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New Similarity Reductions and Compacton Solutions for Boussinesq-Like Equations with Fully Nonlinear Dispersion*

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Abstract *In this paper, similarity reductions of Boussinesq-like equations with nonlinear dispersion (simply called $B(m,n)$ equations) $u_{tt} = (u^n)_{xx} + (u^m)_{xxxx}$, which is a generalized model of Boussinesq equation $u_{tt} = (u^2)_{xx} + u_{xxxx}$ and modified Bousinesq equation $u_{tt} = (u^3)_{xx} + u_{xxxx}$, are considered by using the direct reduction method. As a result, several new types of similarity reductions are found. Based on the reduction equations and some simple transformations, we obtain the solitary wave solutions and compacton solutions (which are solitary waves with the property that after colliding with other compacton solutions, they re-emerge with the same coherent shape) of $B(1,n)$ equations and $B(m,m)$ equations, respectively.*

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1 Introduction

Solving nonlinear evolution equations (NLEEs) is an important but difficult subject in soliton theory. Generally speaking, reducing the high-dimensional NLEEs into low-dimensional NLEEs is a powerful tool to solve NLEEs. There exist many powerful methods to search for similarity reductions of nonlinear partial differential equations: (i) The classical method due to Lie, which works by using the Lie group method of infinitesimal transformation.[1−3] (ii) The nonclassical method due to Blumann and Cole.[4] The method is also called as the method of conditional symmetries or the method of partial symmetries of the first type.[5,6] (iii) The direct method due to Clarkson and Kruskal[7−9] and the improved direct method due to Lou *et al.*[10−15] The method involves no use of group theory and has been used to generate many new similarity reductions and exact solutions for many famous physically significant partial differential equations, such as KdV equation, Burgers equation, mKdV equation, KP equation, Boussinesq equation, modified Boussinesq equation, $K(m,n)$ equations and so on.[7−18]

Recently, Rosenau and Hyman[17] introduced a class of solitary waves with compact support (called compactons[20,21]) that are solutions of a two-parameter family of fully nonlinear dispersive partial differential equations such as $K(2,2)$ equation $u_t+(u^2)_x+(u^2)_{xxx} = 0$ and 3D Kadomtsev–Petviashvili equation with nonlinear dispersion $u_{xt} + (uu_x)_x + (u^2)_{xxxx} + u_{yy} + u_{zz} = 0$. More recently, Lou *et al.*[18] proved that $K(m, m-1)$ equation $u_t + (u^m)_x + (u^{m-1})_{xxx} = 0$ and $K(m, m-2)$ equation $u_t + (u^m)_x + (u^{m-2})_{xxx} = 0$ are Painlevé integrable and gave the compacton solution of $K(m,m)$ equation $u_t + (u^m)_x + (u^m)_{xxx} = 0$. Wang[19] gave some symmetry reductions of $K(m,n)$ equation $u_t + (u^m)_x + (u^n)_{xxx} = 0$.

To understand the role of nonlinear dispersion in the formation of patterns in liquid drops, here we would like to introduce and consider a family of fully Boussinesq equation $B(m,n)$,

$$u_{tt} = (u^n)_{xx} + (u^m)_{xxxx}\,. \tag{1}$$

When $(m,n) = (1,2)$, equation (1) becomes the well-known Boussinesq equation; When $(m,n) = (1,3)$, equation (1) reduces to the modified Boussinesq equation which arises from the famous Fermi–Pasta–Ulam problem[22,23] and is used to investigate the behaviour of systems which are primarily linear but a nonlinearity is introduced as a perturbation; When $(m,n) = (1,n)$, equation (1) is the higher-order modified Boussinesq equation. The similarity reductions and integrability of these three equations have been studied.[7,14,15] However, for the other cases of (m,n), similarity reductions and exact solutions of Eq. (1) were not considered.

