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Functional Separable Solutions of Nonlinear Heat Equations in Non-Newtonian Fluids*

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Abstract *We study the functional separation of variables to the nonlinear heat equation:* $u_t = (A(x)D(u)u_x^n)_x + B(x)Q(u)$, $A_x \neq 0$. *Such equation arises from non-Newtonian fluids. Its functional separation of variables is studied by using the group foliation method. A classification of the equation which admits the functional separable solutions is performed. As a consequence, some solutions to the resulting equations are obtained.*

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1 Introduction

In this paper, we study the functional separation of variables (FSV) of nonlinear heat equation

$$u_t = (A(x)D(u)u_x^n)_x + B(x)Q(u)\,, \tag{1}$$

which is used to describe shear flows of non-Newtonian fluids.[1,2]

It is well known that the method of separation of variables is an efficient way to solve linear PDEs with certain initial/boundary value conditions in mathematical physics. The Lie point symmetry method[3–5] plays an important role in the study of the separation of variables to linear PDEs with variable coefficients.[6] For nonlinear PDEs, a natural question is whether they admit the functional separable solutions or generalized functional separable solutions. So it is of great interest to study the FSV of nonlinear PDEs. In the last decades, there have been several methods proposed to study the FSV, such as the R-matrix method,[7,8] the Ansatz-based method,[9–12] the geometric method,[13,14] the nonclassical method,[15] the generalized conditional symmetry method,[16–22] and the multi-linear variable separation approach.[23,24] Similar to self-similar solutions of nonlinear PDEs, functional separable solutions play an important role in characterizing the behavior of nonlinear phenomena.

Consider an n-th-order nonlinear evolution PDE with one dependent variable u and two independent variables x and t,

$$u_t = E(x, u, u_x, u_{xx}, u_{xxx}, \ldots)\,. \tag{2}$$

Its solutions of the form

$$u = \varphi(x)\psi(t) \quad \text{or} \quad u = \varphi(x) + \psi(t) \tag{3}$$

are said to be the product or additive separable solutions. Usually we call them the ordinary separable solutions. A generalization to the ordinary separable solutions is

$$f(u) = \varphi(x) + \psi(t)\,, \tag{4}$$

with $f(u) \neq u$ and $f(u) \neq \ln u$, we say it to be the functional separable solutions. It has been shown that a number of nonlinear PDEs have this type of separable solutions.[9–20] A further extension to the functional separable solutions is the derivative-dependent functional separable solutions,[21,22] which takes the form

$$f(u, u_x) = \varphi(x) + \psi(t)\,, \tag{5}$$

for some functions $f(u, u_x)$. Galaktionov *et al.*[25–27] introduced the concept of the nonlinear separation of variables in the study of blow-up of nonlinear parabolic equation, where the solutions take the form

$$u = \varphi(x)\psi(t) + \eta(t)\,. \tag{6}$$

A further extension to the nonlinear separable solution is the solution of the form

$$f(u) = \varphi(x)\psi(t) + \eta(t)\,, \tag{7}$$

which we call generalized functional separable solution.

2 Group Foliation Method

The group foliation method was introduced originally by Lie, developed by Vessiot,[28] and in a modern form by Ovsiannikov.[5] It is associated to the Lie point symmetries of the considered PDEs, and has been used successfully to obtain solutions of nonlinear PDEs.[29–34] The first step of the group foliation method is to foliate the solution space of the equation in question into orbits, choosing a symmetry group for the foliation. Each orbit is determined by the automorphic system joined to the original equations, and considered as invariant differential constraints. Due to automorphic property of this system, any of its solution can be obtained from any other solutions by a transformation of the chosen symmetry group. The collection of orbits of all solutions of the original equations is determined by the resolving system. So the problem is reduced to seek exact solutions of the group-resolving system. To obtain the functional separation solutions, our basic idea is to convert the given nonlinear PDEs into an equivalent first-order PDE system which characterizes the functional separation of variables or the generalized functional separable solutions. This system is the so-called resolving system in the group foliation method.

Assume that equation (2) does not depend on time t explicitly, then it is invariant under time translation. It means that the first-order invariants for the time translation are x, u, and

$$u_x = G(x, u)\,, \quad u_t = F(x, u)\,.$$

Using the group foliation method, we have the associated resolving system

$$F_x + GF_u = FG_u\,, \quad F = E(x, u, G, G_x, G_u, \ldots)\,. \tag{8}$$

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To obtain the functional separable solutions, we put $G(x,u) = g(x)h(u)$ and solve $F(x,u)$ from the first one in Eq. (8), we get

$$F(x,u) = h(u)f(\eta),\ \ \eta(t) = \int_0^u \frac{1}{h(s)}\mathrm{d}s - \int_0^x g(y)\mathrm{d}y,$$

where $g(x)$ and $h(u)$ are to be determined and $\eta = \eta(t)$ satisfies $\eta' = f(\eta)$. Substituting the expressions for $G(x,u)$ and $F(x,u)$ into Eq. (2), we obtain an equation depending on $A(x)$, $B(x)$, $D(u)$, $Q(u)$ and $h(u)$, each term in the expression is a product of u-dependent function and x-dependent function. The solutions of the equation depend on the dimension of the space spanned by the x-dependent functions. Consequently, the functional separable solutions take the form

$$H(u) = \int_0^u \frac{1}{h(s)}\mathrm{d}s = \int_0^x g(y)\mathrm{d}y + \eta(t)\,. \tag{9}$$

