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Quantum Tunneling Radiation of Kerr-NUT Black Hole*

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Abstract *Based on particles in a dynamical geometry, extending the Parikh's method of quantum tunneling radiation, we deeply investigate the quantum tunneling radiation of Kerr-NUT black hole. When self-gravitating action, energy conservation, and angular momentum conservation are taken into account, the emission rate of the particle on the event horizon is related to the change of Bekenstein–Hawking entropy and the emission spectrum is not precisely thermal, but is consistent with an underlying unitary theory.*

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1 Introduction

In 1970s, Hawking proved the temperature and thermal radiation of the black hole.[1] Since then, a considerable amount of work has been done relating to static, stationary, and non-stationary black holes' thermodynamic properties by lots of astronomers and physicists.[2−8] However, Hawking radiation spectra are pure thermal ones supposing that the space-time backgrounds of the black holes are fixed. There are two points that are worth while discussing when studying the thermal radiation. The first is the missing of information, that is, the change of pure quantum state into the state of chaos, i.e. the unitarity is lost. The second is the technical defect. At present, although it is known that the black hole radiation is the contribution of the quantum tunneling effect, the causes of the tunneling potential barriers are still not clear. And the related documents are not used as the quantum tunneling language to discuss the thermal radiation problem, which is not the real quantum tunneling method in fact.

Recently, Parikh and Wilczek put forward a semi-classical tunneling method to investigate Hawking radiation of the static Schwarzschild and Reissner–Nordström black holes.[9] The research of Parikh *et al.* indicates that Hawking radiation of a black hole is not pure thermal if the energy conservation is considered and the radiation is viewed as a tunneling process. However, this method is currently limited to discussing the radiation of static spherically symmetric black holes only. In order to deeply research into the tunneling radiation effect, in this paper we extend the Parikh's method to investigate Hawking radiation of stationary axial symmetric Kerr-NUT black hole by using a new coordinate system well-behaved at the event horizon and implementing energy conservation and angular momentum conservation. This is a subject that is worth while studying but has not been studied before.

Kerr-NUT black hole describes an important solution of Einstein–Maxwell equations for electro-vacuum space-time,[10] which owns mass, angular momentum, and UNT parameters. Due to the angular speed of the black hole $\Omega \neq 0$ and UNT parameter $l \neq 0$, the discussion in this paper differs from Parikh's. By calculating the tunneling rate, we derive the corrected spectrum of the Kerr-NUT black hole. The result indicates that the tunneling rate is related to the change of the Bekenstein–Hawking entropy, but satisfies unitary quantum theory. It is a good correction to Hawking pure thermal spectrum. In special cases, the derived result can be reduced to the quantum tunneling radiation of the static Schwarzschild black hole. It also accordingly provides a might explanation to the paradox of the black hole information lost.

2 Event Horizon and Infinite Red-Shift Surface of Black Hole

The space-time metric for the Kerr-NUT black hole can be written as[11]

$$\mathrm{d}s^2 = -\frac{u^2}{\rho^2}(\mathrm{d}t_u - P\,\mathrm{d}\varphi)^2 + \frac{\rho^2}{u^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{\sin^2\theta}{\rho^2}[(F+l^2)\mathrm{d}\varphi - a\,\mathrm{d}t_u]^2\,, \tag{1}$$

in which t_u denotes the time coordinate of the Kerr-NUT black hole and

$$F = r^2 + a^2\,,$$

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$$u^2 = r^2 - 2Mr + a^2 - l^2\,,\quad P = a\sin^2\theta - 2l\cos\theta\,,\quad \rho^2 = r^2 + (l + a\cos\theta)^2\,. \tag{2}$$

We can also write the line element (1) as

$$\begin{aligned}\mathrm{d}s^2 = &-\Big(\frac{u^2 - a^2\sin^2\theta}{\rho^2}\Big)\mathrm{d}t_u^2 + \frac{\rho^2}{u^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2 + \frac{2u^2(a\sin^2\theta - 2l\cos\theta) - 2a\sin^2\theta(r^2 + a^2 + l^2)}{\rho^2}\mathrm{d}t\mathrm{d}\varphi \\ &- \frac{u^2(a\sin^2\theta - 2l\cos\theta)^2 - \sin^2\theta(r^2 + a^2 + l^2)^2}{\rho^2}\mathrm{d}\varphi^2\,.\end{aligned} \tag{3}$$

