

Solutions and Conditional Lie–Bäcklund Symmetries of Quasi-linear Diffusion-Reaction Equations*

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Abstract *New classes of exact solutions of the quasi-linear diffusion-reaction equations are obtained by seeking for the high-order conditional Lie–Bäcklund symmetries of the considered equations. The method used here extends the approaches of derivative-dependent functional separation of variables and the invariant subspace. Behavior to some solutions such as blow-up and quenching is also described.*

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1 Introduction

This paper is devoted to obtaining new classes of exact solutions and conditional Lie–Bäcklund symmetries of the quasi-linear diffusion-reaction equation,

$$u_t = f(u, u_x)u_{xx} + g(u, u_x), \quad (1)$$

where u is a function of two independent variables t and x .

Equations of the form (1) with certain $f(u, u_x)$ and $g(u, u_x)$ arise from several important physical situations such as microwave heating,^[1] theory of chemical reactions,^[2] mathematical biology,^[3] and solid state physics,^[4] etc. It is well-known that exact solutions play an important role in understanding the structure and properties of the general solutions of Eq. (1).^[5] So it is of interest to explore various methods of constructing exact solutions of nonlinear partial differential equation (PDEs). Indeed, several methods have been used to seek for symmetry reductions and construct exact solution of Eq. (1). The most popular method for constructing exact solutions of Eq. (1) is the Lie's classical method.^[6–8] A direct generalization of the Lie's classical method is the nonclassical symmetry method due to Bluman and Cole,^[9] which has been used to construct solutions of Eq. (1),^[8–11] some of which cannot be recovered by the classical symmetry method. A generalization of both classical and nonclassical methods is the so-called conditional Lie–Bäcklund symmetry (CLBS) method,^[12,13] which has been used successfully to construct solutions of Eq. (1) in a number of papers.^[14–28] To construct new solutions of Eq. (1), various approaches related to symmetries have been proposed.^[29–41] King^[34] obtained the polynomial types of solutions of some nonlinear diffusion equations. Galaktionov^[35] obtained the nonlinear separable solutions

of the form $u = \alpha(t) + \beta(t)\phi(x)$ to some nonlinear diffusion equations. In Ref. [36], Galaktionov and Posashkov derived a considerable number of solutions by means of sign-invariant approach. Invariant subspace method was also used by several authors to derive solutions of Eq. (1).^[37,38] In Ref. [17] the CLBS was developed to study functional separable solutions $F(u) = \alpha(t) + \beta(x)$ of nonlinear PDEs. The approach of derivative-dependent functional separation of variables was introduced by Zhang, Lou, and Qu,^[20] which induces the solutions of the form $F(u, u_x) = \alpha(t) + \beta(x)$. The group-foliation method^[41] was also used to obtain solutions of Eq. (1), which is based on the invariants of the considered equation. All these results show that equation (1) admits rich symmetries and exact solutions for certain $f(u, u_x)$ and $g(u, u_x)$.

In this paper, we develop the CLBS approach to construct exact solutions of Eq. (1) of the forms

$$F(u, u_x) = \alpha(t) + \beta(t)x + \gamma(t)x^2, \quad (2)$$

$$F(u, u_x) = \alpha(t) + \beta(t)\cos x + \gamma(t)\sin x, \quad (3)$$

$$F(u, u_x) = \alpha(t) + \beta(t)\cosh x + \gamma(t)\sinh x. \quad (4)$$

Solutions (2) ~ (4) with $F(u, u_x) = u$ were considered in a number of papers.^[15,18,37,38] Indeed, equation (2) with $F = u$ gives polynomial-type solutions of Eq. (1), and equation (3) with $F = u$ gives a special periodic solution of Eq. (1). Solutions of the form (2) ~ (4) are motivated by the approaches of derivative-dependent functional separation of variables^[20] and the invariant subspace.^[15,37,38] By substituting the solution into the equation, it is reduced to a three-dimensional dynamic system for $\alpha(t)$, $\beta(t)$, and $\gamma(t)$. The system cannot always be solved exactly in terms of elementary functions. But in any case,

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the analysis of a finite-dimensional system is more simpler than the considered equation since the equation is regarded as an infinite-dimensional dynamic system.

Let V be an evolutionary vector field with the characteristic η ,

$$V = \sum_{k=0}^{\infty} D_x^k \eta \frac{\partial}{\partial u_k} + \dots, \quad (5)$$

where we have used the following notations

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k},$$

$$D_x^{j+1} = D_x(D_x^j), \quad D_x^0 = 1.$$

Definition 1.1 The evolutionary vector field (5) is said to be a Lie–Bäcklund symmetry of Eq. (1) if and only if,

$$V(u_t - E)|_L = 0,$$

where $E = f(u, u_x)u_{xx} + g(u, u_x)$ and L is the set of all differential consequences of Eq. (1) the equation, that is

$$u_t - E = 0, \quad D_x^j D_t^k (u_t - E) = 0, \quad j, k = 0, 1, 2, \dots$$

Definition 1.2 The evolutionary vector field (5) is said to be a CLBS of Eq. (1) if and only if

$$V(u_t - E)|_{L \cap M} = 0,$$

where M denotes the set of all differential consequences of the equation $\eta = 0$ with respect to x , that is $D_x^j \eta = 0$, $j = 0, 1, 2, \dots$

