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Supersymmetric Sawada–Kotera–Ramani Equation: Bilinear Approach

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Abstract *In this paper, using the Hirota's bilinear method, we consider the $N = 1$ supersymmetric Sawada–Kotera–Ramani equation and obtain the Bäcklund transformation of it. Its one- and two-supersoliton solutions are obtained and N-supersoliton solutions for $N \geq 3$ are given under the condition $k_i\xi_j = k_j\xi_i$.*

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1 Introduction

It is well known that the Bäcklund transformation (BT) is one of the powerful methods in searching for exact solutions of nonlinear evolution equations. BT allows us to derive new solutions from known ones. Furthermore, BT and inverse scattering transform, conservation laws have close relations.[1,2]

On the other hand, the construction of soliton solutions for a given nonlinear evolution equation is an important topic. There have been many works about it.[3−8] In recent years the study of supersymmetric integrable system has been a very interesting subject. The physical interest in the study of these systems is initiated by the seminal paper of Alvarez–Gaume *et al.*[9] concerning the partition function and super-Virasoro constraints of two-dimensional (2D) quantum supergravity. A number of well-known integrable equations have been generalized into the symmetric context. Various methods such as Painlevé test,[10] prolongation structures,[11] Darboux and Bäcklund transformation,[12] Hirota bilinear method,[12−14] and Hamiltonian formalism,[15] etc., have been extended to study supersymmetric integrable systems. And Hirota's bilinear approach is a very powerful method for constructing particular solutions for soliton systems.[16] Although Hirota bilinear method was used to study supersymmetric integrable system in Refs. [12] ∼ [14], the bilinear formalism for the supersymmetric integrable systems is s very little investigated.

In this paper, we consider the $N = 1$ supersymmetric Sawada–Kotera–Ramani equation using Hirota's bilinear method. As a result, a Bäcklund transformation is obtained. Its one- and two-supersoliton solutions are calculated and N-supersoliton solutions for $N \geq 3$ are given under some conditions.

This paper is organized as follows. In Sec. 2, Sawada–Kotera–Ramani equation

$$u_t + 15(u^3 + uu_{xx})_x + u_{5x} = 0 \qquad (1)$$

is supersymmetrized and bilinearized. In Sec. 3, we give the bilinear identities that are used in this paper. In Sec. 4, we construct a Bäcklund transformation for the $N = 1$ supersymmetric Sawada–Kotera–Ramani equation. In Sec. 5, using Hirota's bilinear method, we calculate its supersoliton solutions. At last, we summarize our results.

2 Supersymmetric Sawada–Kotera–Ramani Equation

We consider Sawada–Kotera–Ramani equation

$$u_t + 15(u^3 + uu_{xx})_x + u_{5x} = 0\,, \qquad (2)$$

where the subscripts denote partial derivatives. We extend independent variables to obtain the supersymmetric Sawada–Kotera–Ramani equation. As usual, we choose to extend the variable x to a doublet (x, θ), where θ is a Grassmann variable. Thus, the original independent variables (x, t) are extended to supercase (x, t, θ). And the associated superderivative is

$$D = \partial_\theta + \theta\partial_x\,. \qquad (3)$$

A bosonic superfield is

$$F(x, t, \theta) = u(x, t) + \theta\,\eta(x, t)\,, \qquad (4)$$

or a fermionic superfield

$$\Phi(x, t, \theta) = \eta(x, t) + \theta\,u(x, t)\,. \qquad (5)$$

Let us proceed with a direct extension, namely multiplying each term of Eq. (2) by θ and rewriting the resultant terms in the light of superfields:

$$u_t \to \Phi_t\,, \quad u_{5x} \to D^{10}\Phi\,,$$
$$3(u^3 + uu_{xx}) \to 2\,D\Phi\,D^4\Phi + D^5\Phi\Phi + 3(D\Phi)^2\Phi\,. \qquad (6)$$

The $N = 1$ supersymmetric Sawada–Kotera–Ramani equation is

$$\Phi_t + D^2[10\,D\Phi\,D^4\Phi + 5\,D^5\Phi\Phi$$

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$$+15(D\Phi)^2\Phi] + D^{10}\Phi = 0\,. \tag{7}$$