From the null super-surface equation

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0\,, \tag{4}$$

we can obtain the outer and inner event horizon of the black hole respectively,

$$r_+ = r_h = M + \sqrt{M^2 + l^2 - a^2}\,,\quad r_- = M - \sqrt{M^2 + l^2 - a^2}\,. \tag{5}$$

Firstly we calculate the area of the black hole. When t_u is constant, in the case of $r = r_h$, the line element (3) can be written as

$$\begin{aligned}\mathrm{d}\sigma^2 = &\rho^2\mathrm{d}\theta^2 - \frac{u^2(a\sin^2\theta - 2l\cos\theta)^2}{\rho^2}\mathrm{d}\varphi^2 \\ &+ \frac{\sin^2\theta(r_h^2 + a^2 + l^2)^2}{\rho^2}\mathrm{d}\varphi^2\,.\end{aligned} \tag{6}$$

The determinant of the two-dimensional metric above is

$$g = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33}\end{vmatrix} = \sin^2\theta(r_h^2 + a^2 + l^2)^2\,. \tag{7}$$

So the area of the black hole is

$$A_h = \int \mathrm{d}A = \int \sqrt{g}\,\mathrm{d}\theta\mathrm{d}\varphi = 4\pi(r_h^2 + a^2 + l^2)\,. \tag{8}$$

From $g_{00} = -(u^2 - a^2\sin^2\theta)/\rho^2 = 0$, we can arrive at the following infinite red-shift surfaces,

$$r_\pm^s = M \pm \sqrt{M^2 + l^2 - a^2\cos^2\theta}\,. \tag{9}$$

Clearly the infinite red-shift surfaces are not consistent with event horizons. So performing dragging coordinate transformation is necessary. Setting

$$\dot\varphi = \frac{\mathrm{d}\varphi}{\mathrm{d}t_u} = -\frac{g_{03}}{g_{33}}\,, \tag{10}$$

and substituting Eq. (10) into Eq. (3), we can get

$$\mathrm{d}s^2 = \hat g_{00}\mathrm{d}t_u^2 + \frac{\rho^2}{u^2}\mathrm{d}r^2 + \rho^2\mathrm{d}\theta^2\,, \tag{11}$$

where

$$\begin{aligned}\hat g_{00} &= g_{00} - \frac{g_{03}^2}{g_{33}} \\ &= -\frac{u^2\rho^2\sin^2\theta}{(r^2+a^2+l^2)^2\sin^2\theta - u^2(a\sin^2\theta - 2l\cos\theta)^2}\,.\end{aligned} \tag{12}$$

When $\hat g_{00} = 0$, the infinite red-shift surfaces can be obtained,

$$r_\pm^{irs} = M \pm \sqrt{M^2 + l^2 - a^2}\,. \tag{13}$$

Thus, we get the infinite red-shift surfaces coincide with the event horizons in dragging coordinate system.

3 General Painevé–Kerr-NUT Coordinate Transformation

It is necessary to eliminate coordinate singularity when one analyzes the Hawking radiation effect as tunneling at the event horizon of the black hole. In the expression (11), there still exits coordinate singularity in dragging coordinate system. Accordingly we further make general Painlevé coordinate transformation and set[12]

$$\mathrm{d}t_u = \mathrm{d}t + F(r,\theta)\mathrm{d}r + G(r,\theta)\mathrm{d}\theta\,, \tag{14}$$

in which the integrability condition is

$$\frac{\partial F(r,\theta)}{\partial\theta} = \frac{\partial G(r,\theta)}{\partial r}\,. \tag{15}$$