Proposition 1.3^[12,13] Equation (1) admits the CLBS. (5) if there exists a function $W(t, x, u, \eta)$ such that

$$\frac{\partial \eta}{\partial t} = [E, \eta] + W(t, x, u, \eta), \quad W(t, x, u, 0) = 0, \quad (6)$$

where $[E, \eta] = E'\eta - \eta'E$, the prime denotes the Gateaux derivative and W is an analytic function of t, x, u, u_1, \dots , and $\eta, D_x \eta, D_x^2 \eta, \dots$

It follows from Eq. (6) that equation (1) admits the CLBS with the characteristic η if and only if the characteristic satisfies

$$D_t \eta = 0.$$

It turns out that the CLBS is a natural generalization of the nonclassical symmetry in a similar way that the Lie–Bäcklund symmetry is a generalization of the Lie-point symmetry. Therefore the procedure for calculating the CLBS is essentially the same as for the nonclassical symmetry. The first step is to act with the vector field V on the expression $u_t - E$ considered as a function of the independent variables t, x, u, u_t, u_1, u_2 . The next step is to eliminate u_{tj} , $j = 0, 1, 2, \dots$, and u_N, u_{N+1}, \dots , by using the equation $u_t - E = 0$ and $\eta = 0$ and their differential consequences $D_x^j (u_t - E) = 0$, $D_x^j \eta = 0$, $j = 1, 2, \dots$. Equating the resulting expressions to zero yields a system of a linear PDEs called the system of the determining

equations. Solving this system gives the general form of the CLBSs.

2 Conditions for Eq. (1) Admitting Solutions (2) ~ (4)

To obtain the solutions (2) ~ (4) of Eq. (1), we need to distinguish two cases, (i) $F_1 = 0$ and $F_1 \neq 0$, where and hereafter F_0 and F_1 denote partial derivatives of $F(u, u_x)$ (also for $f(u, u_x)$ and $g(u, u_x)$) with respect to u and u_x respectively. In the case of $F_1 = 0$, it suffices to show that equation (1) admits the CLBS with the characteristic

$$\eta_1 = u_{xxx} + q_1 u_x u_{xx} + q_2 u_x^3 + \epsilon u_x, \quad (7)$$

where $q_1 = 3F_{00}/F_0$ and $q_2 = F_{000}/F_0$ are functions of u and $\epsilon = 0, 1, -1$ correspond to solutions (2), (3), and (4) respectively. In the case of $F_1 \neq 0$, the characteristic of the CLBS admitted by Eq. (1) is

$$\eta_2 = u_{xxxx} + h_1 u_{xx} u_{xxx} + h_2 u_{xx}^3 + h_3 u_{xxx} + h_4 u_{xx}^2 + h_5 u_{xx} + h_6, \quad (8)$$

where

$$h_1 = \frac{3F_{11}}{F_1}, \quad h_2 = \frac{F_{111}}{F_1}, \quad h_3 = \frac{(F_0 + 3u_x F_{01})}{F_1},$$

$$h_4 = \frac{3(u_x F_{011} + F_{01})}{F_1}, \quad h_5 = \frac{3(u_x^2 F_{011} + u_x F_{00})}{F_1} + \epsilon,$$

$$h_6 = \frac{(F_{000} + \epsilon F_0)}{F_1}.$$

Solving $\eta_1 = \eta_2 = 0$, we arrive at the following result.

Theorem 2.1 If equation (1) admits the CLBS with the characteristic (7) or (8), then equation (1) has the solutions (2), (3), and (4), respectively corresponding to $\epsilon = 0, 1, -1$.

A direct computation gives

$$\frac{D\eta_1}{Dt} = f_{111} u_{xx}^4 + \Pi_1 u_{xx}^3 + \Pi_2 u_{xx}^2 + \Pi_3 u_{xx} + \Pi_4, \quad (9)$$

where

$$\begin{aligned} \Pi_1 &= K_1 + q_1 f_1 + u_x q_1 f_{11}, \\ \Pi_2 &= K_2 + u_x f q_1' + q_1 \Phi_1 + u_x q_1 \Psi_1 + 3u_x^2 q_2 f_1 + \epsilon f_1, \\ \Pi_3 &= K_3 + u_x q_1' g + q_1 \Phi_2 + u_x q_1 \Psi_2 + u_x^3 f q_2' \\ &\quad + 3u_x^2 q_2 \Phi_1 + \epsilon \Phi_1, \\ \Pi_4 &= K_4 + u_x q_1 \Psi_3 + u_x^3 g q_2' + 3u_x^2 q_2 \Phi_2 + \epsilon \Phi_2, \\ K_1 &= u_x (f_{110} - 3q_1 f_{11}) + \Psi_{11}, \\ K_2 &= u_x \Psi_{1,0} - 3f_{11} q_2 u_x^3 - 3\epsilon u_x f_{11} - 2u_x q_1 \Psi_1, \\ K_3 &= u_x (\Psi_{2,0} - 2\epsilon \Psi_1 - q_1 \Psi_2) - 2u_x^3 q_2 \Psi_1 + \Psi_{3,1}, \\ K_4 &= u_x (\Psi_{3,0} - \epsilon \Psi_2) - u_x^3 q_2 \Psi_2, \\ \Psi_1 &= \Phi_{1,1} + u_x (f_{10} - 2q_1 f_1), \\ \Psi_2 &= \Phi_{2,1} + u_x (\Phi_{1,0} - q_1 \Phi_1 - 2\epsilon f_1) - 2u_x^3 f_1 q_2, \end{aligned}$$

