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# Geometric perspective of reciprocal transformations induced by conservation laws in integrable systems

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**Abstract**

This paper presents a geometric perspective that connects reciprocal transformations with multidimensional integrable deformations. By interpreting conservation laws as closed 1-forms, we formalize reciprocal transformations as induced local diffeomorphisms on the jet bundle. This allows us to characterize higher-dimensional deformations as systematic fiber bundle extensions, where fiber coordinates are generated by potential functions of the conservation laws. This perspective provides an interpretation for the covariant lifting of Lax pairs to higher dimensions and reveals that auto-Bäcklund transformations are composite diffeomorphisms. These results are applied to several classical integrable models.

Keywords: integrable systems, reciprocal transformation, conservation laws, fiber bundle, Lax pairs, Bäcklund transformation

## 1. Introduction

The connection between integrable systems and differential geometry has wide applications in modern mathematical physics, providing novel insights into a vast array of non-linear phenomena, from fluid mechanics [1] and optics [2] to field theory [3] and general relativity [4]. A central goal is the ongoing quest to discover new integrable models, particularly in higher dimensions and revealing the rich geometric and algebraic structures that govern them, such as the existence of infinite symmetries and conservation laws [5], not only serve as hallmarks of integrability, but also provide methods for constructing unknown systems and their solutions.

A powerful method for generating higher-dimensional Lax integrable systems is the deformation algorithm proposed by Lou, Hao and Jia [6]. The algorithm introduces new dimensions by utilizing conservation laws ($\rho_i$, $J_i$) to deform the derivative operators: $\partial_x \to \partial_x + \sum \rho_i \partial_{x_i}$ and $\partial_t \to \partial_t + \sum \bar{J}_i \partial_{x_i}$. This method preserves essential properties of the original (1+1)-dimensional models, including their Lax pairs and integrable hierarchies. The generality of this algorithm was proven by Casati and Zhang [7] and formalized a generalized version of the algorith, we term it Multidimensional Integrable Deformations. This approach has since been applied to a wide range of (1+1)-dimensional integrable models [8–11], such as the Burgers equation [12], the Camassa–Holm (CH) equation [13].

On one hand, reciprocal transformations [14] remain a central focus in integrable systems research due to their profound connections across different theoretical frameworks. They are instrumental in unifying different spectral problems, such as linking the Ablowitz–Kaup–Newell–Segur (AKNS) and Wadati–Konno–Ichikawa (WKI) hierarchies [15], and in revealing hidden symmetries, as exemplified by the mapping between the Korteweg–de Vries (KdV) and Harry–Dym (HD) hierarchies [16, 17]. From a geometric viewpoint, it is well-established that these transformations can be interpreted as diffeomorphisms [18], where the preservation of integrability corresponds to the invariance of geometric structures [19]. On the other hand, while the deformation algorithm provides an effective method for generating higher-dimensional systems, its underlying geometric principles and its precise relationship to the theory of reciprocal transformations have remained unexplored. The crucial question of how the systematic introduction of new dimensions via operator deformation fits

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into the established geometric pictures has not yet been addressed. We present a natural geometric interpretation for the multidimensional integrable deformations, reveal its innate connection to reciprocal transformations, and explain the emergence of auto-Bäcklund transformations from composite diffeomorphisms.

Section 2 establishes our geometric intepretation, where reciprocal transformations are formalized as diffeomorphisms induced by closed 1-forms In section 3, we apply this method to the Burgers, KdV and AKNS systems to demonstrate the concepts of fiber bundle extensions for higher-dimensional models, Bäcklund transformations and the covariant lifting of their Lax pairs. Finally, section 4 is a summary and discussion.

## 2. The reciprocal transformation and its geometric interpretation

In this section, we establish the geometric framework by interpreting conservation laws as closed 1-forms, we formalize the reciprocal transformation as a local diffeomorphism between manifolds. We demonstrate that this perspective enables us to connect reciprocal transformations, Bäcklund transformations, and higher-dimensional integrable systems.

Considering a (1+1)-dimensional integrable equation defined on the manifold $\mathcal{M}=\mathbb{R}^2_{(x,t)}\times\mathcal{U}$:

$$u_t=F(u,u_x,u_{xx},\ldots,u^{(n)}),\tag{1}$$

where $\mathcal{U}$ is the space of dependent variables $u$. $u^{(n)}$ denotes the derivatives of $u$ up to order $n$ on the $n$th jet bundle $\mathcal{J}^n(\mathcal{M})$. A conservation law for equation (1) is defined on $\mathcal{J}^n(\mathcal{M})$ as a pair of smooth functions $(\rho, J)$, satisfying the continuity equation:

$$D_t\rho=D_xJ,\tag{2}$$

where $D_x$, $D_t$ are total derivatives, $\rho=\rho(u,u_x,\ldots)$ is the conserved density and $J=J(u,u_x,\ldots)$ represents the associated conserved flow.

The condition $D_t\rho-D_xJ=0$ is equivalent to the statement that the 1-form $w=\rho\,\mathrm{d}x+J\mathrm{d}t\in\Omega^1(\mathcal{M})$ is closed, i.e. $\mathrm{d}w=0$, where $\Omega^1(\mathcal{M})$ denotes the space of smooth 1-form on $\mathcal{M}$. By Poincaré Lemma, the closedness of $w$ guarantees the local existence of a potential function $w\in\mathcal{C}^\infty(\mathcal{M})$ such that $w_x=\rho$, $w_t=J$.