By means of Eq. (5), this equation in components reads

$$\eta_t + \eta_{5t} + (10u\eta_{2x} + 5u_{xx}\eta + 15u^2\eta)_x = 0\,, \tag{8a}$$

$$u_t + u_{5t} + (15uu_{xx} + 15u^3 + 2\eta_x\eta_{xx} + \eta\eta_{3x} + 6\eta\eta_x)_x = 0\,. \tag{8b}$$

We reformulate Eq. (7) into Hirota bilinear form and make a dependent variable transformation as follows:

$$\eta = 2D^3 \ln f(x,t,\theta)\,. \tag{9}$$

Then through straightforward manipulations we find that equation (7) is transformed into

$$S_x(D_t + S_x^{10})\, f \cdot f = 0\,, \tag{10}$$

which is equivalent to the form

$$S_x(D_t + D_x^5) f \cdot f = 0\,, \tag{11}$$

where we used the Hirota derivative, which is defined as

$$S_x D_t^m D_x^n\, f \cdot g = (D_{\theta_1} - D_{\theta_2})\Big(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\Big)^m \Big(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\Big)^n f(x_1,t_1,\theta_1)\, g(x_2,t_2,\theta_2)\Big|_{\substack{x_1=x_2=x\\ t_1=t_2=t\\ \theta_1=\theta_2=\theta}}$$

In the sequent sections we mainly study Eq. (11).

3 Some Bilinear Identities

In this section, we list the relevant bilinear identities, which can be proved directly. Here a, b, c, and d are arbitrary even functions of the independent variables x, t, and θ.

$$(S_xD_t a \cdot a)b^2 - a^2(S_xD_t b\cdot b) = 2S_x(D_t a\cdot b)\cdot ba\,, \tag{12}$$

$$\begin{aligned}(S_xD_x^5 a \cdot a)b^2 - a^2(S_xD_x^5 b\cdot b) &= 2S_x[(D_x^5 a\cdot b)\cdot ab - 5(D_x^4 a\cdot b)\cdot (D_x a\cdot b) + 10(D_x^3 a\cdot b)\cdot(D_x^2 a\cdot b)]\\ &-5[(S_xD_x a\cdot a)(D_x^4 b\cdot b) + 2(S_xD_x^3 a\cdot a)(D_x^2 b\cdot b) - 2(S_xD_x^3 b\cdot b)(D_x^2 a\cdot a) - (S_xD_x b\cdot b)(D_x^4 a\cdot a)]\,,\end{aligned} \tag{13}$$

$$\begin{aligned}&(S_xD_x a\cdot a)(D_x^4 b\cdot b) + 2(S_xD_x^3 a\cdot a)(D_x^2 b\cdot b) - 2(S_xD_x^3 b\cdot b)(D_x^2 a\cdot a) - (S_xD_x b\cdot b)(D_x^4 a\cdot a)\\ &= D_x(S_xD_x^3 a\cdot b)\cdot (D_x a\cdot b) + D_x(S_xD_x a\cdot b)\cdot (D_x^3 a\cdot b) + D_x^3(S_xD_x a\cdot b)\cdot (D_x a\cdot b)\\ &-2S_x(D_x^4 a\cdot b)\cdot (D_x a\cdot b) + 4S_x(D_x^3 a\cdot b)\cdot (D_x^2 a\cdot b)\,,\end{aligned} \tag{14}$$

$$D_x(S_x a\cdot b)\cdot (D_x a\cdot b) = S_x(D_x^2 a\cdot b)\cdot ab - D_x(S_xD_x a\cdot b)\cdot ab\,, \tag{15}$$

$$D_x^3(S_x a\cdot b)\cdot (D_x a\cdot b) = S_x(D_x^4 a\cdot b)\cdot ab - D_x(S_xD_x^3 a\cdot b)\cdot ab - 3D_x(S_xD_x a\cdot b)\cdot (D_x^2 a\cdot b)\,, \tag{16}$$

$$D_x(S_x a\cdot b)\cdot (D_x^3 a\cdot b) = S_x(D_x^4 a\cdot b)\cdot ab - D_x^3(S_xD_x a\cdot b)\cdot ab - 3D_x(S_xD_x^2 a\cdot b)\cdot (D_x a\cdot b)\,, \tag{17}$$

$$D_x(S_x a\cdot b)\cdot ab = S_x(D_x a\cdot b)\cdot ab\,, \tag{18}$$

$$\begin{aligned}D_x(S_x a\cdot b)\cdot (D_x^2 a\cdot b) &= S_x(D_x^3 a\cdot b)\cdot ab + S_x(D_x^2 a\cdot b)\cdot (D_x a\cdot b) - D_x(S_xD_x^2 a\cdot b)\cdot ab\\ &- 2D_x(S_xD_x a\cdot b)\cdot (D_x a\cdot b)\,,\end{aligned} \tag{19}$$

$$D_x^3(S_x a\cdot b)\cdot ab = S_x(D_x^3 a\cdot b)\cdot ab - 3S_x(D_x^2 a\cdot b)\cdot (D_x a\cdot b)\,. \tag{20}$$