3 FSV to Nonlinear Heat Equation

We now consider FSV of the nonlinear heat equation (1). Using the group foliation method in Sec. 2 and putting $G(x,u) = g(x)h(u)$, we obtain

$$F(x,u) = h(u)f(\eta),\ \ \eta(t) = \int_0^u \frac{1}{h(s)}\mathrm{d}s - \int_0^x g(y)\mathrm{d}y\,.$$

Substituting $F(x,u)$, $G(x,u)$, and $\eta(t)$ into Eq. (1) implies f satisfying

$$f(\eta) = (Ag^n)_x Dh^{n-1} + Ag^{n+1}(Dh^n)_u + B\frac{Q}{h}, \tag{10}$$

where the functional separable solutions to Eq. (1) are given by Eq. (9). To obtain the functional separable solutions, we need to determine $g(x)$, $A(x)$, $B(x)$, $D(u)$, $Q(u)$, and $h(u)$. Noting that f does not depend on x, and by differentiating Eq. (10) with respect to x, we obtain

$$(Ag^n)_{xx}Dh^{n-1} + A_x g^{n+1}[(Dh^{n-1})_u h + (Dh^n)_u] + Ag^n g_x[nh(Dh^{n-1})_u + (n+1)(Dh^n)_u] + Ag^{n+2}h(Dh^n)_{uu} + B_x\frac{Q}{h} + Bgh\Big(\frac{Q}{h}\Big)_u = 0\,. \tag{11}$$

To proceed, we distinguish the following special cases.

Case 1 $h_{uu} = 0$, $h = u$. In this case, equation (11) leads to the following equation

$$(Ag^n)_{xx}Du^{n-1} + A_x g^{n+1}[(Du^{n-1})_u u + (Du^n)_u] + Ag^n g_x[nu(Du^{n-1})_u + (n+1)(Du^n)_u] + Ag^{n+2}u(Du^n)_{uu} + B_x\frac{Q}{u} + Bgu\Big(\frac{Q}{u}\Big)_u = 0\,. \tag{12}$$

To determine $A(x)$, $B(x)$, $D(u)$, and $Q(u)$, we consider two subcases.

Subcase 1.1 $A_x \neq 0$, $B = b = \text{Const.} \neq 0$. Without loss of generality we put $B = 1$.

In this subcase, equation (12) leads to

$$(Ag^n)_{xx}Du^{n-1} + A_x g^{n+1}[(Du^{n-1})_u u + (Du^n)_u] + Ag^{n+2}u(Du^n)_{uu} + Ag^n g_x[nu(Du^{n-1})_u + (n+1)(Du^n)_u] + gu\Big(\frac{Q}{u}\Big)_u = 0\,. \tag{13}$$

Let Γ denote the space spanned by $(Ag^n)_{xx}$, $Ag^n g_x$, $A_x g^{n+1}$, Ag^{n+2} and g. We solve Eq. (13) depending on the dimension of Γ. Let us restrict our attention to the situations $\dim\Gamma \geq 2$.

i) $\dim\Gamma = 2$. In this situation, A and g satisfy

$$Ag^n g_x = a_1 g + a_2 Ag^{n+2},\ \ A_x g^{n+1} = b_1 g + b_2 Ag^{n+2}. \tag{14}$$

Substituting Eq. (14) into Eq. (13), then the functions $Q(u)$ and $D(u)$ satisfy the following system of the ODE's,

$$[na_2^2(n+1) + a_2(b_2(1+2n) + 2n^2) + b_2(b_2+2n-1)]u^{n-1}D + [a_2(1+2n) + 2b_2]u^n D_u + u(u^n D)_{uu} = 0\,,$$

$$[(a_1(n+1) + b_1)(na_2 + b_2) + b_1(2n-1) + 2a_1 n^2]u^{n-1}D + [a_1(2n+1) + 2b_1]u^n D_u + u\Big(\frac{Q}{u}\Big)_u = 0\,.$$

To solve it, four cases are distinguished.

(a) $b_2 \neq -(n+1)a_2$, $a_2 \neq -(1+b_2)/n$.

$$D = bu^{\alpha+(1-n)} + u^{\beta+(1-n)},$$

$$Q = lu^{\alpha+1} - (na_1 + b_1)u^{\beta+1} + cu\,,$$

where and hereafter b and c denote constants of integration and α and β are constants, and $\alpha = -[b_2 + (n+1)a_2]$, $\beta = -(na_2 + 1 + b_2)$, $l = -b[(b_1 + na_1) + (a_2 - 1)(b_1 + a_1(n+1))/\alpha]$.