The rest of this paper is organized as follows. In Sec. 2, the direct method due to Clarkson and Kruskal and the improved direct method due to Lou *et al.* are extended to Eq. (1) such that three types of symmetry reductions are obtained. In Sec. 3, solitary wave solutions of $B(1,n)$ equation and the compacton solution of $B(m,m)$ equation are found based on the obtained reduction ordinary

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differential equations. Finally, some conclusions are given in Sec. 4.

2 Similarity Reductions of $B(m,n)$ Equation

As is well known, all the similarity solutions of the form

$$u(x,t)=U(x,t,Q(Z)),\qquad Z=Z(x,t)\,, \tag{2}$$

where U and Z are functions of the indicated variables and $Q(Z)$ satisfies a certain ordinary differential equation (ODE), may be obtained by substituting Eq. (2) into Eq. (1). But similar to the direct method used in Refs [7]–[9], it is shown that seeking the similarity reduction of Eq. (1) is also sufficient in the following special form

$$u(x,t)=\alpha(x,t)+\beta(x,t)Q(Z(x,t)) \tag{3}$$

rather than the most general form (2), where $\alpha(x,t)$, $\beta(x,t)$, $Z(x,t)$ are functions to be determined later and $W(Z)$ satisfies a certain ODE.

Setting

$$u^m=\sum_{i=0}^{m}\frac{m!}{i!(m-i)!}\alpha^{m-i}\beta^iQ^i\,, \tag{4}$$

$$u^n=\sum_{i=0}^{n}\frac{n!}{i!(n-i)!}\alpha^{n-i}\beta^iQ^i\,. \tag{5}$$

Substituting Eqs (3) $\sim$ (5) into Eq. (1), we get

$$\begin{aligned}
&-\beta Z_t^2Q''-(\beta Z_{tt}+2\beta_tZ_t)Q'-\beta_{tt}Q+\sum_{i=0}^{n-1}\Big[\frac{n!}{(n-i-2)!i!}\alpha^{n-i-2}\alpha_x^2\beta^iQ^i\\
&+\frac{n!}{(n-i-1)!i!}\alpha^{n-i-1}\alpha_{xx}\beta^iQ^i\Big]+\sum_{i=1}^{n-1}\frac{2n!}{(n-i-1)!(i-1)!}\alpha^{n-i-1}\alpha_x(\beta^{i-1}\beta_xQ^i+\beta^iZ_xW^{i-1}Q')\\
&+\sum_{i=1}^{n}\frac{n!}{(n-i)!(i-1)!}\alpha^{n-i}(\beta^{i-1}\beta_{xx}Q^i+\beta^iZ_{xx}Q^{i-1}Q'+\beta^iZ_x^2W^{i-1}Q'')\\
&+\sum_{i=2}^{n}\frac{n!}{(n-i)!(i-2)!}\alpha^{n-i}(\beta^{i-2}\beta_x^2Q^i+\beta^iZ_x^2W^{i-2}Q'^2)+\sum_{i=1}^{n}\frac{2in!}{(n-i)!(i-1)!}\alpha^{n-i}\beta^{i-1}\beta_xZ_xQ^{i-1}Q'\\
&+\sum_{i=1}^{m-4}\frac{m!}{(m-i-4)!i!}\alpha^{m-i-4}\alpha_x^4\beta^iQ^i+\sum_{i=1}^{m-3}\frac{6m!}{(m-i-3)!i!}\alpha^{m-i-3}\alpha_x^2\alpha_{xx}\beta^iQ^i\\
&+\sum_{i=0}^{m-2}\frac{3m!}{(m-i-2)!i!}\alpha^{m-i-2}\alpha_{xx}^2\beta^iQ^i+\sum_{i=0}^{m-2}\frac{4m!}{(m-i-2)!i!}\alpha^{m-i-2}\alpha_x\alpha_{xxx}\beta^iQ^i\\
&+\sum_{i=0}^{m-1}\frac{m!}{(m-i-1)!i!}\alpha^{m-i-2}\alpha_{xxxx}\beta^iQ^i+\sum_{i=0}^{m-2}\frac{12m!}{(m-2-i)!i!}\alpha^{m-i-2}\alpha_x\alpha_{xx}(\beta^iQ^i)_x\\
&+\sum_{i=0}^{m-3}\frac{4n!}{(m-3-i)!i!}\alpha^{m-i-3}\alpha_x^3(\beta^iQ^i)_x+\sum_{i=0}^{m-1}\frac{4m!}{(m-1-i)!i!}\alpha^{m-i-1}\alpha_{xxx}(\beta^iQ^i)_x\\
&+\sum_{i=0}^{m-2}\frac{6m!}{(m-2-i)!i!}\alpha^{m-i-2}\alpha_x^2(\beta^iQ^i)_{xx}+\sum_{i=0}^{m-1}\frac{6m!}{(m-1-i)!i!}\alpha^{m-i-1}\alpha_{xx}(\beta^iQ^i)_{xx}\\
&+\sum_{i=0}^{m-1}\frac{2m!}{(m-1-i)!i!}\alpha^{m-i-1}\alpha_x(\beta^iQ^i)_{xxx}+\sum_{i=0}^{m}\frac{m!}{(m-i)!i!}\alpha^{m-i}(\beta^iQ^i)_{xxxx}=0
\end{aligned} \tag{6}$$