This potential function naturally induces a coordinate transformation $\phi$: $\mathcal{M}\to\mathcal{M}'$, where $\mathcal{M}'=\mathbb{R}^2_{(x',t')}\times\mathcal{U}'$ is the target manifold:

$$x'=w(x,t),\quad t'=t.\tag{3}$$

The Jacobian matrix of this transformation is

$$J_\phi\equiv\begin{pmatrix}\partial x'/\partial x & \partial x'/\partial t\\ \partial t'/\partial x & \partial t'/\partial t\end{pmatrix}=\begin{pmatrix}\rho & J\\ 0 & 1\end{pmatrix}.\tag{4}$$

The determinant of the Jacobian is $\det(J_\phi)=\rho$. A condition for this transformation to be a local diffeomorphism is that the Jacobian must be non-singular, which requires $\rho\neq0$. In this paper, we focus on systems where this condition holds, ensuring that the map $\phi$ is locally invertible.

For the corresponding cotangent bundle $\mathcal{T}^*\mathcal{M}$, the pullback map $\phi^*$: $\mathcal{T}^*\mathcal{M}'\to\mathcal{T}^*\mathcal{M}$ acts on differentials as

$$\phi^*(\mathrm{d}x')=\rho\,\mathrm{d}x+J\mathrm{d}t,\quad \phi^*(\mathrm{d}t')=\mathrm{d}t.\tag{5}$$

Conversely, for the tangent bundle $\mathcal{TM}$, the pushforward map $\phi_*$: $\mathcal{TM}\to\mathcal{TM}'$ transforms the basis partial derivative operators as:

$$\phi_*(\partial_x)=\rho\partial_{x'},\quad \phi_*(\partial_t)=\partial_{t'}+J\partial_{x'}.\tag{6}$$

The original conservation law ensures that the 1-form $\mathrm{d}x'$ is closed on the manifold $\mathcal{M}'$, this corresponds to a new closed 1-form $w'=\rho'\mathrm{d}x'+J'\mathrm{d}t'\in\Omega^1(\mathcal{M}')$, which in turn generates a conservation law on $\mathcal{M}'$: $D_{t'}\rho'=D_{x'}J'$, where the density and flow satisfy the reciprocal relations:

$$\rho'=\frac{1}{\rho},\quad J'=-\frac{J}{\rho},\quad D_{t'}\rho'=D_{x'}J'.\tag{7}$$

The commutative properties of the reciprocal transformations have been proposed on the basis of the coordinate-free property of the exponential map [20]. The existence of this conservation law and its associated potential function $w'$ implies that the property of having a conservation law is preserved under the transformation. We summarize these properties in the following theorem.

***Theorem.*** *Let $\mathcal{M}=\mathbb{R}^2_{(x,t)}\times\mathcal{U}$ be the manifold of a (1+1)-dimensional integrable system equation (1) equipped with a conservation law equation (2). If the conserved density ρ is non-zero, the reciprocal transformation equation (3) induces a local diffeomorphism ϕ: $\mathcal{M}\to\mathcal{M}'$. Under this map ϕ:*

- *Preservation of Closedness:* The transformation preserves the structure of conservation laws. A closed 1-form $w$ on $\mathcal{M}$ maps to a closed 1-form $w'$ on $\mathcal{M}'$.
- *Covariance of Lax Pair:* If the original system equation (1) is Lax integrable, possessing a Lax pair $(M, N)$ that satisfies the zero-curvature condition $[D_x+M, D_t+N]=0$, then the transformed system on $\mathcal{M}'$ inherits Lax integrability. It possesses a transformed Lax pair $(M', N')$ satisfying the zero-curvature condition $[D_{x'}+M', D_{t'}+N']=0$.

***Proof.*** The conservation law $\rho_t=J_x$ is equivalent to the closedness of the 1-form $w=\rho\mathrm{d}x+J\mathrm{d}t$. Under the map $\phi$, this 1-form is identified with the pullback of $\mathrm{d}x'$, i.e. $w=\phi^*(\mathrm{d}x')$. The exterior derivative on $\mathcal{M}$ is thus $\mathrm{d}w=\mathrm{d}(\phi^*(\mathrm{d}x'))=\phi^*(\mathrm{d}(\mathrm{d}x'))$. Since $d^2=0$ for exterior derivative, $\mathrm{d}(\mathrm{d}x')=0$ on $\mathcal{M}'$. This implies that the closedness is preserved.

The Lax pair defines a flat connection $\nabla=\mathrm{d}+\Gamma$ on a vector bundle over $\mathcal{M}$, where $\Gamma=M\mathrm{d}x+N\mathrm{d}t$ is the connection 1-form. The zero-curvature condition is equivalent to the flatness of this connection, $\Omega=\mathrm{d}\Gamma+\Gamma\wedge\Gamma=0$. Under the diffeomorphism $\phi$, the connection 1-form $\Gamma$ is pulled back to the connection $\Gamma'$ on the bundle over $\mathcal{M}'$. To find the components of $\Gamma'$, we express the original basis 1-forms $(\mathrm{d}x, \mathrm{d}t)$ in terms of the basis $(\mathrm{d}x', \mathrm{d}t')$. Inverting the pullback relations equation (5) gives $\mathrm{d}x = (1/\rho)\mathrm{d}x' - (J/\rho)\mathrm{d}t'$ and $\mathrm{d}t = \mathrm{d}t'$. Substituting these into the expression for $\Gamma$ yields:

$$\Gamma = \left(\frac{M}{\rho}\right)\mathrm{d}x' + \left(N - \frac{M}{\rho}J\right)\mathrm{d}t'.$$