(b) $b_2 \neq -(n+1)a_2$, $a_2 = -(1+b_2)/n$.

$$D = bu^{1-n} + u^{2-n-a_2},$$

$$Q = cu - (2b_1 + 2na_1 + a_1)u^{2-a_2}\,.$$

(c) $b_2 = -(n+1)a_2$, $a_2 \neq -(1+b_2)/n$.

$$D = bu^{1-n} + u^{a_2-n}\,,$$

$$Q = b(a_2 - 1)[(n+1)a_1 + b_1]u\ln u - (na_1 + b_1)u^{a_2} + cu\,.$$

(d) $b_2 = -(n+1)a_2$, $a_2 = -(1+b_2)/n$.

$$D = bu^{1-n} + u^{1-n}\ln u\,,$$

$$Q = cu - (a_1 + 2na_1 + 2b_1)u\ln u\,.$$

ii) $\dim\Gamma = 3$. In this case, A and g fulfill the following system

$$A_x g^{n+1} = a_1 g + a_2 Ag^n g_x + a_3 Ag^{n+2}\,,$$
$$(Ag^n)_{xx} = b_1 g + b_2 Ag^n g_x + b_3 Ag^{n+2}\,. \tag{15}$$

Substituting Eq. (15) into Eq. (13) we arrive at the system

$$[b_2 + a_2(2n-1) + 2n^2]u^{n-1}D + (1 + 2a_2 + 2n)u^n D_u = 0\,,$$

$$[b_1 + a_1(2n-1)]u^{n-1}D + 2a_1 u^n D_u + u\Big(\frac{Q}{u}\Big)_u = 0\,,$$

$$[b_3 + a_3(2n-1)]u^{n-1}D + 2a_3 u^n D_u + u(u^n D)_{uu} = 0\,. \tag{16}$$

Solving Eq. (16) we obtain the solutions as follows.

(a) $a_2 \neq -(2n+1)/2$, $b_2 \neq -(a_2 + 1 + n)$,

$$D = u^\alpha,\ \ \ Q = \frac{b_1 + a_1(2\alpha + 2n - 1)}{\alpha + n - 1}u^{\alpha+n} + bu\,,$$

where $\alpha = -[(2n-1)a_2 + b_2 + 2n^2]/(2a_2 + 2n + 1)$, $b_3 = -[a_3(2n + 2\alpha - 1) + (n+\alpha)(n+\alpha-1)]$.

(b) $a_2 \neq -(2n+1)/2$, $b_2 = -(a_2 + 1 + n)$.

$$D = u^{1-n}\,, \quad Q = bu - (a_1 + b_1)u\ln u\,,$$

where $b_3 = -a_3$.

iii) $\dim\Gamma = 4$. In this case, A and g satisfy

$$(Ag^n)_{xx} = a_1 A g^n g_x + a_2 A_x g^{n+1} + a_3 A g^{n+2} + a_4 g\,.$$

We thus arrive at the system

$$(a_1 + 2n^2)D + (2n+1)uD_u = 0\,,$$
$$(a_2 + 2n - 1)D + 2uD_u = 0\,,$$
$$(a_3 + n^2 - n)D + 2nuD_u + u^2 D_{uu} = 0\,,$$
$$a_4 u^{n-1} D + u\Big(\frac{Q}{u}\Big)_u = 0\,,$$

which is easily solved with respect to the constants a_i.

(a) $a_1 \neq -(n+1)$.

$$D = u^{\alpha}\,, \quad Q = bu - \frac{a_4}{n-1+\alpha}u^{\alpha+n}\,,$$

where $\alpha = -(a_1 + 2n^2)/(2n+1)$ and the constants satisfy $a_2 = -(2n - 1 + 2\alpha)$, $a_3 = -(n+\alpha)(n+\alpha-1)$.

(b) $a_1 = -(n+1)$, $a_2 \neq -1$.

$$D = u^{[1-(2n+a_2)]/2}\,, \quad Q = \frac{2a_4}{1+a_2}u^{(1-a_2)/2} + bu\,,$$

where the constants satisfy $a_1 = a_2(n + 1/2) - 1/2$, $a_3 = (1 - a_2^2)/4$.

(c) $a_1 = -(n+1)$, $a_2 = -1$.

$$D = u^{1-n}\,, \quad Q = bu - a_4 u\ln u\,,$$

where the constants satisfy $a_3 = 0$.

Subcase 1.2 $A_x \neq 0$, $B_x \neq 0$.

Similarly, let Γ denote a linear space spanned by $(Ag^n)_{xx}$, $Ag^n g_x$, $A_x g^{n+1}$, Ag^{n+2}, Bg, and B_x. To solve Eq. (12) and determine Q and D, we consider the following cases.

i) $\dim\Gamma = 2$. In this case, A, B, and g satisfy

$$B_x = a_1 Bg + b_1 A g^{n+2}\,,$$
$$A_x g^{n+1} = a_2 Bg + b_2 A g^{n+2}\,,$$
$$A g^n g_x = a_3 Bg + b_3 A g^{n+2}\,.$$

Then Q and D satisfy

$$\begin{aligned}&[b_2^2 + b_2((2n+1)b_3 + 2n - 1) + nb_3((n+1)b_3 + 2n) + b_1(a_2 + na_3)]u^n D\\ &+ [(2n+1)b_3 + 2b_2]u^{n+1}D_u + u^2(Du^n)_{uu} + b_1 Q = 0\,,\\ &[b_2((n+1)a_3 + a_2) + nb_3(a_2 + (n+1)a_3) + a_1(a_2 + na_3) + (2n-1)a_2 + 2n^2 a_3]u^n D\\ &+ [(2n+1)a_3 + 2a_2]u^{n+1}D_u + a_1 Q + u^2\Big(\frac{Q}{u}\Big)_u = 0\,. \end{aligned} \tag{17}$$

We derive the solutions of Eq. (17) as follows.

