\int_0^\rho F(\rho')\mathrm{d}\rho']$, where $\chi_L$ is a two-dimensional spinor. The chiral zero modes of quarks are also coupled to the gluon potential $G_\mu^a$, bringing in a two-dimensional gauge anomaly on the string. To cancel this gauge anomaly, Callan and Harvey find an effective action [19]

$$S_{\mathrm{C-S}} = -\frac{g^2}{16\pi^2}\int \mathrm{d}^4x\, \partial_\mu\sigma\, K^\mu, \tag{27}$$

where $K^\mu \equiv 2\epsilon^{\mu\nu\lambda\rho}\,\mathrm{Tr}(G_\nu\partial_\lambda G_\rho + \frac{2}{3}G^\nu G^\lambda G^\rho)$ is the Chern–Simons current of gluons. The cancellation happens because the massive quark modes that live off the vortex mediate an effective interaction between the quark condensate and the gluon field, hence inducing a vacuum current

$$\langle J^{\mu a}_{\mathrm{ind}}\rangle = \frac{g^2 N_f}{8\pi^2}\epsilon^{\mu\nu\rho\tau}\partial_\nu\sigma G^a_{\rho\tau}. \tag{28}$$

Converting it to an effective action one obtains (27). With this topological term included, the effective Lagrangian becomes

$$\begin{aligned}\mathcal{L}^1_{\mathrm{eff}} = &-\frac{1}{4}\,\kappa(F)\quad G^a_{\mu\nu}G^{a\mu\nu} - \partial^\mu F\partial_\mu F - V(F^2)\\ &+ \bar{q}[\mathrm{i}\gamma^\mu(\partial_\mu - \mathrm{i}gG^a_\mu T^a) - g_Y F\mathrm{e}^{\mathrm{i}\sigma\gamma^5}]q - F^2\partial^\mu\sigma\partial_\mu\sigma\\ &- \frac{g^2}{16\pi^2}\,\partial_\mu\sigma\, K^\mu.\end{aligned} \tag{29}$$

How the topological term (the third line in equation (29)) is related to an alternative solution to the $U_A(1)$ problem has been discussed in [15–18]. Due to the decomposition (8), this term can be rewritten as,

$$\partial_\mu\sigma\, K^\mu = \partial_\mu\vartheta\, K^\mu + X_\mu\, K^\mu. \tag{30}$$

It is tempting to consider the connection between the $\vartheta$ field and the usual $\theta$ parameter in QCD. After integration by parts, the first term in the above equation becomes the familiar form of the topological charge,

$$\partial_\mu\vartheta\, K^\mu \sim \vartheta\, \partial_\mu K^\mu \sim \vartheta\, G^a_{\mu\nu}\tilde{G}^{a\mu\nu}. \tag{31}$$

However, we recall that $\vartheta$ is a well-defined smooth function modulo $2\pi$, and transforms according to (10), thus any particular value has no physical meaning for $\vartheta$. This can also be seen from the fact that $\partial_\mu\vartheta\, K^\mu$ is a derivative coupling which is invariant under a shift of $\vartheta$ by any constant, i.e., $\vartheta \to \vartheta + \vartheta_0$. This can shed light on finding a new solution to the Strong CP problem, since if other possible topological defects, e.g. instantons [39, 40, 41], also contribute such a term but with a constant coefficient, like $\theta_0 G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$, then we can use the shift symmetry to absorb it. This novel absorbing mechanism results clearly from the derivative type coupling in equation (30).

The vorticity term $X_\mu\, K^\mu$ plays a crucial role in canceling the lower dimensional gauge anomaly on the axionic string via the anomaly inflow mechanism [19]. Note that in our model, $X_\mu$ is not a derivative and the term $X_\mu K^\mu$ cannot be integrated by parts in the same way as the term $\partial_\mu\vartheta\, K^\mu$ does. However, when studying its gauge variation of the color symmetry, we integrate by parts the other way around, such that a derivative acts on $X_\mu$ (or $\partial_\mu\sigma$) in the form of

$$\partial_\mu X_\nu - \partial_\nu X_\mu = [\partial_\mu, \quad \partial_\nu]\sigma = X_{\mu\nu} \neq 0, \tag{32}$$

which functions as a projection of a higher-dimensional quantity into its lower dimensional correspondent. We now demonstrate it via the so-called *descent equation* [42, 43], which relates the chiral anomaly in $(2n+2)$-dimensions to the gauge anomaly in $2n$-dimensions. For simplicity in notations, we express the terms in the coupling (30) via differential forms