This has the form $\Gamma' = M'\mathrm{d}x' + N'\mathrm{d}t'$, which defines the transformed Lax pair $(M', N')$. Since a diffeomorphism preserves the flatness of a connection, the condition $\Omega = 0$ on $\mathcal{M}$ implies that the curvature of the pulled back connection $\Gamma'$ on $\mathcal{M}'$ is also zero. This guarantees that the transformed system satisfies the zero-curvature condition $[D_{x'} + M', D_{t'} + N'] = 0$, thus proving the covariance of the Lax pair. □

The framework established above provides a geometric interpretation for reciprocal transformations in (1+1)-dimensional Lax integrable systems. Furthermore, a natural question arises that does this geometric picture can be used to explain the deformation of higher-dimensional integrable systems from lower-dimensional ones. In the subsequent sections, we demonstrate that the multidimensional deformations can be understood as a fiber bundle extension of the original manifold $\mathcal{M}$, connecting both concepts under this geometric perspective.

## 3. Applications of reciprocal transformations to integrable systems

In this section, we apply the geometric perspective established in section 2 to several classical integrable models. We derive the corresponding reciprocal equation via the diffeomorphism map $\phi$, and demonstrate how this framework provides a geometric interpretation for higher-dimensional deformed integrable systems, the transformed equations and their associated Lax pairs are also derived.

### *3.1. The reciprocal Burgers equation and its auto-Bäcklund transformation*

The Burgers equation is a fundamental model has been widely applied in various fields, such as plasma physics [21] and fluid mechanics [22], is given by:

$$u_t - u_{xx} - 2uu_x = 0 \tag{8}$$

This equation (8) resides on the jet bundle $\mathcal{J}^2(\mathcal{M})$ with coordinates $(x, t, u, u_x, u_{xx})$ and a conservation law:

$$(u)_t = (u^2 + u_x)_x, \quad \rho = u, \quad J = u^2 + u_x, \tag{9}$$

which defines the closed 1-form $w = u\mathrm{d}x + (u^2 + u_x)\mathrm{d}t$. The condition $\mathrm{d}w = (u_t - u_{xx} - 2uu_x)\mathrm{d}t \wedge \mathrm{d}x = 0$ is equivalent to the Burgers equation (8).

We consider the case where the transformed solution $U$ as a function of the coordinates $(w, t')$ with $w$ replacing $x$ as the spatial variable and $t' = t$. There locally exists a potential function $w \in C^\infty(\mathcal{J}^2(\mathcal{M}))$, such that

$$u = U(w, t'), \quad w_x = u, \quad w_t = u^2 + u_x. \tag{10}$$

In terms of the tangent vector fields, the pushforward map $\phi_*$ transforms the derivative operators as:

$$\partial_x = U\partial_w, \quad \partial_t = \partial_{t'} + (U^2 + w_x U_w)\partial_w.$$

Applying this transformation to the Burgers equation (8), using the chain rule, we have:

$$\begin{aligned} u_x &= \phi_*(\partial_x)U = UU_w, \\ u_t &= \phi_*(\partial_t)U = U_{t'} + (U^2 + w_x U_w)U_w. \end{aligned} \tag{11}$$

Substituting these expressions equation (11) back into equation (8) yields:

$$[U_{t'} + (U^2 + UU_w)U_w] - [U(U_w^2 + UU_{ww})] - 2U^2U_w = 0.$$

Simplifying this expression and relabeling the coordinates $(w, t')$ as $(y, t)$ and the function $U$ as $u$, we obtain the first-type reciprocal Burgers equation reads:

$$u_t = u^2(u_y + u_{yy}). \tag{12}$$

This equation (12) is equivalent to the reduction of the (2+1)-dimensional deformed Burgers system when the solution is independent of the original coordinate $x$ [12].

The reciprocal transformation provides a method to construct an auto-Bäcklund transformation. For the reciprocal Burgers equation (12), possessing a conservation law:

$$(u^{-1})_t = -(u + u_y)_x, \tag{13}$$

this conservation law equation (13) can be used to define an inverse reciprocal transformation $\phi^{-1}$. The corresponding potential $w'$ with $w'_y = u^{-1}$, $w'_t = -u - u_y$. Reversing the transformation via $\partial_y = u^{-1}\partial_x$ and $\partial_t = \partial_{t'} - (u + u_y)\partial_y$ leads to

$$U_{t'} - U_{w'w'} - 2UU_{w'} = 0. \tag{14}$$

Equation (14) is indeed the original Burgers equation (8), therefore, the composition of the forward transformation $\phi$ (induced by $\rho = u$) and the inverse transformation (induced by $\rho' = u^{-1}$) act as an auto-Bäcklund transformation for the Burgers equation (8).

### *3.2. Fiber bundle extension of the Burgers equation and its Lax pair*

The deformation algorithm enables us to transform a (1+1)-dimensional integrable equation into a higher one [6], a conservation law corresponds to a potential vector $w^{(n)}$. We interpret this deformation as a fiber bundle extension. We lift the solution $u(x, t)$ on the base manifold $\mathcal{M}$ to a function $U(x, w, t)$ on the extended manifold $\mathcal{M}_2 = \mathcal{M} \times \mathbb{R}_y$, where the fiber coordinate is $y = w$, namely, $u = U(x, w, t)$.