(a) $b_1 \neq 0$.

$$D = u^{\alpha} + bu^{\beta} + cu^{\gamma}\,, \quad Q = -\Big[(a_2 + na_3)u^{\alpha+n} + \frac{b}{2b_1}m_1 u^{\beta+n} + \frac{c}{2b_1}m_2 u^{\gamma+n}\Big]\,,$$

where α, β, γ, m_1, and m_2 fulfill the following system,

$$\begin{aligned}&\alpha = -[b_2 + n(1+b_3)]\,, \quad \beta = l_1 - \frac{l_2}{2}\,, \quad \gamma = l_1 + \frac{l_2}{2}\,, \quad l_1 = \frac{\alpha}{2} - \frac{n}{2} + \frac{\alpha + n + b_2}{2n} + 1 - \frac{a_1}{2}\,,\\ &l_2 = \Big[\Big(1 + \frac{1}{n}\Big)(\alpha + n + b_2)\Big(\Big(1 + \frac{1}{n}\Big)(\alpha + n + b_2) + 2(a_1 - b_2)\Big) + (b_2 - a_1)^2 + 4b_1(a_2 + na_3 + a_3)\Big]^{1/2}\,,\\ &m_1 = l_3 - l_2(a_1 + \alpha + n - 1)\,, \quad m_2 = l_3 + l_2(a_1 + \alpha + n - 1)\,. \end{aligned} \tag{18}$$

(b) $b_1 = 0$, $a_1 \neq b_2 + (n+1)b_3$.

$$D = u^{\alpha} + bu^{\beta}\,, \quad Q = \frac{m_3}{\alpha + a_1 + (n-1)}u^{\alpha+n} - b(na_3 + a_2)u^{\beta+n} + cu^{1-a_1}\,,$$

where α, β, and m_3 satisfy

$$\alpha = -[b_2 - 1 + n + b_3(1+n)]\,, \quad \beta = -[b_2 + n(b_3 + 1)]\,,$$
$$m_3 = \beta[(n+1)a_3 + a_2] - \alpha[2a_2 + (2n+1)a_3] - (a_2 + na_3)(1 - n - a_1)\,.$$

(c) $b_1 = 0$, $a_1 = b_2 + (n+1)b_3$, $b_3 \neq 1$.

$$D = u^{\alpha} + bu^{\beta}\,, \quad Q = [(n+1)a_3 + a_2]u^{\alpha+n}\ln u - b(a_2 + na_3)u^{\beta+n} + cu^{\alpha+n}\,,$$

where $\alpha = -[b_2 - 1 + n + (1+n)b_3]$, $\beta = -[b_2 + n(1+b_3)]$.

(d) $b_1 = 0$, $a_1 = b_2 + (n+1)b_3$, $b_3 = 1$.

$$D = u^{-(2n+b_2)} + bu^{-(2n+b_2)}\ln u\,, \quad Q = -[(2n+1)a_3 + 2a_2]u^{-(b_2+n)}\ln u + bu^{-(b_2+n)}\,.$$

ii) $\dim\Gamma = 3$. In this case, A, B, and g satisfy the following system,

$$B_x = a_1 Bg + b_1 A g^{n+2} + c_1 A g^n g_x\,, \quad A_x g^{n+1} = a_2 Bg + b_2 A g^{n+2} + c_2 A g^n g_x\,, \quad (Ag^n)_{xx} = a_3 Bg + b_3 A g^{n+2} + c_3 A g^n g_x\,.$$

It implies that Q and D satisfy

$$\begin{aligned}&(a_1 - 1)Q + uQ_u + [(2n-1)a_2 + a_3]u^n D + 2a_2 u^{n+1}D_u = 0\,,\\ &b_1 Q + [(2n-1)b_2 + b_3 + n(n-1)]D + 2(n + b_2)D_u + u^{n+2}D_{uu} = 0\,,\\ &c_1 Q + (2c_2 + 2n + 1)u^{n+1}D_u + [c_3 + (2n-1)c_2 + 2n^2]u^n D = 0\,. \end{aligned} \tag{19}$$

We obtain the solutions of Eq. (19) as follows:

(a) $c_1 \neq 0$.

$$D = u^{\alpha} + bu^{\beta}\,, \quad Q = \frac{1}{2c_1^2}(lbu^{\beta+n} - mu^{\alpha+n})\,,$$

where the constants satisfy $h_4 = m(\alpha + a_1 - 1 + n)$, $h_5 = -l(\beta + a_1 - 1 + n)$, and α, β, m, and l are given by

$$\alpha = -\frac{1}{2c_1}(h_1 - h_2)\,, \quad \beta = -\frac{1}{2c_1}(h_1 + h_2)\,, \quad m = h_2(2n + 2c_2 + 1) + h_3\,,$$
$$l = h_2(2n + 2c_2 + 1) - h_3\,, \quad h_1 = c_1(2b_2 + 2n - 1) - b_1(1 + 2n + 2c_2)\,,$$
$$h_2 = \big[4b_1^2\big(c_2 + n + \tfrac{1}{2}\big)^2 - 4b_1c_1\big((1 + 2n + 2c_2)b_2 - c_3 - \tfrac{1}{2}\big) - 4c_1^2\big(b_3 - b_2^2 - \tfrac{1}{4}\big)\big]^{1/2}\,,$$
$$h_3 = 4b_1\big(c_2 + n + \tfrac{1}{2}\big)^2 - 4c_1\big[b_2\big(n + c_2 + \tfrac{1}{2}\big) - \tfrac{1}{4} - \tfrac{c_3}{2}\big]\,,$$
$$h_4 = 2c_1^2[a_2(2\alpha + 2n - 1) + a_3]\,, \quad h_5 = 2c_1^2[a_2(2\beta + 2n - 1) + a_3]\,.$$