$$\begin{aligned}\Psi_3 &= u_x(\Phi_{2,0} - \epsilon\Phi_1) - u_x^3 q_2 \Phi_1, \\ \Phi_1 &= u_x(f_0 - q_1 f) + g_1, \\ \Phi_2 &= u_x(g_0 - \epsilon f) - f q_2 u_x^3,\end{aligned}$$

where the prime denotes the derivative with respect to u .

$$\begin{aligned}\frac{D\eta_2}{Dt} &= \Lambda_1 u_{xx}^5 + \Lambda_2 u_{xx}^4 + \Lambda_3 u_{xx}^3 u_{xxx} + \Lambda_4 u_{xx}^2 u_{xxx} \\ &\quad + \Lambda_5 u_{xx} u_{xxx}^2 + \Lambda_6 u_{xx} u_{xxx} + \Lambda_7 u_{xxx}^2 + \Lambda_8 u_{xx}^3 \\ &\quad + \Lambda_9 u_{xx}^2 + \Lambda_{10} u_{xxx} + \Lambda_{11} u_{xx} + \Lambda_{12},\end{aligned}\quad (10)$$

where

$$\begin{aligned}\Lambda_1 &= J_1 + h_1 \Gamma_1 + f_1 h_{2,1} + 3h_2 f_{11}, \\ \Lambda_2 &= J_2 + h_1 \Gamma_2 + f h_{2,0} + \tilde{f} h_{2,1} + 3h_2 \hat{f} \\ &\quad + h_3 \Gamma_1 + f_1 h_{4,1} + 2h_4 f_{11}, \\ \Lambda_3 &= J_3 + f_1 h_{1,1} + f_{11} h_1 + h_1 \Gamma_3 + 6h_2 f_1, \\ \Lambda_4 &= J_4 + h_{1,1} \tilde{f} + h_1 \Gamma_5 + 3\tilde{f} h_2 \\ &\quad + f_1 h_{3,1} + h_3 \Gamma_3 + 4f_1 h_4, \\ \Lambda_5 &= J_5 + 2h_1 f_1 + h_1 \Gamma_4, \\ \Lambda_6 &= J_6 + g h_{1,0} + \tilde{g} h_{1,1} + \hat{g} h_1 + h_1 \Gamma_6 \\ &\quad + f h_{3,0} + \tilde{f} h_{3,1} + h_3 \Gamma_5 + 2h_4 \tilde{f} + 2h_5 f_1, \\ \Lambda_7 &= J_7 + h_1 \tilde{f} + h_3 \Gamma_4, \\ \Lambda_8 &= J_8 + h_1 \Gamma_7 + g h_{2,0} + \tilde{g} h_{2,1} + 3h_2 \hat{g} + h_3 \Gamma_2 \\ &\quad + f h_{4,0} + h_{4,1} \tilde{f} + 2h_4 \hat{f} + f_1 h_{5,1} + h_5 f_{11}, \\ \Lambda_9 &= J_9 + h_1 \Gamma_8 + 3h_2 \tilde{g}_0 + h_3 \Gamma_7 + g h_{4,0} + h_{4,1} \tilde{g} + 2h_4 \hat{g} \\ &\quad + f h_{5,0} + h_{5,1} \tilde{f} + \hat{f} h_5 + f_1 h_{6,1}, \\ \Lambda_{10} &= J_{10} + h_1 \tilde{g}_0 + \tilde{g} h_{3,1} + h_3 \Gamma_6 + h_5 \tilde{f}, \\ \Lambda_{11} &= J_{11} + h_1 \Gamma_9 + h_3 \Gamma_8 + 2h_4 \tilde{g}_0 + g h_{5,0} \\ &\quad + \tilde{g} h_{5,1} + \hat{g} h_5 + f h_{6,0} + \tilde{f} h_{6,1}, \\ \Lambda_{12} &= J_{12} + h_3 \Gamma_9 + h_5 \tilde{g}_0 + g h_{6,0} + \tilde{g} h_{6,1}, \\ J_1 &= \Gamma_{1,1} - h_2 \Gamma_3, \\ J_2 &= \Gamma_{2,1} + \Gamma_{1,0} u_x - h_4 \Gamma_3 - h_2 \Gamma_5, \\ J_3 &= 4\Gamma_1 + \Gamma_{3,1} - h_1 \Gamma_3 - 2h_2 \Gamma_4, \\ J_4 &= 3\Gamma_2 + \Gamma_{3,0} u_x - h_3 \Gamma_3 - 2h_4 \Gamma_4 + \Gamma_{5,1} - h_1 \Gamma_5, \\ J_5 &= 2\Gamma_3 + \Gamma_{4,1} - 2h_1 \Gamma_4, \\ J_6 &= \Gamma_{5,0} u_x - 2h_5 \Gamma_4 - h_3 \Gamma_5 + \Gamma_{6,1} - h_1 \Gamma_6 + 2\Gamma_7,\end{aligned}$$