4 Bäcklund Transformation

BT is a useful concept and an effective tool for seeking solution as well as a characteristic of integrability. In this section, we derive a bilinear BT for Eq. (11). Our results are summarized in the following.

Proposition Suppose that f is a solution of Eq. (12), and g satisfies the following relations

$$S_xD_x\, f\cdot g - \lambda\, S_x\, f\cdot g = 0\,, \tag{21}$$

$$S_xD_x^2\, f\cdot g - \lambda^2\, S_x\, f\cdot g = 0\,, \tag{22}$$

$$S_xD_x^3\, f\cdot g - \lambda^3\, S_x\, f\cdot g = 0\,, \tag{23}$$

$$(D_t + 5\,\lambda^4\, D_x - 10\,\lambda^3\, D_x^2 + 10\,\lambda^2 D_x^3 - 5\,\lambda\, D_x^4 + D^5)\, f\cdot g = 0\,, \tag{24}$$

where λ is an arbitrary constant. Then g is a new solution of Eq. (11)

Proof We consider

$$Q = (S_x(D_t + D_x^5)\, f\cdot f)\, g g - f f(S_x(D_t + D_x^5)\, g\cdot g)\,. \tag{25}$$

We will use various bilinear identities, which for convenience, are presented in Sec. 3.

$$\begin{aligned}Q \xlongequal{(12),(13)} &2S_x[(D_t + D_x^5) f\cdot g]\cdot fg - 10S_x(D_x^4 f\cdot g)\cdot (D_x f\cdot g) + 20S_x(D_x^3 f\cdot g)\cdot (D_x^2 f\cdot g)\\ &- 5(S_xD_x f\cdot f)(D_x^4 g\cdot g) + 5(S_xD_x g\cdot g)(D_x^4 f\cdot f) - 10(S_xD_x^3 f\cdot f)(D_x^2 g\cdot g) + 10(S_xD_x^3 g\cdot g)(D_x^2 f\cdot f)\end{aligned}$$