The total derivative operators $D_x$ and $D_t$ are replaced by the pushforward operators acting on the extended tangent bundle $\mathcal{TM}_2$:

$$\begin{aligned} D_x &= \phi_*(\partial_x) = u\partial_w + \partial_x, \\ D_t &= \phi_*(\partial_t) = (u^2 + u_x)\partial_w + \partial_t, \end{aligned}$$

applying these operators with the chain rule yields $u_x = U_x + UU_w$ and $u_t = U_t + (U^2 + w_xU_x)U_w$, substituting these equations to the original Burgers equation (8), we have:

$$U_t = U_{xx} + U^2U_w + U^2U_{ww} + 2UU_x + 2UU_{xw},$$

relabeling the coordinatex $(x, w, t)$ as $(x, y, t)$ and the function $U$ as $u$, we obtain the (2+1)-dimensional Burgers equation:

$$u_t = u_{xx} + u^2(u_y + u_{yy}) + 2uu_x + 2uu_{xy}. \tag{15}$$

This equation (15) resides on the extended jet bundle $\mathcal{J}^2(\mathcal{M}_2)$. The derivation implies that the higher-dimensional equation emerges from the lifting procedure. Some properties of the (2+1)-dimensional Burgers equation (15) have been proposed in [12], such as the Lie point symmetry, travelling wave solutions, higher-order symmetries, etc.

This lifting procedure also covariantly transforms the Lax pair. The Lax pair for the Burgers equation (8) defined on the trivial vector bundle $\mathcal{M} \times \mathbb{C}$ is given by:

$$M\psi \equiv \psi_x + (u + \lambda)\psi, \quad N\psi \equiv \psi_t - \psi_{xx} - 2u\psi_x, \tag{16}$$

where $\lambda$ is a spectral parameter and defining a flat connection $\Gamma = M\mathrm{d}x + N\mathrm{d}t$ satisfies the zero-curvature condition $[M, N] = 0$.

To find the Lax pair for the (2+1)-dimensional Burgers equation (15), substituting the transformed solutions $u = U(x, w, t)$ and $\psi = \phi(x, w, t)$ into the Lax pair equation (16) with potential function equation (10), we obtain the lifting of the Lax pair to the extended bundle $\mathcal{M}_2$:

$$\begin{aligned} M'\phi &\equiv \phi_x + \phi_w U + U\phi + \lambda\phi, \\ N'\phi &\equiv \phi_t - \phi_{xx} - U^2(\phi_{ww} + \phi_w) - 2U(\phi_x + \phi_x w), \end{aligned}$$

relabeling the coordinates $(x, w, t)$ as $(x, y, t)$ and the functions $U$ as $u$, $\phi$ as $\psi$, the pushforward map $\phi_*$ transforms the connection covariantly to

$$\begin{aligned} M'\phi &\equiv \phi_x + u\phi_y + (u + \lambda)\phi, \\ N'\phi &\equiv \phi_t - \phi_{xx} - u^2(\phi_y + \phi_{yy}) - 2u(\phi_x + \phi_{xy}), \end{aligned} \tag{17}$$

the zero-curvature condition is preserved due to the diffeomorphism invariance, the explicit form of the compatibility condition is

$$[M', N'] = (u_t - u_{xx} - u^2u_y - u^2u_{yy} - 2uu_x - 2uu_{xy}) \times \left(1 + \frac{\partial}{\partial y}\right) = 0. \tag{18}$$

This leads to the (2+1)-dimensional Burgers equation (15). For $u$ is $y$-independent, equation (15) reduces to the original Burgers equation (8), when $u$ is $x$-independent, equation (15) becomes to the reciprocal Burgers equation (12).

### *3.3. Reciprocal transformations and composite diffeomorphisms of the KdV–HD systems*

The KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{19}$$

was first introduced by Boussinesq [23] and rediscovered by Diederik Korteweg and Gustav de Vries [24], which describes the evolution of one-dimensional long waves in many physical phenomenon, including shallow water waves with weak nonlinear restoring forces, the two-dimensional quantum gravity [25], etc. The KdV equation (19) can be solved by the inverse scattering transform or other methods applied to integrable systems [26].

The KdV equation (19) possesses infinitely many conservation laws, in this section, two of them are of concern to us:

$$\rho_1 = u, \quad J_1 = u_{xx} + 3u^2, \tag{20}$$

and

$$\rho_2 = u^2, \quad J_2 = 2uu_{xx} - u_x^2 + 4u^3. \tag{21}$$

The conservation law equation (20) defines the closed 1-form

$$w_1 = u\,\mathrm{d}x + (u_{xx} + 3u^2)\mathrm{d}t, \tag{22}$$

including a diffeomorphism map $\phi_1$: $\mathcal{M} \to \mathcal{M}_1'$, where the local potential function $w_1$ satisfies $w_{1,x} = u$, $w_{1,t} = u_{xx} + 3u^2$. The closed 1-form of the second one equation (21) is $w_2 = u^2\mathrm{d}x + (2uu_{xx} - u_x^2 + 4u^3)\mathrm{d}t$, defining $\phi_2$: $\mathcal{M} \to \mathcal{M}_2'$, and $w_{2,x} = u^2$, $w_{2,t} = 2uu_{xx} - u_x^2 + 4u^3$.