$$\begin{aligned}J_7 &= \Gamma_{4,0} - 2h_3 \Gamma_4 + \Gamma_5, \\ J_8 &= \Gamma_{2,0} - h_5 \Gamma_3 - h_4 \Gamma_5 - h_2 \Gamma_6 + \Gamma_{7,1}, \\ J_9 &= \Gamma_{7,0} - h_6 \Gamma_3 - h_5 \Gamma_5 - h_4 \Gamma_6 + \Gamma_{8,1}, \\ J_{10} &= \Gamma_{6,0} - 2h_6 \Gamma_4 - h_3 \Gamma_6 + \Gamma_8, \\ J_{11} &= \Gamma_{9,1} - h_6 \Gamma_5 - h_5 \Gamma_6 + \Gamma_{8,0} u_x, \\ J_{12} &= \Gamma_{9,0} - h_6 \Gamma_6, \\ \Gamma_1 &= 2f_1, \quad \Gamma_2 = f_{110} u_x - 2f_1 h_4 - \hat{f}_1 - \tilde{f} h_2, \\ \Gamma_3 &= 5f_{11} - 2f_1 h_1, \quad \Gamma_4 = 2f_1, \\ \Gamma_5 &= -2\tilde{f} h_3 + \tilde{f}_1 - \tilde{f} h_1 + 2\hat{f} + 2f_{10} u_x, \\ \Gamma_6 &= \tilde{f}_0 u_x - \tilde{f} h_3, \quad \Gamma_7 = \hat{g}_1 - 2f_1 h_5 - \tilde{f} h_4 - \hat{f}_0 u_x, \\ \Gamma_8 &= \tilde{g}_{01} - 2f_1 h_6 - \tilde{f} h_5 + \hat{g}_0 u_x, \quad \Gamma_9 = \tilde{g}_{00} - \tilde{f} h_6, \\ \hat{f} &= f_{10} u_x + \tilde{f}_1, \quad \hat{g} = \tilde{f}_0 u_x + \tilde{g}_1, \\ \tilde{f} &= f_0 u_x + g_1, \quad \tilde{g} = u_x g_0.\end{aligned}$$

From Proposition 1.2., we have the following main results.

Theorem 2.2 Equation (1) admits the CLBS with the characteristic (7), if the functions f , g , and F satisfy the following equations

$$f_{111} = 0, \quad \Pi_i = 0, \quad i = 1, 2, 3, 4. \quad (11)$$

Theorem 2.3 Equation (1) admits the CLBS with the characteristic (8), if the functions f , g , and F satisfy the following equations

$$\Lambda_i = 0, \quad i = 1, 2, \dots, 12. \quad (12)$$

It is noted that the system (12) is quite complicated and seems very difficult to solve. However, we can solve the system (11) explicitly for any $F(u)$. For brevity, we restrict our attention to a few illustrate examples in the following section.

3 Quasi-linear Differential Equations Admitting the CLBSs with Charateristic (7)

In this section we consider Eq. (1) with admitting the CLBSs (8). It follows from Theorems 2.1 and 2.2 that equation (1) possesses the solution of the form

$$u = [\alpha(t) + \beta(t)x + \gamma(t)x^2]^{1/s}, \quad (13)$$

where and hereafter s is a real constant, if and only if the coefficient functions in Eq. (1) satisfy the system,

$$\begin{aligned}f_{111} &= 0, \\ 9(1-s)f_1 + u(3f_{01} + g_{111}) + 15(1-s)u_x f_{11} + 3u u_x f_{011} &= 0, \\ 3g_{01} u^4 - 7s u_x f_{01} + 2s u^2 u_x [2(s-1)f - g_{11}] + u^3 u_x [5(1-s)f_0 + f_{00} + 2g_{11}] \\ &\quad + u^4 u_x g_{011} 3u^2 u_x^2 [(9s-2)(s-1)f_1 - 12f_{11}] + 21u^3 u_x^2 f_{01} + 3u^4 u_x^2 f_{011} - 6s(s-3)u^2 u_x^3 f_{11} = 0, \\ 3(1-s)u^2 u_x g + 3(s-1)u^3 u_x g_0 + 3u^4 u_x g_{00} + 3(s-1)u^2 u_x^2 g_1\end{aligned}$$