with

$$(\beta^iQ^i)_x=(\beta^i)_xQ^i+i\beta^iQ^{i-1}Q'Z_x\,, \tag{7}$$

where primes are derivatives with respect to Z. Equation (6) is rewritten as

$$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{2}\sum_{l=0}^{4}\sum_{s=0}^{N}F_{ijkls}Q^s(Q')^l(Q'')^k(Q''')^j(Q'''')^i=0,\qquad N=\max\{m,n\}\,, \tag{8}$$

where $F_{ijkls}=F_{ijkls}(\alpha,\beta,Z,\ldots)$ are coefficients of terms $Q^s(Q')^l(Q'')^k(Q''')^j(Q'''')^i$. In order to make Eq. (6) become an ODE of Z, it is needed that the ratios of coefficient derivatives and powers of $Q(Z)$ are functions of only Z, the details are discussed as follows.

Case 1 When $Z_x \neq 0$, if we choose $F_{1000(m-1)} = m\beta^m Z_x^4$, i.e., the coefficient of the term $Q^{m-1}Q''''$, as the normalizing coefficient and therefore require that the other coefficients are of the form $m\beta^m Z_x^4 \Gamma(Z)$, where $\Gamma(Z)$ being a function of Z to be determined later. Therefore we have

$$F_{ijkls} = m\beta^m Z_x^4 \Gamma_{ijkls}(Z), \qquad i,j = 0,1; \quad k = 0,1,2; \quad l = 0,1,2,3,4; \quad s = 0,1,2,\ldots,N\,, \tag{9}$$

where $\Gamma_{ijkls}(Z)$ is the function of only Z to be determined later.

In order to work on the determinations of $\alpha(x,t)$, $\beta(x,t)$, $Z(x,t)$, $Q(Z)$ and $\Gamma(Z)$, for convenience, we can apply the following remarks without loss of generality.

Remark (a) If $\alpha(x,t)$ has the form $\alpha(x,t) = \alpha_0(x,t) + \beta(x,t)W(Z)$, then we can take $W(Z) = 0$ (by substituting $Q(Z) \to Q(Z) - W(Z)$).

Remark (b) If $\beta(x,t)$ is determined by the form $\beta(x,t) = \beta_0(x,t)W(Z)$, then we can take $W(Z) = \text{const.} = W_0$ (by substituting $Q(Z) \to Q(Z)W_0/W(Z)$).

Remark (c) If $Z(x,t)$ is given by the form $Z_0(x,t) = W(Z)$, then we can take $W(Z) = Z$ (by taking $Z \to W^{-1}(Z)$).

Remark (d) We reserve uppercase letters for undetermined function of Z so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.) the result can be denoted by the same letter [e.g.,the derivative of $\Gamma(Z)$ will be called $\Gamma(Z)$].