Substituting Eqs. (14) into Eqs. (11), we have

$$\begin{aligned}\mathrm{d}s^2 = &\hat g_{00}\mathrm{d}t^2 + \Big[\hat g_{00}F^2(r,\theta) + \frac{\rho^2}{u^2}\Big]\mathrm{d}r^2 \\ &+ \big[\hat g_{00}G^2(r,\theta) + \rho^2\big]\mathrm{d}\theta^2 + 2\hat g_{00}F(r,\theta)\mathrm{d}t\mathrm{d}r \\ &+ 2\hat g_{00}G(r,\theta)\mathrm{d}t\mathrm{d}\theta + 2\hat g_{00}F(r,\theta)G(r,\theta)\mathrm{d}r\mathrm{d}\theta\,.\end{aligned} \tag{16}$$

Considering that constant-time slices are flat Euclidean space in radial, we get

$$\hat g_{00}F^2(r,\theta) + \frac{\rho^2}{u^2} = 1\,, \tag{17}$$

namely

$$F(r,\theta) = \pm\sqrt{\frac{1 - \frac{\rho^2}{u^2}}{\hat g_{00}}}\,. \tag{18}$$

from which we can get the space-time line element in general Painlevé–Kerr-NUT coordinate,

$$\begin{aligned}\mathrm{d}s^2 = &\hat g_{00}\mathrm{d}t^2 + \mathrm{d}r^2 + 2\sqrt{\hat g_{00}\Big(1 - \frac{\rho^2}{u^2}\Big)}\mathrm{d}t\mathrm{d}r \\ &+ [\hat g_{00}G^2(r,\theta) + \rho^2]\mathrm{d}\theta^2 + 2\hat g_{00}G(r,\theta)\mathrm{d}t\mathrm{d}\theta \\ &+ 2\sqrt{\hat g_{00}\Big(1 - \frac{\rho^2}{u^2}\Big)}G(r,\theta)\mathrm{d}r\mathrm{d}\theta\,.\end{aligned} \tag{19}$$

According to Landau's condition of coordinate clock synchronization[13]

$$\frac{\partial}{\partial x^i}\Big(-\frac{g_{0j}}{\hat g_{00}}\Big) = \frac{\partial}{\partial x^j}\Big(-\frac{g_{0i}}{\hat g_{00}}\Big)\,, \tag{20}$$

we obtain

$$\frac{\partial F(r,\theta)}{\partial\theta} = \frac{\partial G(r,\theta)}{\partial r}\,, \tag{21}$$

which is equivalent to Eq. (15), then we can infer that the space-time line element in new coordinate satisfies the Landau's condition of coordinate clock synchronization. This is the essential condition to study the tunneling effect of the black hole. Moreover, according to expression

(19), the space-time line element in general Painlevé–Kerr-NUT coordinate possesses a number of nice features, such as no singularity at event horizon, coincidence between its event (outer) horizon, and outer infinite red-shift surface, and constant-time slices are flat Euclidean space in radial. All these are very advantageous for us to investigate the quantum tunneling radiation of black hole.

4 Quantum Tunneling Radiation Characteristics

From the metric (19), the radial null geodesics is given by

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\hat{g}_{00}\Big(1-\frac{\rho^2}{u^2}\Big)} \pm \sqrt{-\hat{g}_{00}\frac{\rho^2}{u^2}} = \frac{\pm\rho^2\sin\theta - \sqrt{\rho^2\sin^2\theta(\rho^2-u^2)}}{\sqrt{(r^2+a^2+l^2)\sin^2\theta - u^2(a\sin^2\theta-2l\cos\theta)^2}}\,. \tag{22}$$

where $\pm$ signs correspond to outgoing and ingoing geodesics respectively. Consider that a pair of virtual particles spontaneously create just inside the horizon, the positive energy virtual particle can tunnel out and the negative energy particle is absorbed by the black hole. Under considering self-gravitating action, energy conservation and angular momentum conservation, the particle is as a shell (an ellipsoid shell) of energy ω and angular momentum ωa. When a particle tunnels out, the black hole's mass will become $M-\omega$ and the angular momentum of the black hole will become $(M-\omega)a$. Meanwhile, the event horizon will shrink. Accordingly the line element (19) and Eq. (22) should replace M with $(M-\omega)$, and the event horizon and angular speed are, respectively,

$$r'_h = (M-\omega) + \sqrt{(M-\omega)^2+l^2-a^2}\,,$$
$$r'_- = (M-\omega) - \sqrt{(M-\omega)^2+l^2-a^2}\,,$$
$$\Omega'_h = \dot{\varphi} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{a}{a^2+r'^2_h+l^2}\,. \tag{23}$$