$$\begin{aligned}
& -20u^2u_x^2f_0 - 3(s-1)u^3u_x^2g_{01} + 3u^4u_x^2g_{001} + 6s(s-1)(s-4)uu_x^3f \\
& - u^2u_x^3[s(s+21)f_0 + 3(s-1)(s-2)g_{11} + 6sf_{00}] \\
& + 6u^3u_x^3f_{00} + u^4u_x^3f_{000} + (s-1)(s-2)(21s-13)uu_x^4f_1 - 9(s-1)(s-2)u^2u_x^4f_{01} = 0, \\
& - 2(s-1)(s-2)uu_x^3g + 2(s-1)(s-2)u^2u_x^3g_0 + 3(s-1)u^3u_x^3g_0 \\
& + u^4u_x^3g_{000} + 2(s-1)(s-2)uu_x^4g_1 - 3(s-1)(s-2)u^2u_x^4g_{01} \\
& - 6s(s-1)(s-2)u_x^5f + 3(s-1)(s-2)(s+1)uu_x^5f_0 \\
& - 3(s-1)(s-2)u^2u_x^5f_{00} + 3(s-1)^2(s-2)^2u_x^6f_1 = 0.
\end{aligned} \tag{14}$$

Solving Eq. (14), we arrive at

$$\begin{aligned}
f(u, u_x) &= \left(\frac{c_1}{s-1}u^{2s-2} + \frac{2c_2}{s-2}u^{3s-2}\right)u_x^2 + \left(\frac{c_3}{s-1}u^{s-1} + \frac{2c_4}{s-2}u^{2s-1}\right)u_x + \frac{2c_5}{s-2}u^{2s} + \frac{c_6}{s-1} + c_7u^s, \\
g(u, u_x) &= (c_1u^{2s-3} + c_2u^{3s-3})u_x^4 + (c_3u^{s-2} + c_4u^{2s-2})u_x^3 \\
&+ (c_5u^{2s-1} + c_6u^{-1} + c_8u^{s-1})u_x^2 + c_{11}u_x + c_9u + c_{10}u^{1-s},
\end{aligned} \tag{15}$$

where c_i , $i = 1, 2, \dots$, are constants.

Substituting Eq. (13) into Eq. (1), and we get the three-dimensional dynamical system for α , β , and γ :

$$\begin{aligned}
\alpha'(t) &= \frac{2c_1}{s^2(s-1)}\beta^2\gamma + \left[\frac{c_2}{s}\beta^2 + \frac{c_4}{s(s-2)}\beta + \frac{c_5}{s-2}\alpha\right](4\alpha\gamma - \beta^2) + \frac{2c_3}{s(s-1)}\beta\gamma \\
&+ \frac{2c_6}{s-1}\gamma + c_7\left(\frac{1-s}{s}\beta^2 + 2\alpha\gamma\right) + \frac{c_8}{s}\beta^2 + sc_9\alpha + sc_{10} + c_{11}\beta, \\
\beta'(t) &= \frac{8c_1}{s^2(s-1)}\beta\gamma^2 + \left[\frac{4c_2}{s}\beta\gamma + \frac{2c_4}{s(s-2)}\gamma + \frac{c_5}{s-2}\beta\right](4\alpha\gamma - \beta^2) + \frac{4c_3}{s(s-1)}\gamma^2 \\
&+ \frac{2[(s+2)c_7 + 2c_8]}{s}\beta\gamma + sc_9\beta + 2c_{11}\gamma, \\
\gamma'(t) &= \frac{8c_1}{s^2(s-1)}\gamma^3 + \left(\frac{4c_2}{s}\gamma + \frac{c_5}{s-2}\right)(4\alpha\gamma - \beta^2)\gamma + \frac{2[(s+2)c_7 + 2c_8]}{s}\gamma^2 + sc_9\gamma.
\end{aligned}$$

It is impossible to derive the general solution of this system. Here we only consider the case $c_i = 0$, $i \neq 1, 2$, the system can be solved

$$\alpha = \frac{a_1^2}{4}\gamma + a_2, \quad \beta = a_1\gamma, \quad \gamma = \pm 2b_1(t_0 - t)^{-1/2},$$

where $b_1 = 8c_1/(s(s-1)) + 16c_2a_2/s$. Namely, we have shown that the equation

$$u_t = \left(\frac{c_1}{s-1}u^{2s-2} + \frac{2c_2}{s-2}u^{3s-2}\right)u_x^2u_{xx} + (c_1u^{2s-3} + c_2u^{3s-3})u_x^4,$$

has the solution (13). Notice that when $s > 0$, the solution (13) blows up at $t = t_0$.

From Theorems (2.1) and (2.2), we know that equation (1) possesses the spacial periodic solution