By using Remarks (a) ∼ (d), we can obtain the following results from system (9),

$$\begin{aligned}
&\Gamma_{0001(m-4)}(Z) = (m-1)(m-2)(m-3)\,,\\
&\Gamma_{0012(m-3)}(Z) = 6(m-1)(m-2), \qquad \Gamma_{0101(m-2)}(Z) = 4(m-1)\,,\\
&\Gamma_{1000(m-1)}(Z) = \Gamma_{0010(n-1)}(Z) = 1\,,\\
&\Gamma_{0020(m-2)}(Z) = 3(m-1), \qquad\qquad \Gamma_{0001(n-2)}(Z) = n-1\,,\\
&\Gamma_{00001}(Z) = -\frac{2A^2(2n-m+1)}{n-m}\Big(\frac{m}{n}\Big)^{(m-2)/(m-n)}\,,\\
&\Gamma_{00010}(Z) = -\frac{5n-4m-1}{n-m}\Big(\frac{m}{n}\Big)^{(m-2)/(m-n)} A(AZ+B)\,,\\
&\Gamma_{00100}(Z) = -(AZ+B)^2\,,\\
&\Gamma_{ijkls}(Z) = 0, \qquad ijkls \neq 0001(m-4),\ 0012(m-3),\ 0101(m-2), 1000(m-1),\ 0020(m-2),\\
&\qquad\qquad\qquad\qquad\quad 0001(n-2), 00001, 00010, 00100\,,\\
&Z(x,t) = x\theta(t) + \phi(t), \qquad \alpha = 0\,,\\
&\beta(x,t) = \Big(\frac{m\theta^2(t)}{n}\Big)^{1/(n-m)}\,,
\end{aligned} \tag{10}$$

where $\theta(x,t)$ and $\phi(x,t)$ satisfy

$$\theta_t = Am^{m/2}n^{(1-m)/2}\theta^{(3n-2m-1)/(n-m)}\,, \tag{11}$$

$$\phi_t = m^{m/2}n^{(1-m)/2}\theta^{(2n-m-1)/(n-m)}(A\phi + B)\,, \tag{12}$$

and A, B are arbitrary constants.

Therefore, we can get the following similarity reduction of Eq. (1),

$$u(x,t) = \Big(\frac{m}{n}\Big)^{1/(n-m)}\theta(t)^{2/(n-m)}Q(Z), \qquad Z(x,t) = \theta(t)x + \phi(t)\,, \tag{13}$$

where $\theta(x,t)$ and $\phi(x,t)$ satisfy Eqs (11) and (12) and $Q(Z)$ satisfies

$$\begin{aligned}
&Q^{m-1}Q'''' + 4(m-1)Q^{m-2}Q'Q''' + 3(m-1)Q^{m-2}Q''^2 + 6(m-1)(m-2)(m-3)Q^{m-3}Q'^2Q''\\
&\quad + (m-1)(m-2)(m-3)Q^{m-4}Q'^4 + Q^{n-1}Q'' + (n-1)Q^{n-2}Q'^2 - (AZ+B)^2Q''
\end{aligned}$$

$$-\frac{5n-4m-1}{n-m}\Big(\frac{m}{n}\Big)^{(m-2)/(m-n)}A(AZ+B)Q'-\frac{2A^2(2n-m+1)}{n-m}\Big(\frac{m}{n}\Big)^{(m-2)/(m-n)}Q=0\,. \tag{14}$$

Nowadays, we consider two special types, respectively.