(b) $c_1 = 0$, $c_2 \neq -(2n + 1)/2$.

$$D = u^{\alpha}\,, \quad Q = \lambda u^{\alpha+n}\,,$$

where $\alpha = [c_2(1 - 2n) - c_3 - 2n^2]/(2n + 1 + 2c_2)$, $\lambda = -[(n + \alpha)(n + \alpha - 1 + 2b_2) - b_2 + b_3]/b_1$, and the constants satisfy $a_3 = -[\lambda(\alpha + n + a_1 - 1) + a_2(2n + 2\alpha - 1)]$.

iii) $\dim\Gamma = 4$. In this case, A, B, and g satisfy

$$B_x = a_1Bg + a_2Ag^{n+2} + a_3Ag^ng_x + a_4A_xg^{n+1}\,, \quad (Ag^n)_{xx} = b_1Bg + b_2Ag^{n+2} + b_3Ag^ng_x + b_4A_xg^{n+1}\,. \tag{20}$$

Substituting Eq. (20) into Eq. (12), we obtain the system

$$b_1u^nD + (a_1 - 1)Q + uQ_u = 0, \quad a_2Q + (b_2 + n^2 - n)u^nD + 2nu^{n+1}D_u + u^{n+2}D_{uu} = 0\,,$$
$$a_3\frac{Q}{u} + (b_3 + 2n^2)u^nD + (2n + 1)u^{n+1}D_u = 0\,, \quad a_4Q + (b_4 + 2n - 1)u^nD + 2u^{n+1}D_u = 0\,. \tag{21}$$

The solutions of Eq. (21) are as follows.

(a) $a_3 \neq 0$, $a_3 \neq [(2n + 1)a_4]/2$.

$$D = u^{\alpha}\,, \quad Q = ku^{\alpha+n}\,,$$

where $\alpha = [a_3(2n - 1 + b_4) - a_4(2n^2 + b_3)]/[(2n + 1)a_4 - 2a_3]$, $k = -[(2n+1)b_4 - 2b_3 - 1]/[(2n+1)a_4 - 2a_3]$, and the constants in Eq. (21) satisfy $b_1 = -k(n - 1 + a_1 + \alpha)$, $b_2 = a_2k - (n + \alpha - 1)(n + \alpha)$.

(b) $a_3 = 0$, $a_4 \neq 0$.

$$D = u^{\alpha}\,, \quad Q = ku^{\alpha+n}\,,$$

where $\alpha = -(2n^2 + b_3)/(1 + 2n)$, $k = (2b_3 + 1 - b_4 - 2nb_4)/(a_4 + 2na_4)$, and the constants satisfy $b_1 = -(n + \alpha + a_1 - 1)k$, $b_2 = -[(n + \alpha)(n + \alpha - 1) + ka_2]$.

(c) $a_3 = a_4 = 0$, $a_2 \neq 0$.

$$D = u^{\alpha}, \quad Q = ku^{\alpha+n}\,,$$

where $\alpha = (1 - 2n - b_4)/2$, $k = -(b_4^2 + 4b_2 - 1)/(4a_2)$, and the constants satisfy $b_1 = -k(n - 1 + a_1 + \alpha)$, $b_3 = -(2n^2 + 2n\alpha + \alpha)$.

(d) $a_2 = a_3 = a_4 = 0$, $b_3 \neq (2n + 1)a_1 - n - 1$.

$$D = u^{\alpha}, \quad Q = ku^{\alpha+n} + bu^{1-a_1}\,,$$

where $\alpha = -(2n^2 + b_3)/(2n+1)$, $k = -b_1/(\alpha + n - 1 + a_1)$, and the constants satisfy $b_4 = 1 - 2\alpha - 2n$, $b_2 = -(\alpha + n)(\alpha + n - 1)$.

(e) $a_2 = a_3 = a_4 = 0$, $b_3 = (2n + 1)a_1 - n - 1$.

$$D = u^{-a_1}, \quad Q = -b_1u^{n-a_1}\ln u + bu^{n-a_1}\,,$$

where the constants satisfy $b_2 = -(n - a_1)(n - 1 - a_1)$, $b_4 = 2a_1 - 2n + 1$.

Case 2 $h_{uu} = 0$, $h = 1$. In this case, equation (1) has additive separable solutions,

$$H(u) = u = \int_0^x g(y)\,\mathrm{d}y + \eta(t)\,,$$

and equation (11) implies

$$(Ag^n)_{xx}D + 2A_xg^{n+1}D_u + Ag^ng_x(2n + 1)D_u + Ag^{n+2}D_{uu} + B_xQ + BgQ_u = 0\,. \tag{22}$$

As in Case 1, we consider the following subcases.