The pullback map acts as $\phi^*(\mathrm{d}w_1) = u\mathrm{d}x + (u_{xx} + 3u^2)\mathrm{d}t$ and $\phi^*(\mathrm{d}t') = \mathrm{d}t$. Utilizing the transformation with the transformed solution $u = U(w_1, t)$ to the KdV equation (19) with the chain rule:

$$\begin{aligned} u_x &= \frac{\partial U}{\partial w_1}\frac{\partial w_1}{\partial x} = U_{w_1}u, \\ u_t &= \frac{\partial U}{\partial t} + \frac{\partial U}{\partial w_1}\frac{\partial w_1}{\partial t} = U_t + U_{w_1}J_1, \end{aligned}$$

yielding $U_t = [U^3(U_{ww} + 1)]_w$, this is the first type of the HD equation:

$$u_t = [u^3(u_{yy} + 1)]_y, \tag{23}$$

the closed 1-forms of equation (23) are

$$w_1' = u\,\mathrm{d}y + u^3(u_{yy} + 1)\mathrm{d}t, \tag{24}$$

$$w_2' = u^{-1}\mathrm{d}y - (uu_{yy} + u_y^2 + 3u)\mathrm{d}t. \tag{25}$$

Utilizing the second conservation law equation (21), we obtain the second type of the HD equation via the diffeomorphism map $\phi_2$:

$$u_t = \left(u^6u_{zz} + \frac{1}{2}u^4\right)_z + 3u^4u_z^3, \tag{26}$$

whose closed 1-form are

$$w_1'' = u^{-1}\mathrm{d}z - u^2(u^2u_{zz} + uu_z^2 + 1)\mathrm{d}t, \tag{27}$$

$$w_2'' = u^{-2}\mathrm{d}z - u(2u^2u_{zz} + 3uu_z^2 + 4)\mathrm{d}t. \tag{28}$$

Furthermore, we can also get the third type of the HD equation via the conservation law equation (20) and lifting $u$ to $U(x, w_1)$ with coordinate $y = w_1$:

$$\left(u_{xx} + \frac{3}{2}u^2u_{yy} + 3u_xu_y + u^2\right)_x + \left(u^3u_{yy} + \frac{3}{2}u^2u_{xy} + 3uu_{xx} + 3u_x^2 + u^3\right)_y = 0. \tag{29}$$

**Table 1.** Summary of auto-Bäcklund transformations.

| Starting equation | Transformation | Induced by | Resulting equation |
|---|---|---|---|
| Burgers (8) | $\phi \circ \phi^{-1}$ | Laws (9), (13) | Burgers (8) |
| KdV (19) | $\phi_1 \circ \phi_1' \circ \phi_2''$ | 1-forms (22), (24), (28) | KdV (19) |
| KdV (19) | $\phi_2 \circ \phi_2''$ | 1-forms (21), (28) | KdV (19) |
| $HD_1$ (23) | $\phi_1' \circ \phi_1''$ | 1-forms (24), (25) | $HD_1$ (23) |
| $HD_2$ (26) | $\phi_2'' \circ \phi_2$ | 1-forms (28), (21) | $HD_2$ (26) |

These reciprocal transformations can be composed to form auto-Bäcklund transformations as composite diffeomorphisms:

**I.** The Bäcklund transformation is realized as a sequence of the diffeomorphism $\{\phi_i\}$, induced by the 1-forms of conservation laws equations (20), (24) and (28), respectively,

- $\phi_1$: $\mathcal{M} \to \mathcal{M}_1$, mapping the KdV equation (19) to the first type of the HD equation (23) via $y = w_1$, preserving the closedness of $w_1 = u\mathrm{d}x + (u_{xx} + 3u^2)\mathrm{d}t$.
- $\phi_1'$: $\mathcal{M}_1' \to \mathcal{M}_1''$, mapping the first type of the HD equation (23) to the second type of the HD equation (26) via $z = w_1'$, preserving the closedness of $w_1' = u\,\mathrm{d}y + u^3(u_{yy} + 1)\,\mathrm{d}t$.
- $\phi_2''$: $\mathcal{M}_1'' \to \mathcal{M}$, mapping the second type of the HD equation (26) to the KdV equation (19) via $x = w_2''$, preserving the closedness of $w_2'' = u^{-2}\,\mathrm{d}z - u(2u^2 u_{zz} + 3uu_z^2 + 4)\,\mathrm{d}t$.

Especially, these processes can be obtained through the composition $\phi_1 \circ \phi_1' \circ \phi_2''$, which emphasizes the geometric consistency of the reciprocal transformations, due to the preservation of structures under the induction of the conservation laws. The transformed HD equations themselves possess new conservation laws. These laws can generate further reciprocal transformations. This leads to an auto-Bäcklund transformation cycle:

$$\mathrm{KdV} \xrightarrow{\phi_1(\text{via } w_1)} \mathrm{HD}_1 \xrightarrow{\phi_1'(\text{via } w_1')} \mathrm{HD}_2 \xrightarrow{\phi_2''(\text{via } w_2'')} \mathrm{KdV}.$$

Then we give the other three sets of the auto-Bäcklund transformations:

**II.**

$$\mathrm{KdV} \xrightarrow{\phi_2(\text{via } w_2)} \mathrm{HD}_2 \xrightarrow{\phi_2''(\text{via } w_2'')} \mathrm{KdV}.$$

**III.**

$$\mathrm{HD}_1 \xrightarrow{\phi_1'(\text{via } w_1')} \mathrm{HD}_2 \xrightarrow{\phi_1''(\text{via } w_1'')} \mathrm{HD}_1.$$

**IV.**

$$\mathrm{HD}_2 \xrightarrow{\phi_2''(\text{via } w_2'')} \mathrm{KdV} \xrightarrow{\phi_2(\text{via } w_2)} \mathrm{HD}_2.$$

These sequences of transformations imply that the composition of diffeomorphism $\phi_i$ act as an identity map on the space of solutions, where a sequence of diffeomorphisms induced by a chain of conservation laws maps an integrable system back onto itself.