$$\begin{aligned}&\partial_\mu\sigma\mathrm{d}x^\mu \to \mathrm{d}\sigma, \qquad \partial_\mu\vartheta\mathrm{d}x^\mu \to \mathrm{d}\vartheta,\\ &X_\mu\mathrm{d}x^\mu \to X, \qquad K^\mu \to \mathcal{K}^0_3,\end{aligned} \tag{33}$$

where $\mathcal{K}^0_3$ is the Chern–Simons 3-form and all others are 1-forms Under a gauge transformation of the color symmetry group

$$\begin{aligned}\delta\int_M \mathrm{d}\sigma\wedge\mathcal{K}^0_3 &= \int_M \mathrm{d}\sigma\wedge\mathrm{d}\mathcal{K}^1_2\\ &= -\int_M \mathrm{d}^2\sigma\wedge\mathcal{K}^1_2\\ &= -\int_M \mathrm{d}X\wedge\mathcal{K}^1_2,\end{aligned} \tag{34}$$

where we have used the decomposition $\mathrm{d}\sigma = \mathrm{d}\vartheta + X$, the smooth condition of the function $\vartheta(x)$, i.e. $\mathrm{d}^2\vartheta = 0$,[8] and the equation

$$\delta\mathcal{K}^0_3 = \mathrm{d}\,\mathcal{K}^1_2, \tag{35}$$

which connects the Chern–Simons 3-form $\mathcal{K}^0_3$ to the lower dimensional gauge anomaly $\mathcal{K}^1_2$. Equation (35) is one of the descent equations [42, 43]

$$\delta\mathcal{K}^{i-1}_{2n-i} = \mathrm{d}\,\mathcal{K}^i_{2n-i-1}. \tag{36}$$

For the straight vortex configuration centered on the $z$-axis (denoted by $S$), its vorticity is given by

$$\mathrm{d}X = 2\pi\delta(x)\delta(y)\mathrm{d}x\wedge\mathrm{d}y, \tag{37}$$

where the projecting feature is carried by the $\delta$-functions. Thus, we obtain the cancellation of the lower dimensional gauge anomaly

$$\delta\int_M \mathrm{d}\sigma\wedge\mathcal{K}^0_3 = -\int_S \mathcal{K}^1_2, \tag{38}$$

such that the whole theory contains no gauge anomaly.

The quark zero modes are also coupled to the electromagnetic field $A_\mu$, which leads to the gauge anomaly associated with electromagnetism. The same anomaly inflow mechanism applies, therefore we have another effective coupling term

$$\mathcal{S}_{\mathrm{MCS}} = -\frac{e^2}{8\pi^2}\int \mathrm{d}^4x\, \partial_\mu\sigma\, K^\mu_{\mathrm{MCS}}, \tag{39}$$

where $K^\mu_{\mathrm{MCS}} = \epsilon^{\mu\nu\rho\tau}A_\nu F_{\rho\tau}$ is the Maxwell–Chern–Simons

[8] Note that in contrast to the identity $\mathrm{d}^2\vartheta = 0$ for the smooth function $\vartheta$, $\mathrm{d}^2\sigma \neq 0$ holds due to the singularity contained in the axionic field $\sigma$, more precisely, $\mathrm{d}^2\sigma = \mathrm{d}X$.

current. Including all relevant terms we finally obtain

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & -\kappa(F)/4 \quad G^{a}_{\mu\nu}G^{a\mu\nu} - 1/4 \quad F_{\mu\nu}F^{\mu\nu} \\ & - \partial^{\mu}F\partial_{\mu}F - V(F^2) \\ & + \bar{q}[\mathrm{i}\gamma^{\mu}(\partial_{\mu} - \mathrm{i}gG^{a}_{\mu}T^{a} - \mathrm{i}eA_{\mu}) - g_{Y}F\mathrm{e}^{\mathrm{i}\sigma\gamma^5}] \\ & \times q - F^2\partial^{\mu}\sigma\partial_{\mu}\sigma \\ & - (g^2/16\pi^2)\,\partial_{\mu}\sigma\, K^{\mu} - (e^2/8\pi^2)\,\partial_{\mu}\sigma\, K^{\mu}_{\text{MCS}}\,, \end{aligned} \quad (40)$$

which completes our effective Lagrangian for phenomenologically describing an axionic QCD string. Note that as we emphasized earlier, an abelian projection of the gluon fields is needed to explore confinement phenomena, which is not the focus of the present paper (see [15, 16, 38] for details).