Subcase 2.1 $A_x \neq 0$, $B = b \neq 0$. Without loss of generality we put $B = 1$. In this subcase, equation (22) can be rewritten as

$$(Ag^n)_{xx}D + 2A_xg^{n+1}D_u + Ag^ng_x(2n + 1)D_u + Ag^{n+2}D_{uu} + gQ_u = 0\,.$$

Let Γ be the space spanned by $(Ag^n)_{xx}$, Ag^ng_x, A_xg^{n+1}, Ag^{n+2}, and g. Similarly we consider three cases.

i) $\dim\Gamma = 2$. In this case, A and g fulfill the following system

$$Ag^ng_x = a_1g + a_2Ag^{n+2}\,, \quad A_xg^{n+1} = b_1g + b_2Ag^{n+2},$$

and Q and D satisfy

$$[(na_2 + b_2)(b_1 + (n + 1)a_1)]D + [(2n + 1)a_1 + 2b_1]D_u + Q_u = 0\,,$$
$$[(n^2 + n)a_2^2 + (2n + 1)b_2a_2 + b_2^2]D + [(2n + 1)a_2 + 2b_2]D_u + D_{uu} = 0\,. \tag{23}$$

Solving Eq. (23) we obtain the solutions as follows.

(a) $b_2 \neq -(n + 1)a_2$.

$$D = \mathrm{e}^{\lambda u} + b\,\mathrm{e}^{(\lambda + a_2)u}\,,$$
$$Q = k\,\mathrm{e}^{\lambda u} - b(na_1 + b_1)\,\mathrm{e}^{(\lambda + a_2)u} + c\,,$$

where $\lambda = -[(1 + n)a_2 + b_2]$, $k = [a_2(na_1 + b_1 + a_1) - \lambda(na_1 + b_1)]/\lambda$.

(b) $b_2 = -(n + 1)a_2$, $a_2 \neq 0$

$$D = b + \mathrm{e}^{a_2u}\,,$$
$$Q = ba_2(a_1 + na_1 + b_1)u - (na_1 + b_1)\,\mathrm{e}^{a_2u} + c\,.$$

(c) $a_2 = 0$, $b_2 \neq 0$.

$$D = \mathrm{e}^{-b_2u} + bu\,\mathrm{e}^{-b_2u}\,, \quad Q = mu\,\mathrm{e}^{-b_2u} + k\,\mathrm{e}^{-b_2u} + c\,,$$

where $m = -(na_1 + b_1)$, $k = b(b_1 + a_1 + na_1)/b_2 - na_1 - b_1$.

(d) $a_2 = b_2 = 0$.

$$D = u\,, \quad Q = -(2b_1 + a_1 + 2na_1)u + b\,.$$

ii) $\dim\Gamma = 3$. In this case, A and g satisfy

$$A_xg^{n+1} = a_1g + a_2Ag^ng_x + a_3Ag^{n+2}\,,$$
$$(Ag^n)_{xx} = b_1g + b_2Ag^ng_x + b_3Ag^{n+2}\,,$$

and Q and D fulfill the system

$$b_3D + 2a_3D_u + D_{uu} = 0\,, \quad b_1D + 2a_1D_u + Q_u = 0\,,$$
$$b_2D + (2a_2 + 2n + 1)D_u = 0\,. \tag{24}$$

Solving Eq. (24) we obtain the solutions as follows.

(a) $a_3^2 - b_3 > 0$, $b_3 \neq 0$. We have the solutions given by

$$D = \mathrm{e}^{\lambda_1 u} + b\,\mathrm{e}^{\lambda_2 u},$$
$$Q = \frac{2a_1b_3 - b_1\lambda_1}{b_3}\,\mathrm{e}^{\lambda_1 u} + \frac{2a_1b_3 - b_2\lambda_2}{b_3} b\,\mathrm{e}^{\lambda_2 u} + c\,,$$

where $\lambda_1 = -a_3 + (a_3^2 - b_3)^{1/2}$, $\lambda_2 = -a_3 - (a_3^2 - b_3)^{1/2}$, and the coefficients fulfill $b_2 = 0$, $a_2 = -(n+1/2)$.

(b) $a_3^2 - b_3 = 0$, $a_3 \neq 0$. We have the solutions given by

$$D = b\,\mathrm{e}^{-a_3 u} + u\,\mathrm{e}^{-a_3 u},$$
$$Q = k_1\,\mathrm{e}^{-a_3 u} + k_2 u\,\mathrm{e}^{-a_3 u} + c\,,$$

where $k_1 = b_1/a_3^2 + b(b_1 - 2a_1a_3)/a_3$, $k_2 = b_1/a_3 - 2a_1$ and the constants satisfy $b_2 = a_3(2a_2 + 1 + 2n)$, $a_2 = -(2n+1)/2$.

(c) $a_3^2 - b_3 < 0$, $a_3 \neq 0$. We have the solutions given by

$$D = [\cos(\tau u) + b\sin(\tau u)]\,\mathrm{e}^{-a_3 u}\,,$$
$$Q = \frac{b_1 - 2a_1a_3}{a_3}[\cos(\tau u) + b\sin(\tau u)]\,\mathrm{e}^{-a_3 u} + c\,,$$

where $\tau = (b_3 - a_3^2)^{1/2}$, and the constants satisfy $b_2 = a_3(2a_2 + 1 + 2n)$.

iii) $\dim\Gamma = 4$. In this case, A and g satisfy

$$(Ag^n)_{xx} = a_1Ag^ng_x + a_2A_xg^{n+1} + a_3Ag^{n+2} + a_4g\,,$$

and Q and D fulfill the system

$$a_3D + D_{uu} = 0\,,\quad a_2D + 2D_u = 0\,,$$
$$a_4D + Q_u = 0\,,\quad a_1D + (2n+1)D_u = 0\,. \tag{25}$$

Solving Eq. (25) we obtain the solutions as follows.

$$D = \mathrm{e}^{-a_2u/2},\quad Q = \frac{2a_4}{a_2}\,\mathrm{e}^{-a_2u/2} + b\,,$$

and the constants satisfy $a_3 = -a_2^2/4$, $a_1 = (n+1/2)a_2$.