Table 1: Auto-Bäcklund transformations realized as composite diffeomorphisms. Each row represents a sequence of transformations induced by conservation laws and the specified 1-forms, that maps an equation onto itself.

### *3.4. Fiber bundle extension to the (3+1)-dimensional KdV–HD system*

To construct the (3+1)-dimensional KdV equation via fiber bundle extension over the base manifold $\mathcal{M}$, we extend the manifold by two fiber dimensions, yielding $\mathcal{M}_3 = \mathcal{M} \times \mathbb{R}_y \times \mathbb{R}_z$ with two potential functions $w$ and $m$, corresponding to the fiber coordinates $y = w$, $z = m$, where $w_x = u$, $m_x = u^2$, $w_t = u_{xx} + 3u^2$ and $m_t = 2uu_{xx} - u_x^2 + 4u^3$.

We lift $u(x, t)$ to $U(x, w, m, t) \in \mathcal{C}^\infty(\mathcal{M}_3)$. The push-forward map transforms derivatives as

$$\begin{aligned} \phi_*(\partial_x) &= \partial_x + u\partial_w + u^2\partial_m, \\ \phi_*(\partial_t) &= \partial_t + (u_{xx} + 3u^2)\partial_w \\ &\quad + (2uu_{xx} - u_x^2 + 4u^3)\partial_m, \end{aligned}$$

which prolongs the KdV equation (19) onto the bundle $\mathcal{J}^3(\mathcal{M}_3)$ and allows the original system within a higher dimensional fiber bundle leads to the following (3+1)-dimensional KdV equation:

$$\begin{aligned} &u_t + u_{xxx} + u^3 u_{yyy} + u^6 u_{zzz} + 3u(u^4 u_{yzz} + u^3 u_{xzz} \\ &+ u^3 u_{yyz} + 2u^2 u_{xyz} + uu_{xxz} + uu_{xyy} + u_{xxy}) \\ &+ 3(2u^3 u_{zz} + 3u^2 u_{yz} + 2uu_{xz} + uu_{yy} + u_{xy}) \\ &(u^2 u_z + uu_y + u_x) + 3u_x^2 u_z \\ &+ 6u(uu_z^2 + u_y u_z + 1)u_x + 3u^2 u_y(2uu_z^2 \\ &+ u_y^2 u_z + 1)u_y + u^3 u_z(3uu_z^2 + 2) = 0. \end{aligned} \tag{30}$$

The structure of equation (30) is evident in its various dimensional reductions, which recover the original KdV equation, its reciprocal links, and coupled (2+1)-dimensional systems. The main reduction cases are as follows:

- If the solution $u$ is independent of both $x$ and $z$, the (3+1)-dimensional KdV equation (30) reduces to the first HD equation (23).
- If $u$ is independent of both $x$ and $y$, equation (30) becomes the second HD equation equation (26), which is a reciprocal link of the KdV equation (19).
- If $u$ is independent of both fiber coordinates $y$ and $z$, equation (30) returns to the original KdV equation (19).
- When the solution $u$ is independent of the fiber coordinate $y$, equation (30) becomes an integrable (2+1)-dimensional equation in $(x, z, t)$ that couples the

KdV equation (19) with its second reciprocal link equation (26).
- When $u$ is independent of the fiber coordinate $z$, equation (30) reduces to an integrable (2+1)-dimensional equation in $(x, y, t)$ that couples the KdV equation (19) with its first reciprocal link equation (23).

Its Lax pair also can be regarded as a connection on the cotangent bundle $\mathcal{T}^*\mathcal{M}$, the Lax pair of the KdV equation (19) is defined on the trivial bundle is

$$\begin{aligned} M\psi &\equiv \psi_{xx} + (u - \lambda)\psi, \\ N\psi &\equiv \psi_t - 4\psi_{xxxx} - 6u\psi_x - 3u_x\psi. \end{aligned} \tag{31}$$

Utilizing the conservation laws $(\rho_i, J_i)$, we can construct the diffeomorphism map $\phi_i$, the lifted connections can be derived via the pushforward map $\phi_{i*}$ and the pullback map $\phi_i^*$. Through the chain rule of covariant derivatives, the original Lax pair equation (31) is lifted to $(M_i, N_i)$.