Type 1 When $A=0$, the general solutions of Eqs (11) and (12) are written as

$$\theta(t)=\theta_0, \qquad \phi(t)=Bm^{m/2}n^{(1-m)/2}\theta_0^{(2n-m-1)/(n-m)}t+\phi_0\,, \tag{15}$$

where θ_0 and ϕ_0 are integration constants. Thus we get the following similarity reduction of Eq. (1),

$$u(x,t)=\Big(\frac{m}{n}\Big)^{1/(n-m)}\theta_0^{2/(n-m)}Q(Z)\,, \tag{16a}$$

$$Z(x,t)=\theta_0x+Bm^{m/2}n^{(1-m)/2}\theta_0^{(2n-m-1)/(n-m)}t+\phi_0\,, \tag{16b}$$

and $Q(Z)$ satisfying

$$\begin{aligned}Q^{m-1}Q''''&+4(m-1)Q^{m-2}Q'Q'''+3(m-1)Q^{m-2}Q''^2+6(m-1)(m-2)(m-3)Q^{m-3}Q'^2Q''\\&+(m-1)(m-2)(m-3)Q^{m-4}Q'^4+Q^{n-1}Q''+(n-1)Q^{n-2}Q'^2-B^2Q''=0\,.\end{aligned} \tag{16c}$$

Integrating Eq. (16c) with respect to Z twice yields a second-order nonlinear ordinary differential equation

$$\frac{1}{m}(Q^m)''+\frac{1}{n}Q^n-B^2Q+c_2Z+c_1=0\,, \tag{16d}$$

where c_1 and c_2 are arbitrary constants.

If we take $c_1=c_2=0$, equation (16d) becomes

$$\frac{1}{m}(Q^m)''+\frac{1}{n}Q^n-B^2Q=0\,. \tag{17}$$

Setting $Q^m=W$ and integrating Eq. (17) with respect to W yields

$$\frac{1}{2m}W'^2+\frac{m}{n(n+m)}W^{(n+1)/m}-\frac{B^2m}{m+1}W^{(m+1)/m}-c_3=0\,, \tag{18}$$

which has the general solution

$$\int^W\frac{\mathrm{d}W}{\sqrt{2mc_3-\dfrac{2m^2}{n(n+m)}W^{(n+1)/m}+\dfrac{2B^2m^2}{m+1}W^{(m+1)/m}}}=Z-Z_0\,,$$

where c_3 is a constant.

Type 2 When $A\neq0$, equations (11) and (12) have the general solutions

$$\theta(t)=\Big[Am^{m/2}n^{(1-m)/2}\frac{1-n}{n-m}(t-t_0)\Big]^{(n-m)/(1-n)}\,, \tag{19}$$

$$\phi(t)=-\frac{B}{A}+\frac{c_4}{A}\exp\Big\{m^{-m/2}n^{(m-1)/2}\frac{1-n}{n-m}\Big[Am^{m/2}n^{(1-m)/2}\frac{1-n}{n-m}(t-t_0)\Big]^{(n-m)/(1-n)}\Big\}\,. \tag{20}$$

Thus we get the following similarity reduction of Eq. (1),

$$u(x,t)=\Big(\frac{m}{n}\Big)^{1/(n-m)}\Big[Am^{m/2}n^{(1-m)/2}\frac{1-n}{n-m}(t-t_0)\Big]^{2/(1-n)}Q(Z)\,, \tag{21a}$$

$$\begin{aligned}Z(x,t)=&\Big[Am^{m/2}n^{(1-m)/2}\frac{1-n}{n-m}(t-t_0)\Big]^{(n-m)/(1-n)}x-\frac{B}{A}\\&+\frac{c_4}{A}\exp\Big\{m^{-m/2}n^{(m-1)/2}\frac{1-n}{n-m}\Big[Am^{m/2}n^{(1-m)/2}\frac{1-n}{n-m}(t-t_0)\Big]^{(n-m)/(1-n)}\Big\}\,,\end{aligned} \tag{21b}$$

where $Q(Z)$ satisfies Eq. (14).