Subcase 2.2 $A_x \neq 0$, $B_x \neq 0$. Similarly, let Γ denote a linear space spanned by $(Ag^n)_{xx}$, Ag^ng_x, A_xg^{n+1}, Ag^{n+2}, Bg, and B_x. To solve Eq. (22) and determine Q and D, we consider the following cases.

i) $\dim\Gamma = 2$. In this case, A, B, and g satisfy

$$B_x = a_1Bg + b_1Ag^{n+2}\,,$$
$$A_xg^{n+1} = a_2Bg + b_2Ag^{n+2}\,,$$
$$Ag^ng_x = a_3Bg + b_3Ag^{n+2}\,.$$

Then Q and D satisfy

$$[b_2^2 + (2n+1)b_3b_2 + nb_3^2(n+1) + b_1(a_2 + na_3)]D + [(2n+1)b_3 + 2b_2]D_u + D_{uu} + b_1Q = 0\,,$$
$$[((n+1)a_3 + a_2)(b_2 + nb_3) + a_1(a_2 + na_3)]D + [(2n+1)a_3 + 2a_2]D_u + a_1Q + Q_u = 0\,. \tag{26}$$

We derive the solutions of Eq. (26) as follows.

(a) $b_1 \neq 0$.

$$D = \mathrm{e}^{\lambda_1 u} + b\,\mathrm{e}^{\lambda_2 u} + c\,\mathrm{e}^{\lambda_3 u},\quad Q = -\Big[(na_3 + a_2)\,\mathrm{e}^{\lambda_1 u} + \frac{bm_1}{2b_1}\,\mathrm{e}^{\lambda_2 u} + \frac{cn_1}{2b_1}\,\mathrm{e}^{\lambda_3 u}\Big],$$

where λ_1, λ_2, λ_3, m_1, and n_1 fulfill the following system,

$$\lambda_1 = -(b_2 + nb_3)\,,\quad \lambda_2 = l_1 + \frac{l_2}{2}\,,\quad \lambda_3 = l_1 - \frac{l_2}{2}\,,$$
$$l_1 = -\frac{b_2 + (n+1)b_3 + a_1}{2}\,,\quad m_1 = l_3 + l_2(a_1 + \lambda_1)\,,\quad n_1 = l_3 - l_2(a_1 + \lambda_1)\,,$$
$$l_2 = [b_3^2(n+1)^2 + (b_2 - a_1)(b_2 - a_1 + 2b_3(n+1)) + 4b_1((n+1)a_3 + a_2)]^{1/2}\,. \tag{27}$$

(b) $b_1 = 0$, $b_2 \neq a_1 - (n+1)b_3$.

$$D = \mathrm{e}^{\lambda_1 u} + b\,\mathrm{e}^{\lambda_2 u},\quad Q = c\,\mathrm{e}^{-a_1 u} - \frac{m_3}{\lambda_1 + a_1}\,\mathrm{e}^{\lambda_1 u} - b(na_3 + a_2)\,\mathrm{e}^{\lambda_2 u}\,,$$

where $\lambda_1 = -[b_2 + (1+n)b_3]$, $\lambda_2 = -(b_2 + nb_3)$, $m_3 = \lambda_2[a_2 + (n+1)a_3] - \lambda_1[(2n+1)a_3 + 2a_2] - a_1(a_2 + na_3)$.

(c) $b_1 = 0$, $b_2 = a_1 - (n+1)b_3$, $b_3 \neq 0$.

$$D = u^{\lambda_1 u} + bu^{\lambda_2 u}\,,\quad Q = bb_3[(n+1)a_3 + a_2]u\,\mathrm{e}^{\lambda_1} - (a_2 + na_3)\,\mathrm{e}^{\lambda_2 u} + c\,\mathrm{e}^{\lambda_1 u}\,,$$

where $\lambda_1 = -[b_2 + (n+1)b_3]$, $\lambda_2 = -(b_2 + nb_3)$.

(d) $b_1 = 0$, $b_2 = a_1 - (n+1)b_3$, $b_3 = 0$.

$$D = \mathrm{e}^{-b_2 u} + bu\,\mathrm{e}^{-b_2 u}\,,\quad Q = \mathrm{e}^{-b_2 u} - b[(2n+1)a_3 + 2a_2]u\,\mathrm{e}^{-b_2 u}\,.$$

ii) $\dim\Gamma = 3$. In this case, A, B and g satisfy the following system,

$$B_x = a_1Bg + b_1Ag^{n+2} + c_1Ag^ng_x\,,\quad A_xg^{n+1} = a_2Bg + b_2Ag^{n+2} + c_2Ag^ng_x\,,\quad (Ag^n)_{xx} = a_3Bg + b_3Ag^{n+2} + c_3Ag^ng_x\,.$$

It implies that Q and D satisfy

$$a_1Q + Q_u + 2a_2D_u + a_3D = 0\,,\quad b_1Q + b_3D + 2b_2D_u + D_{uu} = 0\,,\quad c_1Q + c_3D + (2c_2 + 2n + 1)D_u = 0\,. \tag{28}$$

We obtain the solutions of Eq. (28) as follows.