$$u = [\alpha(t) + \beta(t)\cos x + \gamma(t)\sin x]^{1/s}, \tag{16}$$

if f and g satisfy the following system

$$\begin{aligned}
& f_{111} = 0, \\
& 9(1-s)f_1 + u(3f_{01} + g_{111}) + 15(1-s)u_xf_{11} + 3uu_xf_{011} = 0, \\
& - 3u^4(f_1 - g_{01}) + 12s(s-1)u^2u_xf - 3(s-1)u^3u_x(5f_0 + 2g_{11}) \\
& + 3u^4u_x(f_{00} - 2f_{11} + g_{011}) + 3(s-1)(9s-2)u^2u_x^2f_1 - 21(s-1) \\
& \times u^3u_x^2f_{01} + 3u^4u_x^2f_{001} - 6(s-1)(s-2)u^2u_x^3f_{11} = 0, \\
& - 3(s-1)u^2u_xg + 3(s-1)u^3u_x(2f + g_0) - 3u^4u_x(2f_0 - g_{00} + g_{11}) \\
& + 3(s-1)u^2u_x^2g_1 + 3(s-1)u^3u_x^2(7f_1 - g_{01}) - 3u^4u_x^2(3f_{01} - g_{001}) \\
& - (s-1)u^2u_x^3[(s-20)f_0 + 3(s-2)g_{11}] - 6(s-1)u^3u_x^3f_{00} + u^4u_x^3f_{000} \\
& + (s-1)(s-2)(21s-13)uu_x^4f_1 - 9(s-1)(s-2)u^2u_x^4f_{01} = 0,
\end{aligned}$$

$$\begin{aligned}
& 3u^4u_x^2(f_1 - g_{01}) - 2(s-1)(s-2)uu_x^3g - 6(s-1)u^2u_x^3f + 2(s-1)(s-2) \\
& \quad \times u^2u_x^3g_0 + 3(s-1)u^3u_x^3(f_0 + g_{00}) - u^4u_x^3(3f_{00} - g_{000}) + 2(s-1)(s-2) \\
& \quad \times uu_x^4g_1 + 3(s-1)(s-2)u^2u_x^4(2f_1 - g_{01}) - 6s(s-1)(s-2)u_x^5f + 3(s-1) \\
& \quad \times (s-2)(s+1)uu_x^5f_0 - 3(s-1)(s-2)u^2u_x^5f_{00} + 3(s-1)^2(s-2)^2u_x^6f_1 = 0.
\end{aligned} \tag{17}$$

Solving the system, we obtain the general solution of Eq. (17) given by

$$\begin{aligned}
f(u, u_x) &= \left(\frac{2c_1}{s-2}u^{3s-2} + \frac{c_2}{s-1}u^{2s-2} \right) u_x^2 + (c_3u^{2s-1} + c_4u^{s-1})u_x \\
& \quad + \frac{2c_1}{s^2(s-2)}u^{3s} + \frac{1}{s-2} \left[-\frac{3c_2}{s(s-1)} + 2c_7 \right] u^{2s} + c_5u^s + \frac{c_6}{s-1}, \\
g(u, u_x) &= (c_1u^{3s-3} + c_2u^{2s-3})u_x^4 + \left[\left(\frac{s}{2} - 1 \right) c_3u^{2s-2} + (s-1)c_4u^{s-2} \right] ru_x^3 \\
& \quad + \left[\frac{2(s-1)c_1}{s^2(s-2)}u^{3s-1} + c_7u^{2s-1} + c_8u^{s-1} + \frac{c_6}{u} \right] u_x^2 + \left(\frac{c_3}{2s}u^{2s} + \frac{c_4}{s}u^s + c_9 \right) u_x \\
& \quad + \frac{c_1}{s^3(s-2)}u^{3s+1} + \frac{1}{s(s-2)} \left[-\frac{c_2(s+1)}{s^2(s-1)} + c_7 \right] u^{2s+1} + \frac{c_5 + c_8}{s^2}u^{s+1} + \frac{c_{10}}{s}u + c_{11}u^{1-s}.
\end{aligned}$$

The substitution of Eq. (16) into Eq. (1) implies that α , β , and γ satisfy the system

$$\begin{aligned}
\alpha'(t) &= \frac{c_1}{s^2(s-2)}(\alpha^2 - \beta^2 - \gamma^2)(\alpha^2 + \beta^2 + \gamma^2) - \frac{c_2}{2s^2}\alpha(\alpha^2 + 3\beta^2 + 3\gamma^2) + \frac{2c_2}{s(s-1)}\alpha^3 \\
& \quad + \frac{3c_2}{2s(s-2)}\alpha(\beta^2 + \gamma^2) + c_5 \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{s} - \beta^2 - \gamma^2 \right) + \frac{c_7}{s-2}\alpha(\alpha^2 - \beta^2 - \gamma^2) \\
& \quad + \frac{c_8}{s}(\alpha^2 + \beta^2 + \gamma^2) + c_{10}\alpha + sc_{11}, \\
\beta'(t) &= \frac{2c_1}{s^2(s-2)}\alpha\beta(\alpha^2 - \beta^2 - \gamma^2) - \frac{c_2}{2s^2}\beta(3\alpha^2 + \beta^2 + \gamma^2) + \frac{c_2}{s(s-1)}\beta(3\alpha^2 - \beta^2 - \gamma^2) \\
& \quad + \frac{3c_2}{2s(s-2)}\beta(\alpha^2 + \beta^2 + \gamma^2) + \left(\frac{c_3}{2s}\gamma + \frac{c_7}{s-2}\beta \right) (\alpha^2 - \beta^2 - \gamma^2) + \frac{c_4}{s}\alpha\gamma \\
& \quad + \left[\left(\frac{2}{s} - 1 \right) c_5 + \frac{2c_8}{s} \right] \alpha\beta + c_9\gamma + \left(c_{10} - \frac{c_6}{s-1} \right) \beta, \\
\gamma'(t) &= \frac{2c_1}{s^2(s-2)}\alpha\gamma(\alpha^2 - \beta^2 - \gamma^2) - \frac{c_2}{2s^2}\gamma(3\alpha^2 + \beta^2 + \gamma^2) + \frac{c_2}{s(s-1)}\gamma(3\alpha^2 - \beta^2 - \gamma^2) \\
& \quad + \frac{3c_2}{2s(s-2)}\gamma(\alpha^2 + \beta^2 + \gamma^2) - \left(\frac{c_3}{2s}\beta - \frac{c_7}{s-2}\gamma \right) (\alpha^2 - \beta^2 - \gamma^2) - \frac{c_4}{s}\alpha\beta \\
& \quad + \left[\left(\frac{2}{s} - 1 \right) c_5 + \frac{2c_8}{s} \right] \alpha\gamma - c_9\beta + \left(c_{10} - \frac{c_6}{s-1} \right) \gamma.
\end{aligned}$$