The lifting of the Lax pairs to the extended bundle $\mathcal{M}_3$ are achieved by their extended pushforward counterparts $D_x$ and $D_t$, we provide three Lax pairs correspond to equation (23), equation (26) and equation (29), respectively,

$$\begin{aligned} M_1\phi &\equiv u^2\phi_{yy} + uu_y\phi_y + (u - \lambda)\phi, \\ N_1\phi &\equiv \phi_t - 4u^3\phi_{yyy} - 12u^2u_y\phi_{yy} \\ &\quad - 3u(uu_{yy} + u_y^2 + u)\phi_y - 3uu_y\phi, \end{aligned}$$

$$\begin{aligned} M_2\phi &\equiv u^4\phi_{zz} + 2u^3u_z\phi_z + (u - \lambda)\phi, \\ N_2\phi &\equiv \phi_t - 4u^6\phi_{zzz} - 24u^5u_z\phi_{zz} \\ &\quad - u^3(6u^2u_{zz} + 21uu_z^2 + 2)\phi_z - 3u^2u_z\phi \end{aligned}$$

and

$$\begin{aligned} M_3\phi &\equiv \phi_{xx} + (u - \lambda)\phi + u^2\phi_{yy} + 2u\phi_{xy} + (u_x + uu_y)\phi_y, \\ N_3\phi &\equiv 4\phi_{xxx} + 12u\phi_{xxy} + 12u^2\phi_{xyy} + 4u^3\phi_{yyy} \\ &\quad + 6u\phi_x + 3(u_x + uu_y)(\phi + 4u\phi_{yyy} + 4\phi_{xy}) \\ &\quad + 3[u^2 + (\partial_x + u\partial_y)(u_x + uu_y)]\phi_y. \end{aligned}$$

The Lax pair of the (3+1)-dimensional KdV equation (30) is not be presented here because of its complexity, whose operator form is:

$$\begin{aligned} M_k\psi &\equiv (D_x^2 + u - \lambda)\psi = 0, \\ N_k\psi &\equiv (D_t - 4D_x^3 - 6uD_x - 3D_xu)\psi = 0. \end{aligned}$$

The Lax integrability of the (3+1)-dimensional KdV equation (30) and equations (23), (26), (29) with the compatibility conditions $[M_k, N_k] = 0$, $[M_1, N_1] = 0$, $[M_2, N_2] = 0$ and $[M_3, N_3] = 0$, respectively.

### *3.5. Fiber bundle extension to the AKNS system and its reductions*

As a final example, we consider the AKNS system, which serves as a general form for the nonlinear Schrödinger (NLS) equation, describing many physical phenomena, including oceanic waves [27], superfluids [28], economics [29], Bose–Einstein condensates [30], quantum optics [31], etc. The system is given by the pair of coupled equations:

$$u_t = u_{xx} + 2u^2v, \quad v_t = -v_{xx} - 2v^2u. \tag{32}$$

We utilize the conservation law:

$$(uv)_t = (vu_x - uv_x)_x, \tag{33}$$

where the conserved density is $\rho = uv$ and the flow is $J = vu_x - uv_x$. This conservation law defines a closed 1-form $\mathrm{d}w = uv\mathrm{d}x + (vu_x - uv_x)\mathrm{d}t$.

Following the principle of fiber bundle extension, we lift the solutions $u(x, t)$ and $v(x, t)$ to functions $U(x, w, t)$ and $V(x, w, t)$ on the extended manifold $\mathcal{M}_2 = \mathcal{M} \times \mathbb{R}_y$, where the fiber coordinate $y$ is defined by the potential function $w$ with $w_x = uv$, $w_t = vu_x - uv_x$. Then the (2+1)-dimensional AKNS system has the form:

$$\begin{aligned} &U_t - U_{xx} - 2U^2V - 2UU_wV_x - 2UVU_{xw} \\ &\quad - 2U^2V(U_wV_w) - U^2V^2U_{ww} = 0, \\ &V_t + V_{xx} + 2UV^2 + 2UVV_{xw} + 2VU_xV_w \\ &\quad + 2UV^2U_wV_w + U^2V^2V_{ww} = 0, \end{aligned}$$

relabeling the coordinates $(x, w, t)$ as $(x, y, t)$ and the functions $U$ as $u$, $V$ as $v$, we obtain the (2+1)-dimensional AKNS system:

$$\begin{aligned} &u_t - u_{xx} - u^2v(2 + vu_{yy} + 2u_yv_y) \\ &\quad - 2u(vu_{xy} + v_xu_y) = 0, \\ &v_t + v_{xx} + uv^2(2 + uv_{yy} + 2u_yv_y) \\ &\quad + 2v(uv_{xy} + u_xv_y) = 0, \end{aligned} \tag{34}$$

which resides on the extended jet bundle $\mathcal{J}^2(\mathcal{M} \times \mathbb{R}_y)$ with the fiber coordinate $y = w$.

The Lax pair for this extended system is obtained by covariantly lifting the original (1+1)-dimensional Lax pair. The explicit form of the Lax pair for the (2+1)-dimensional AKNS system equation (34) is:

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix}_x + uv\begin{pmatrix}\psi\\ \phi\end{pmatrix}_y - \begin{pmatrix}\lambda & -v\\ u & -\lambda\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} = 0,$$

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix}_t - (uv_x - vu_x + u^2vv_y - v^2uu_y)\begin{pmatrix}\psi\\ \phi\end{pmatrix}_y - \begin{pmatrix}-2\lambda^2 - uv & v_x + uvv_y + 2\lambda v\\ u_x + vuu_y - 2\lambda u & 2\lambda^2 + uv\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix} = 0.$$

The deformed system equation (34) contains the original and reciprocal systems as special reductions:

- When the fields are independent of the fiber coordinate $y$, the system equation (34) and its Lax pair reduce to the (1+1)-dimensional AKNS system equation (32).
- When the fields are independent of the original coordinate $x$, the system equation (34) reduces to the reciprocal AKNS system:

$$\begin{aligned} &u_t - u^2v(2 + vu_{yy} + 2u_yv_y) = 0, \\ &v_t + uv^2(2 + uv_{yy} + 2u_yv_y) = 0. \end{aligned} \tag{35}$$

This reciprocal system equation (35) possesses a conservation law $(u^{-1}v^{-1})_t = (uv_y - vu_y)_y$, which can be used to define a further reciprocal transformation, establishing Bäcklund transformations between these systems.