Since the outer event horizon coincides with outer infinite red-shift surface, geometrical optics limit exists. According to WKB approximation tunneling rate and the action of the particle satisfies[14,15]

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,S}\,. \tag{24}$$

The imaginary part of particle's action is

$$\mathrm{Im}\,S = \mathrm{Im}\int_{t_i}^{t_f}(L - P_\varphi\dot{\varphi})\mathrm{d}t = \mathrm{Im}\Big[\int_{r_i}^{r_f} P_r\,\mathrm{d}r - \int_{\varphi_i}^{\varphi_f} P_\varphi\,\mathrm{d}\varphi\Big] = \mathrm{Im}\Big[\int_{r_i}^{r_f}\int_0^{P_r}\mathrm{d}P'_r\,\mathrm{d}r - \int_{\varphi_i}^{\varphi_f}\int_0^{P_\varphi}\mathrm{d}P'_\varphi\,\mathrm{d}\varphi\Big]\,, \tag{25}$$

in which φ as an ignorable coordinate in the Lagrange function is considered and Eq. (25) is just the expression of action after amendments. Taking Hamilton equation into account, then

$$\dot{r} = \frac{\mathrm{d}H}{\mathrm{d}P_r}\Big|_{(r;\varphi,P_\varphi)} = \frac{\mathrm{d}(M-\omega')}{\mathrm{d}P_r}\,,$$
$$\dot{\varphi} = \frac{\mathrm{d}H}{\mathrm{d}P_\varphi}\Big|_{(\varphi;r,P_r)}\,,$$
$$\mathrm{d}H_{(\varphi;r,P_r)} = \Omega'_h\,\mathrm{d}\hat{J} = \Omega'_h a\,\mathrm{d}(M-\omega')\,. \tag{26}$$

so

$$\mathrm{Im}\,S = \mathrm{Im}\int_0^{\omega}\int_{r_i}^{r_f} -(1-a\Omega'_h)\frac{\mathrm{d}\omega'}{\dot{r}}\mathrm{d}r\,. \tag{27}$$

Taking integration on r, then

$$\int_{r_i}^{r_f}\frac{\sqrt{(r^2+a^2+l^2)^2\sin^2\theta - u'^2(a\sin^2\theta-2l\cos\theta)^2}}{\rho^2\sin\theta - \sqrt{\rho^2\sin^2\theta(\rho^2-u'^2)}}\mathrm{d}r = \int_{r_i}^{r_f}\frac{\big[\rho^2\sin\theta - \sqrt{\rho^2\sin^2\theta(\rho^2-u'^2)}\,\big]\sqrt{(r^2+a^2+l^2)^2\sin^2\theta - u'^2(a\sin^2\theta-2l\cos\theta)^2}}{\rho^2\sin^2\theta(r-r'_h)(r-r'_-)}\mathrm{d}r = -\,2\pi\mathrm{i}\,\frac{(r'^2_h+a^2+l^2)}{r'_h-r'_-}\,, \tag{28}$$

where $u'^2 = r^2+a^2-l^2-2(M-\omega')r$, $r_i = r_h-\varepsilon$ is the initial position corresponding to the tunneling particle, $r_f = r'_h+\varepsilon'$ is the final position, ε and ε' are very small quantities. We get this result by deforming the contour around the pole with $r = r'_h$. Then carrying on integrality on ω', we obtain

$$\mathrm{Im}\,S = \mathrm{Im}\int_0^{\omega}(1-a\Omega'_h)2\pi\mathrm{i}\,\frac{(r'^2_h+a^2+l^2)}{r'_h-r'_-}\mathrm{d}\omega' = 2\pi\int_0^{\omega}\frac{2\big[(M-\omega')^2-a^2+l^2\big]+a^2+2(M-\omega')\sqrt{(M-\omega')^2-a^2+l^2}}{\sqrt{(M-\omega')^2-a^2+l^2}}\mathrm{d}\omega' = -\pi\big[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2+l^2}-M\sqrt{M^2-a^2+l^2}\,\big]\,. \tag{29}$$