$$
\begin{aligned}
&\stackrel{(14)}{=\!=} 2S_x[(D_t+D_x^5)f\cdot g]fg - 5D_x(S_xD_x^3f\cdot g)(D_xf\cdot g) - 5D_x^3(S_xD_xf\cdot g)(D_xf\cdot g) - 5D_x(S_xD_xf\cdot g)(D_x^3f\cdot g)\\
&\stackrel{(21),(23)}{=\!=} 2S_x[(D_t+D_x^5)f\cdot g]fg - 5\lambda^3D_x(S_xf\cdot g)(D_xf\cdot g) - 5\lambda D_x^3(S_xf\cdot g)(D_xf\cdot g) - 5\lambda D_x(S_xf\cdot g)(D_x^3f\cdot g)\\
&\stackrel{(16),(17)}{=\!=} 2S_x[(D_t+D_x^5)f\cdot g]fg - 5\lambda^3D_x(S_xf\cdot g)(D_xf\cdot g) - 5\lambda[S_x(D_x^4f\cdot g)\cdot fg - D_x(S_xD_x^3f\cdot g)\cdot fg\\
&\quad - 3D_x(S_xD_xf\cdot g)\cdot(D_x^2f\cdot g))] - 5\lambda[S_x(D_x^4f\cdot g)\cdot fg - D_x^3(S_xD_xf\cdot g)\cdot fg - 3D_x(S_xD_x^2f\cdot g)(D_xf\cdot g)]\\
&\stackrel{(21)\sim(23)}{=\!=} S_x[(2D_t+2D_x^5-10\lambda\,D_x^4)f\cdot g]fg + 5\,\lambda^4D_x(S_xf\cdot g)\cdot fg + 10\,\lambda^3D_x(S_xf\cdot g)(D_xf\cdot g)\\
&\quad + 15\,\lambda^2\,D_x(S_xf\cdot g)\cdot(D_x^2f\cdot g) + 5\,\lambda^2D_x^3(S_xf\cdot g)\cdot fg\\
&\stackrel{(19),(20)}{=\!=} S_x[(2D_t+2D_x^5-10\lambda\,D_x^4)f\cdot g]fg + 5\,\lambda^4D_x(S_xf\cdot g)\cdot fg + 10\,\lambda^3D_x(S_xf\cdot g)(D_xf\cdot g)\\
&\quad + 15\,\lambda^2\,[S_x(D_x^3f\cdot g)\cdot fg - D_x(S_xD_x^2f\cdot g) - 2D_x(S_xD_xf\cdot g)\cdot(D_xf\cdot g) + S_x(D_x^2f\cdot g)\cdot(D_xf\cdot g)]\\
&\quad + 5\,\lambda^2\,[S_x(D_x^3f\cdot g)\cdot fg - 3S_x(D_x^2f\cdot g)\cdot(D_xf\cdot g)]\\
&\stackrel{(22),(23)}{=\!=} S_x[(2D_t+2D_x^5-10\lambda\,D_x^4+20\lambda^2D_x^3)f\cdot g]fg - 10\lambda^4D_x(S_xf\cdot g)\cdot fg - 20\lambda^3D_x(S_xf\cdot g)\cdot(D_xf\cdot g)\\
&\stackrel{(15)}{=\!=} S_x[(2D_t+2D_x^5-10\lambda\,D_x^4+20\lambda^2D_x^3)f\cdot g]fg - 10\lambda^4D_x(S_xf\cdot g)\cdot fg\\
&\quad - 20\lambda^3[S_x(D_x^2f\cdot g)\cdot fg - D_x(S_xD_xf\cdot g)\cdot fg]\\
&\stackrel{(21),(18)}{=\!=} S_x[(2D_t+2D_x^5-10\lambda\,D_x^4+20\lambda^2D_x^3-20\lambda^3D_x^2+10\lambda^4D_x)f\cdot g]fg\\
&=\!= 2S_x[(D_t-5\lambda\,D_x^4+10\lambda^2D_x^3-10\lambda^3D_x^2+5\lambda^4D_x+D_x^5)f\cdot g]fg \stackrel{(24)}{=\!=} 0\,.
\end{aligned}
$$

This proves our result.

5 Soliton Solutions

For a given system, Hirota's bilinear method is ideal for the construction of particular solutions. In this section, we calculate solition solutions of Eq. (11).

As usual, we expand f in some arbitrary parameter ε as follows:

$$f = 1 + \varepsilon\, f^{(1)} + \varepsilon^2\, f^{(2)} + \varepsilon^3\, f^{(3)} + \cdots\,, \tag{26}$$

where $f^{(i)}$ are even functions. Equating the power of ε we find

$$\text{for}\quad \varepsilon\quad D_x(\partial_t+\partial_x^5)f^{(1)} = 0\,, \tag{27}$$

$$\text{for}\quad \varepsilon^2\quad 2D_x(\partial_t+\partial_x^5)f^{(2)} = -S_x(D_t+D_x^5)f^{(1)}\cdot f^{(1)}\,, \tag{28}$$

$$\text{for}\quad \varepsilon^3\quad D_x(\partial_t+\partial_x^5)f^{(3)} = -S_x(D_t+D_x^5)f^{(1)}\cdot f^{(2)}\,, \tag{29}$$

$$\text{for}\quad \varepsilon^4\quad 2D_x(\partial_t+\partial_x^5)f^{(4)} = -S_x(D_t+D_x^5)(2f^{(1)}\cdot f^{(3)} + f^{(2)}\cdot f^{(2)})\,,\quad \cdots\,. \tag{30}$$

To obtain the one-soliton solution we introduce

$$f^{(1)} = \mathrm{e}^{kx+\omega t+\theta\,\xi+x^0}\,, \tag{31}$$

where k, ω, and x^0 are constants, ξ is an odd (fermionic) constant. Substituting Eq. (31) into Eq. (27) we obtain the dispersion relation

$$\omega + k^5 = 0\,.$$

It is easy to see that the right-hand side of Eq. (28) is identically zero. Thus we take $f^{(2)}=0$, $f^{(3)}=0,\ldots$, and series (26) truncates. So, the one-supersoliton solution is given by

$$f = 1 + \mathrm{e}^{kx-k^5t+\theta\xi+x^0} \tag{32}$$

for every k and ξ.