The coupling term $\sim\partial_{\mu}\sigma\, K^{\mu}_{\text{MCS}}$ in the effective Lagrangian (40) can be immediately applied to the chiral magnetic effect (CME) (see e.g. [44–46])—charge separation can occur in a background (electromagnetic) magnetic field, and consequently, an electromagnetic current is generated along the magnetic field. One can readily derive a modified Maxwell equation based on the effective Lagrangian (40),

$$\partial_{\mu}F^{\mu\nu} = J^{\nu} - \frac{e^2}{2\pi^2}\,\partial_{\mu}\sigma\,\tilde{F}^{\mu\nu}, \quad (41)$$

and the induced current $\vec{J} \sim -\dot{\sigma}\vec{B}^{\mathrm{M}}$ agrees with the results from [44–46].

## 4. Discussions

In [15, 16] it has been explored how to address the $U_A(1)$ problem using QCD strings. One of the possible solutions follows in a similar way as the large-$N$ approach [47, 48]. It will be interesting to consider the relation between the axionic excitation and the $\eta_0$ field, if we use the chiral field to accommodate the axionic excitation together with the usual nonet particles ($N_f = 3$),

$$U = \langle\bar{q}q\rangle \exp[\mathrm{i}\sigma(x) + \mathrm{i}\sqrt{2}/F_{\pi}(\pi_a\tau_a + \eta_0/\sqrt{N_f})], \quad (42)$$

where $\tau_a$ are the $SU(N_f)$ generators and $\eta_0$ is the $SU(N_f)$ singlet field. Note that here the $\sigma(x)$ field is the axionic excitation, not the sigma meson (referred as the $f_0(500)$ resonance, see [49] for a review), and should be distinguished from the $\eta_0$ field as well, since the latter is an ordinary quark–antiquark meson, while the axionic excitation $\sigma(x)$ can be thought of as the ‘ripples’ of the quark condensate, and probably involves multi-quark configurations, similar to that the sigma meson $f_0(500)$ might be related to tetraquark and meson-molecule configurations, instead of being intuitively made of a quark–antiquark pair as mentioned in [49–51]. In other words, our axionic excitation and the $f_0(500)$ resonance might be considered ‘partners’ in describing the fluctuations of the phase and the magnitude of quark condensate, respectively. Therefore, they should possess more condensate features like order parameter evolution, collective motions and elementary excitations, and less meson natures like what we have traditionally pictured for quark–antiquark bound states. In the present work, we focused on vortex configurations in the quark condensate for building our string or flux-tube type model, so both magnitude and phase excitations fluctuate with respect to the vortex or string configurations, not a trivial constant vacuum.

The common feature between the Peccei–Quinn mechanism and our model is that both methods start with promoting the Standard Model $\theta$ parameter to a field, but then our approaches differ significantly: Peccei and Quinn added the axion as a new particle to the Standard Model, while we consider it as an excitation in the quark condensate, associated with vorticity in the QCD vacuum and hadron structures; To apply the Peccei–Quinn mechanism, a new symmetry (a global axial $U_{PQ}(1)$ symmetry) is introduced and its breaking at low energy scale allow the axion to pick a vanishing value after the QCD phase transition, while we use an emergent symmetry from the decomposition of the derivative of the axionic excitation without including any new symmetry; Also the topological term in our effective Lagrangian is quite distinctive in the sense that it is a coupling between the axionic current and the Chern–Simons current ($\sim\frac{1}{F^2}J^{A}_{\mu}\, K^{\mu}$). Although it can be connected to the usual topological charge term through integration by parts, this coupling has a gauge variation which is exactly needed for canceling the gauge anomaly located on the QCD string.

There are at least the following limitations of the present work and open questions: First, we have not included the condensation of gluons and those possible excitations from the gluon condensate; Second, as mentioned earlier, for condensates of different types of quarks, multiple axionic excitations might be taken into account, and in general, a matrix form is needed to accommodate them; We only considered vortex or string-like configurations in the quark condensate, while there might be other possibilities, e.g. domain walls and monopoles; Last but not least, we have not suggested specific experimental searches to examine the assumptions made in this work, and numerical simulation and lattice computation would be desired. We leave these interesting and challenging questions to future investigations.

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