## 4. Summary and discussion

In summary, this paper has established a geometric perspective between reciprocal transformations and multidimensional integrable deformations. We have shown that by interpreting conservation laws as closed 1-forms on the jet bundle, reciprocal transformations are formalized as local diffeomorphisms and the deformation algorithm is characterized as a systematic fiber bundle extension. This perspective clarifies the covariant behavior of Lax pairs under these mappings and reveals the structure of auto-Bäcklund transformations as composite diffeomorphisms.

Although we have modified the algorithm [13], challenges arise when extending this geometric framework to (2+1)-dimensional systems. The multidimensional integrable deformation requires a (1+1)-dimensional local evolution equation as its starting point. The Kadomtsev–Petviashvili (KP) equation serves as an example of these difficulties, which is given by:

$$u_t + u_{xxx} - 6uu_x + 6v_{yy} = 0, \quad u = 2v_x.$$

A conservation law for the KP equation can be written in the form $D_t\rho = D_xJ_1 + D_yJ_2$. Following the procedure from section 2, we attempt to define a 1-form $w = \rho \mathrm{d}x + J_1\mathrm{d}t + J_2\mathrm{d}y$.

For our applications, $w$ must be closed, which would guarantee the existence of a single potential function to serve as a new coordinate. Then we check this condition by computing the exterior derivative of $w$: $\mathrm{d}w = \mathrm{d}\rho \wedge \mathrm{d}x + \mathrm{d}J_1 \wedge \mathrm{d}t + \mathrm{d}J_2 \wedge \mathrm{d}y$. Expanding this gives: $\mathrm{d}w = (\partial_t\rho \mathrm{d}t + \partial_y\rho \mathrm{d}y) \wedge \mathrm{d}x + (\partial_xJ_x\mathrm{d}x + \partial_yJ_1\mathrm{d}y) \wedge \mathrm{d}t + (\partial_xJ_x\mathrm{d}x + \partial_tJ_2\mathrm{d}t) \wedge \mathrm{d}y$, collecting the terms for the 2-forms yields:

$$\mathrm{d}w = (D_xJ_1 - D_t\rho)\mathrm{d}x \wedge \mathrm{d}t + (D_yJ_1 - D_tJ_2)\mathrm{d}y \wedge \mathrm{d}t + (D_xJ_2 - D_y\rho)\mathrm{d}x \wedge \mathrm{d}y.$$

Using the conservation law ($D_t\rho = D_xJ_1 + D_yJ_2$) to substitute for the first term's coefficient, which becomes ($-D_yJ_2$). For the KP equation, these coefficients are generally non-zero. Therefore, the non-closedness of $w$ is a fundamental obstruction, as it implies that a single, scalar potential function $w$ for the entire system does not exist, breaking the geometric interpretation in (1+1)-dimensions. This issue stems from the fact that (2+1)-dimensional systems possess a richer structure, with flow distributed over multiple spatial directions and involve non-local terms A potential method may require a more sophisticated geometric structure than a single-potential fiber bundle, involving multiple, compatible potential functions or alternative geometric frameworks such as the $\bar{\partial}$-dressing method [32, 33]. Therefore, developing a multidimensional integrable deformation for (2+1)-dimensions remains a significant open problem.

Furthermore, the deformation process, while preserving Lax integrability, fundamentally alters other properties. A direct consequence is the loss of Painlevé integrability [34]. This occurs because the embedded reciprocal transformations can map simple pole singularities in the original solution to more complex logarithmic or essential singularities in the transformed coordinates. This necessitates another method for constructing higher-dimensional, Painlevé integrable systems [35].

This increased complexity also obstructs classical solution methods. For instance, the Hirota bilinear method [36] is not directly applicable because the deformed equations are complex hybrids of a model and its reciprocal counterparts, these systems generally lack the shared, simple bilinear representation required for a single $\tau$-function. Similarly, the Cole–Hopf (CH) transformation [37] fails to linearize the deformed Burgers equation. The new nonlinear terms introduced by the deformation are algebraically incompatible with the structure of the CH transformation. Consequently, developing solution methods for these deformed models is a critical challenge. Further open problems regarding these higher-dimensional systems are discussed in [38].

Crucially, the deformed systems remain Lax integrable and retain their entire integrable hierarchy of commuting flows. Our method acts as a systematic replacement of derivative operators ($\partial_x \to D_x$). Since the entire integrable hierarchy is generated algebraically by a recursion operator, this replacement can be applied to every flow in the hierarchy. This procedure lifts the commutation relations from the original flows to the deformed ones, thus preserving the system's infinite symmetries [39].

From the geometric perspective of this paper, deeper algebraic structures also merit consideration, particularly from the viewpoint of Hopf algebras and quantum groups [40]. The observed duality between reciprocally transformed equations suggests a more fundamental algebraic duality, motivating an investigation into whether the symmetry algebras of these paired systems are dual in the Hopf-algebraic sense. Furthermore, the deformation operator ($D_x = \partial_x + \rho\partial_y$) can be naturally viewed as a deformed derivative, whose action might be governed by a non-trivial coproduct structure. These open questions suggest a potential connection between the geometric and algebraic foundations of integrability, which is worthy of further study.

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