Hence we have shown that for $c_i = 0$, ($i \neq 7$), the equation

$$u_t = \frac{2c_7}{s-2}u^{2s}u_{xx} + \frac{c_7}{s(s-2)}u^{2s+1},$$

has the solution (16) with α , β , and γ given by

$$\alpha = a_1\gamma, \quad \beta = a_2\gamma, \quad \gamma = \pm\sqrt{2a_0(t_0 - t)},$$

where $a_0 = c_7(a_1^2 - a_2^2 - 1)/(s-2)$. The solution (16) when $s > 0$ extinguish at finite time t_0 .

Equation (1) possesses the solution

$$u = [\alpha(t) + \beta(t) \cosh x + \gamma(t) \sinh x]^{1/s}, \tag{18}$$

if f and g satisfy

$$\begin{aligned}
& f_{111} = 0, \\
& 9(1-s)f_1 + u(3f_{01} + g_{111}) + 15(1-s)u_xf_{11} + 3uu_xf_{011} = 0, \\
& 3u^4(f_1 + g_{01}) + 12s(s-1)u^2u_xf - 3(s-1)u^3u_x(5f_0 + 2g_{11})
\end{aligned}$$

$$\begin{aligned}
& + 3u^4u_x(f_{00} + 2f_{11} + g_{011}) + 3(s-1)(9s-2)u^2u_x^2f_1 - 21(s-1) \\
& \times u^3u_x^2f_{01} + 3u^4u_x^2f_{001} - 6(s-1)(s-2)u^2u_x^3f_{11} = 0, \\
& - 3(s-1)u^2u_xg - 3(s-1)u^3u_x(2f - g_0) + 3u^4u_x(2f_0 + g_{00} + g_{11}) \\
& + 3(s-1)u^2u_x^2g_1 - 3(s-1)u^3u_x^2(7f_1 + g_{01}) + 3u^4u_x^2(3f_{01} + g_{001}) \\
& - (s-1)u^2u_x^3[(s-20)f_0 + 3(s-2)g_{11}] - 6(s-1)u^3u_x^3f_{00} + u^4u_x^3f_{000} \\
& + (s-1)(s-2)(21s-13)uu_x^4f_1 - 9(s-1)(s-2)u^2u_x^4f_{01} = 0, \\
& 3u^4u_x^2(f_1 + g_{01}) - 2(s-1)(s-2)uu_x^3g + 6(s-1)u^2u_x^3f + 2(s-1)(s-2) \\
& \times u^2u_x^3g_0 - 3(s-1)u^3u_x^3(f_0 - g_{00}) + u^4u_x^3(3f_{00} + g_{000}) + 2(s-1)(s-2) \\
& \times uu_x^4g_1 - 3(s-1)(s-2)u^2u_x^4(2f_1 + g_{01}) - 6s(s-1)(s-2)u_x^5f + 3(s-1) \\
& \times (s-2)(s+1)uu_x^5f_0 - 3(s-1)(s-2)u^2u_x^5f_{00} + 3(s-1)^2(s-2)^2u_x^6f_1 = 0.
\end{aligned} \tag{19}$$

Solving system (19), we obtain

$$\begin{aligned}
f(u, u_x) &= \left(\frac{2c_1}{s-2}u^{3s-2} + \frac{c_2}{s-1}u^{2s-2} \right) u_x^2 + (c_3u^{2s-1} + c_4u^{s-1})u_x \\
& - \frac{2c_1s}{(s-1)(s-2)}u^{3s} + \frac{1}{s-2} \left[\frac{3c_2}{s(s-1)} + 2c_6 \right] u^{2s} + c_7u^s + \frac{c_5}{s-1}, \\
g(u, u_x) &= (c_1u^{3s-3} + c_2u^{2s-3})u_x^4 + \left[\left(\frac{s}{2} - 1 \right) c_3u^{2s-2} + (s-1)c_4u^{s-2} \right] u_x^3 \\
& + \left[\frac{2c_1(1-s)}{s^2(s-2)}u^{3s-1} + c_5u^{-1} + c_6u^{2s-1} + c_9u^{s-1} \right] u_x^2 \\
& - \left(\frac{c_3}{2s}u^{2s} + \frac{c_4}{s}u^s - c_8 \right) u_x + \frac{c_1}{s^3(s-2)}u^{3s+1} \\
& - \left[\frac{c_2(s+1)}{s^3(s-1)(s-2)} + \frac{c_6}{s(s-2)} \right] u^{2s+1} - \frac{c_7 + c_9}{s^2}u^{s+1} + \frac{c_{10}}{s}u + c_{11}u^{1-s}.
\end{aligned} \tag{20}$$