(a) $c_1 \neq 0$.

$$D = \mathrm{e}^{\lambda_1 u} + b\,\mathrm{e}^{\lambda_2 u}\,,\quad Q = \frac{1}{2c_1^2}(lb\,\mathrm{e}^{\lambda_2 u} - m\,\mathrm{e}^{\lambda_1 u})\,,$$

where the coefficients satisfy $2c_1^2(2\lambda_1a_2 + a_3) - m(a_1 + \lambda_1) = 0$, $2c_1^2(2\lambda_2a_2 + a_3) - l(a_1 + \lambda_2) = 0$, and λ_1, λ_2, m, and l are given by

$$\lambda_1 = \frac{1}{2c_1}(h_1 + h_2)\,,\quad \lambda_2 = \frac{1}{2c_1}(h_1 - h_2)\,,\quad m = h_2(2n + 2c_2 + 1) + h_3\,,$$

$$l = h_2(2n + 2c_2 + 1) - h_3\,, \quad h_1 = b_1(2n + 2c_2 + 1) - 2b_2c_1\,,$$
$$h_2 = [b_1^2((2n + 1)^2 + 4c_2(2n + 1 + c_2)) - 4b_1b_2c_1(2n + 1 + 2c_2) + 4c_1^2(b_2^2 - b_3) + 4c_1c_3b_1]^{1/2}\,,$$
$$h_3 = (2n + 2c_2 + 1)^2 - 2c_1[b_2(2c_2 + 1 + 2n) - c_3]\,.$$

(b) $c_1 = 0$, $b_1 \neq 0$, $c_2 \neq -(2n + 1)/2$.

$$D = \mathrm{e}^{\lambda u}\,, \quad Q = a\,\mathrm{e}^{\lambda u}\,,$$

where $\lambda = -c_3/(2n+1+2c_2)$, $a = c_3[2b_2(2c_2+1+2n) - c_3]/[b_1(2c_2+2n+1)^2] - b_3/b_1$, and the constants satisfy $a_3 = -[2\lambda a_2 + a(a_1 + \lambda)]$.

(c) $c_1 = 0$, $b_1 = 0$, $c_2 = -(2n + 1)/2$,

$$D = \mathrm{e}^{\lambda u}\,, \quad Q = a\,\mathrm{e}^{\lambda u} + b\,\mathrm{e}^{-a_1 u}\,,$$

where $\lambda = -c_3/(2n+1+2c_2)$, $a = -\lambda(a_3+2a_2)/(\lambda+a_1)$, and the constants satisfy $b_3 = -\lambda(\lambda + 2b_2)$.

iii) $\dim\Gamma$=4. In this case, A, B, and g satisfy

$$B_x = a_1Bg + a_2Ag^{n+2} + a_3Ag^ng_x + a_4A_xg^{n+1}\,,$$
$$(Ag^n)_{xx} = b_1Bg + b_2Ag^{n+2} + b_3Ag^ng_x + b_4A_xg^{n+1}\,. \quad (29)$$

Substituting Eq. (29) into Eq. (22), we obtain the system

$$b_1D + a_1Q + Q_u = 0\,, \quad a_2Q + b_2D + D_{uu} = 0\,,$$
$$a_3Q + b_3D + (2n + 1)D_u = 0,\ a_4Q + b_4D + 2D_u = 0. \quad (30)$$

The solutions of Eq. (30) are as follows.

(a) $a_4 \neq 0$, $a_3 \neq [(2n + 1)a_4]/2$.

$$D = \mathrm{e}^{\lambda u}\,, \quad Q = a\,\mathrm{e}^{\lambda u}\,,$$

where $\lambda = (a_3b_4 - a_4b_3)/[(2n + 1)a_4 - 2a_3]$, $a = -(2\lambda + b_4)/a_4$, and the constants satisfy

$$a_1b_4 - b_1a_4 + \lambda(b_4 + 2\lambda + 2a_1) = 0\,,$$
$$a_2b_4 - b_2a_4 - \lambda(a_4\lambda + 2a_2) = 0\,.$$

(b) $a_3 = a_4 = 0$, $a_2 \neq 0$,

$$D = \mathrm{e}^{-b_4u/2}\,, \quad Q = -\frac{b_4^2 + 4b_2}{4a_2}\,\mathrm{e}^{-b_4u/2}\,,$$

where the constants satisfy $b_3 = (n + 1/2)b_4$, $8(b_1a_2 - a_1b_2) + b_4(b_4^2 - 2a_1b_4 + 4b_2) = 0$.

(c) $a_2 = a_3 = a_4 = 0$,

$$D = \mathrm{e}^{-b_4u/2}\,, \quad Q = b\,\mathrm{e}^{-a_1u} - \frac{2b_1}{2a_1 - b_4}\,\mathrm{e}^{-b_4u/2}\,,$$

where the constants satisfy $b_2 = -b_4^2/4$, $b_3 = b_4(n + 1/2)$.

4 Concluding Remarks

In this paper, we have developed the group foliation method to study the functional separation of variables of the nonlinear heat equations arising from non-Newtonian fluids. It was shown that for certain diffusion and source terms of Eq. (1), it admits functional separation of variables. In general, the functional separable solutions can not be obtained within the framework of the Lie's classical method. A question is that whether equation (1) admits other non-Lie point symmetries such as the conditional Lie Bäcklund symmetries with second-order characteristic.

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