Next, we construct the two-soliton solution. We take $f^{(1)} = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2}$, where $\eta_i = k_ix - k_i^5t + \theta\xi_i + x_i^0$. Equation (28) becomes

$$2D_x(\partial_t+\partial_x^5)f^{(2)} = 5k_1k_2(k_1-k_2)(k_1^2-k_1k_2+k_2^2) \times (k_2-k_1-2\xi_1\xi_2)]\,\mathrm{e}^{\eta_1+\eta_2}\,. \tag{33}$$

Taking into account that f is a Grassmann even function, $f^{(2)}$ must be even. The above equation yields the following expression for $f^{(2)}$,

$$f^{(2)} = A_{12}\Big[1 + 2\theta\frac{k_2\xi_1-k_1\xi_2}{k_1-k_2}\Big]\,\mathrm{e}^{\eta_1+\eta_2}\,, \tag{34}$$

where

$$A_{12} = \frac{(k_1-k_2)(k_1^3+k_2^3)}{(k_1+k_2)(k_1^3-k_2^3)}\Big[1 + 2\frac{k_1-k_2}{(k_1+k_2)^2}\xi_1\xi_2\Big]\,. \tag{35}$$

Substituting the above forms for $f^{(1)}$ and $f^{(2)}$ into Eq. (29), one obtains $f^{(3)} = 0$ and then $f^{(4)} = 0$, $f^{(5)} = 0, \ldots$, i.e. the series truncates. So the two-supersoliton solution is given by

$$f = 1 + \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + A_{12}\Big(1 + 2\theta\frac{k_2\xi_1 - k_1\xi_2}{k_1 - k_2}\Big)\,\mathrm{e}^{\eta_1+\eta_2}\,, \quad (36)$$

where

$$A_{12} = \frac{(k_1 - k_2)(k_1^3 + k_2^3)}{(k_1 + k_2)(k_1^3 - k_2^3)}\Big[1 + 2\frac{k_1 - k_2}{(k_1 + k_2)^2}\xi_1\xi_2\Big] \quad (37)$$

for every k_i, ξ_i, and $x_i^0\,(i = 1,\, 2\,)$.

In order to find the three-supersoliton solution we consider

$$f^{(1)} = \mathrm{e}^{\eta_1} + \mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_3}\,, \quad (38)$$

and the equation for $f^{(3)}$, equation (29) becomes

$$D_x(\partial_t + \partial_x^5)f^{(3)} = -S_x(D_t + D_x^5)(\,\mathrm{e}^{\eta_1}\cdot A_{23}\,\mathrm{e}^{\eta_2+\eta_3} + \mathrm{e}^{\eta_2} \cdot A_{13}\,\mathrm{e}^{\eta_1+\eta_3} + \mathrm{e}^{\eta_3}\cdot A_{12}\,\mathrm{e}^{\eta_1+\eta_2}\,)\,. \quad (39)$$

The solution $f^{(3)}$ (which is very complicated) of this equation cannot let the right-hand side of Eq. (30) be zero. We must impose the condition $k_i\xi_j = k_j\xi_i$ for every i and j. Under this condition and by directly calculation, we can obtain the following N-soliton solution

$$f^{(N)} = \sum_{\mu=0,1} \exp\Big(\sum_{i=1}^{N} \mu_i\,\theta_i + \sum_{i<j} A_{ij}\mu_i\mu_j\Big)\,, \quad (40)$$

where

$$\theta_i = k_ix - k_i^5 + \theta\xi_i + x_i^0\,, \quad (41)$$

$$\exp\; A_{ij} = \frac{(k_i - k_j)^3\,(k_i^3 + k_j^3)}{(k_i + k_j)^3\,(k_i^3 - k_j^3)} \quad (42)$$

for every k_i, ξ_i, and $x_i^0\,(i = 1,\, 2\,, \ldots\,, N)$.

6 Conclusion

In this paper, the $N = 1$ supersymmetric Sawada–Kotera–Ramani equation is studied within the framework of Hirota's bilinear method. A Bäcklund transformation and one- and two-supersoliton solutions are calculated. N-supersoliton solutions for $N \geq 3$ are given under the condition $k_i\xi_j = k_j\xi_i$. We believe that these are important in discussing the integrability and other properties about this system.

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