Substituting Eqs. (18) and (20) into Eq. (1) implies that α , β , and γ satisfy the system

$$\begin{aligned}
\alpha'(t) &= \frac{c_1}{s^2(s-2)}[\alpha^4 - (\beta^2 - \gamma^2)^2] + \frac{c_2}{s} \left[\frac{\alpha}{2s}(3\gamma^2 - 3\beta^2 - \alpha^2) + \frac{2\alpha^3}{s-1} + \frac{3\alpha(\beta^2 - \gamma^2 - \alpha^2)}{2(s-2)} \right] \\
& + \frac{c_6\alpha}{s-2}(\beta^2 - \gamma^2 - \alpha^2) + c_7 \left(\frac{\gamma^2 - \alpha^2 - \beta^2}{s} + \beta^2 - \gamma^2 \right) + \frac{c_9}{s}(\gamma^2 - \beta^2 - \alpha^2) + c_{10}\alpha + sc_{11}, \\
\beta'(t) &= \frac{2c_1\alpha\beta}{s^2(s-2)}(\alpha^2 - \beta^2 + \gamma^2) + \frac{c_2}{s} \left[\frac{\beta}{2s}(\gamma^2 - \beta^2 - 3\alpha^2) \right. \\
& \left. + \frac{\beta}{s-1}(\gamma^2 - \beta^2 - 3\alpha^2) + \frac{3\beta(\beta^2 - \gamma^2 - \alpha^2)}{2(s-2)} \right] + \frac{c_3}{2s}\gamma(\beta^2 - \gamma^2 - \alpha^2) \\
& + \frac{c_5}{s-1}\beta + \frac{c_6}{s-2}\beta(\beta^2 - \gamma^2 - \alpha^2) - \left[\frac{c_4}{s} + \left(\frac{2}{s} - 1 \right) c_7 - \frac{2}{s}c_9 \right] \alpha\beta + c_8\gamma + c_{10}\beta, \\
\gamma'(t) &= \frac{2c_1\alpha\gamma}{s^2(s-2)}(\alpha^2 - \beta^2 + \gamma^2) + \frac{c_2}{s} \left[\frac{\gamma}{2s}(\gamma^2 - \beta^2 - 3\alpha^2) + \frac{\gamma}{s-1}(\gamma^2 - \beta^2 - 3\alpha^2) \right. \\
& \left. + \frac{3\gamma}{2(s-2)(\beta^2 - \gamma^2 - \alpha^2)} \right] + \frac{c_3}{2s}\beta(\beta^2 - \gamma^2 - \alpha^2) + \frac{c_5}{s-1}\gamma \\
& + \frac{c_6}{s-2}\gamma(\beta^2 - \gamma^2 - \alpha^2) - \left[\frac{c_4}{s} + \left(\frac{2}{s} - 1 \right) c_7 - \frac{2}{s}c_9 \right] \alpha\gamma + c_8\beta + c_{10}\gamma.
\end{aligned}$$

Considering the case $c_i = 0$, ($i \neq 1$), equation (1) becomes

$$u_t = \left[\frac{2c_1}{s-2}u^{3s-2}u_x^2 - \frac{2c_1s}{(s-1)(s-2)}u^{3s} \right] u_{xx} + c_1u^{3s-3}u_x^4 + \frac{2c_1(1-s)}{s^2(s-2)}u_x^2.$$

It has the solution (18) with α , β , and γ given implicitly by

$$\alpha = \pm \sqrt{(a_1^2 - 1)\gamma^2 + a_2\gamma}, \quad \beta = a_1\gamma,$$

$$t - \frac{[(a_1^2 - 1)\gamma + a_2][2(a_1^2 - 1)\gamma - a_2]}{3a_2^2 c_1 \gamma \sqrt{(a_1^2 - 1)\gamma^2 + a_2\gamma}} + a_3 = 0.$$

4 Conclusion and Remarks

In this paper, we have developed the CLBS method to obtain new classes of exact solutions of the form (2) ~ (4) of nonlinear parabolic equations. Those solutions are reduced to solve the finite dimensional dynamic systems, and most of them cannot be obtained in the framework within the Lie's classical method and the conditional symmetry method.

It is of interest to construct solutions of Eq. (1) in general form

$$F(u, u_x) = \sum_{i=1}^m \alpha_i(t) \phi_i(x),$$

where $\alpha_i(t)$ and $\phi_i(x)$ are some smooth functions of t and x respectively. This problem will be studied in our future work.

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