The tunneling rate of outgoing particle is

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,S} = \mathrm{e}^{2\pi\big[(M-\omega)^2-M^2+(M-\omega)\sqrt{(M-\omega)^2-a^2+l^2}-M\sqrt{M^2-a^2+l^2}\big]} = \mathrm{e}^{\Delta S_{\mathrm{BH}}}\,. \tag{30}$$

Namely we have

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,S} = \mathrm{e}^{(A'_h - A_h)/4} = \mathrm{e}^{S_{\mathrm{BH}}(M-\omega) - S_{\mathrm{BH}}(M)} = \mathrm{e}^{\Delta S_{\mathrm{BH}}}\,, \tag{31}$$

where $A'_h = 4\pi(r'^2_h + a^2 + l^2)$, A_h and A'_h are the areas of the Kerr-NUT black hole before and after radiation respectively, and $S_{\mathrm{BH}} = A_h/4$ is Bekenstein–Hawking entropy of the Kerr-NUT black hole.[16] Obviously, the spectrum of the black hole radiation described by Eq. (31) is not a pure thermal one.

5 Discussion

Expanding the tunneling rate in $\omega - \omega_0$ we have

$$\Gamma \sim \mathrm{e}^{\Delta S_{\mathrm{BH}}} = \exp\Big\{-\frac{\omega-\omega_0}{T}\Big[1-(\omega-\omega_0)\frac{r_h^2+a^2+l^2}{r_h^4}\Big(M+\sqrt{M^2-a^2+l^2}-\frac{M(a^2-l^2)}{2(M^2-a^2+l^2)}\Big)+\cdots\Big]\Big\}\,, \tag{32}$$

where $\omega_0 = M\Omega_h = Ma/(r_h^2 + a^2 + l^2)$. Neglecting high-order term of ω, we can obtain

$$\Gamma \sim \mathrm{e}^{\Delta S_{\mathrm{BH}}} = \mathrm{e}^{-(\omega-\omega_0)/T}\,. \tag{33}$$

Expression (33) is just the emission rate of the Hawking radiation which ignores self-gravitation action. So we come to the conclusion that the real radiation spectrum of Kerr-NUT black hole is not precisely thermal when energy conservation and angular momentum conservation are taken into account. The tunneling rate we obtained is more accurate and is a good correction to Hawking pure thermal spectrum.

When $l = 0$, Kerr-NUT black hole transforms back into Kerr black hole. According to the expression (30),

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,S} = \mathrm{e}^{2\pi\big[(M-\omega)^2 - M^2 + (M-\omega)\sqrt{(M-\omega)^2-a^2} - M\sqrt{M^2-a^2}\big]} = \mathrm{e}^{\Delta S_{\mathrm{BH}}}\,. \tag{34}$$

This is the tunneling rate of Kerr black hole.[17] When $l = a = 0$, the tunneling rate is

$$\Gamma \sim \mathrm{e}^{-2\,\mathrm{Im}\,S} = \mathrm{e}^{-8\pi\omega[M-(\omega/2)]} = \mathrm{e}^{\Delta S_{\mathrm{BH}}}\,. \tag{35}$$

This is the tunneling rate of Schwarzschild black hole, which is consistent with Parikh's.[9] Its radiation spectrum is consistent with pure spectrum of Hawking radiation only neglecting the terms of ω^2. The result we derived above shows that the black hole radiation causes the space-time background geometry to be varied. Because of self-gravitation and energy conservation and angular momentum conservation, the event horizon of black hole varies with black hole radiation, namely when the particle outgoes, the event horizon will contract and the two turning points pre-contraction and post contraction are the two points of barrier. The tunneling rate of particle is relevant to the mass M, the angle momentum a and UNT parameters l of Kerr-NUT black hole, and satisfies the underlying unitary theory.

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