Case 2 When $Z_x=0$, namely, $Z=Z(t)$, we can take $Z(t)=t$ simply by taking $t\to Z^{-1}(t)$ with Z^{-1} being the inverse function of $Z(t)$. For the case $Z(t)=t$, equation (8) is rewritten as

$$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=0}^{N}H_{ijk}Q^k(Q')^j(Q'')^i=0, \qquad N=\max\{m,n\}\,, \tag{22}$$

where $H_{ijk} = H_{ijk}(\alpha, \beta, \alpha_x, \beta_x, \ldots)$ are the coefficients of the terms $Q^k(Q')^j(Q'')^i$. In order to make Eq. (22) become an ODE, it is needed that the ratios of coefficient derivatives and powers of $Q(Z)$ are functions of only Z. If we choose the coefficient of term Q'' (i.e., β) as the normalizing coefficient and therefore require that the other coefficients are of the form $\beta\Gamma(Z)$, where $\Gamma(Z)$ is a function of Z to be determined later. Therefore we have

$$H_{ijk} = \beta\Gamma_{ijk}(Z), \qquad i, j = 0, 1; \quad k = 0, 1, 2, \ldots, N = \max\{m, n\}\,, \tag{23}$$

where $\Gamma_{ijk}(Z)$ is the function of only Z to be determined later.

Similar to case 1, we can derive the following reduction via using the above Remarks (a) ∼ (d) with the condition $m = 2n - 1$,

$$u(x, t) = (x - x_0)^{2/(n-1)} Q(t)\,, \tag{24}$$

and $W(t)$ satisfies

$$Q'' - \frac{2(n-1)^2}{(2n-1)(4n-1)} Q^n - p^{-1} Q^{2n-1} = 0\,, \tag{25}$$

where

$$p = \frac{2n-1}{(n-1)^3}\Big[24(n-3)^2 + 32(n-3)(n-4) + 96(2n-3)(3-n) + 32(2n-3)(2n-4) + \frac{2(n-3)(n-4)(n-5)}{n-1}\Big]\,.$$

It is easy to know that the general solution of Eq. (25) is

$$\pm\int^{Q} \frac{\mathrm{d}s}{\sqrt{c_5 + \dfrac{4(n-1)^2}{(2n-1)(4n-1)(n+1)} s^{n+1} + \dfrac{1}{2nb} s^{2n}}} = t - t_0\,, \tag{26}$$

where c_5 is a constant of integration.

3 Solitary Wave Solutions and Compacton Solutions

Case 1 When $(m, n) = (1, n)$, $B(1, n)$ becomes

$$u_{tt} + (u^n)_{xx} + u_{xxxx} = 0\,. \tag{27}$$

The similarity reduction of Eq. (27) with $Z_x \neq 0$ is

$$u(x, t) = \theta_0^{2/n} Q(Z), \qquad Z(x, t) = \theta_0 x + B\theta^2 t + \phi_0\,, \tag{28a}$$

$$Q'' + \frac{1}{n} Q^n - B^2 Q = 0\,. \tag{28b}$$

Setting $Q = V^{2/(n-1)}$, equation (28b) is rewritten as

$$\frac{2}{n-1} V V'' + \frac{2(3-n)}{(n-1)^2} V'^2 + \frac{1}{n} V^4 - B^2 V^2 = 0\,. \tag{28c}$$

Suppose that equation (28c) has the following formal solution

$$V = V(Z) = a\,\mathrm{sech}\,(kZ)\,, \tag{29}$$

substituting Eq. (29) into Eq. (28c) and setting the coefficients of $\mathrm{sech}^4(kZ)$, $\mathrm{sech}^2(kZ)$ and the constant term to zero, respectively, yield a system of algebraic equations

$$-\frac{4k^2}{(n-1)^2} + \frac{a^2}{n} - \frac{2k^2}{n-1} = 0\,,$$

$$\frac{4k^2}{(n-1)^2} - \frac{2a^2}{n} + \frac{4k^2}{n-1} + B^2 = 0\,,$$

$$\frac{a^2}{n} - \frac{2k^2}{n-1} + B^2 = 0\,,$$

which read

$$a = \frac{\sqrt{2n(n+1)}}{n-1} k, \qquad B = \frac{2}{n-1} k\,. \tag{30}$$

Thus according to Eqs (28c), (29) and (30), the bell-shaped solitary wave solution of $B(1,n)$ is

$$u(x,t)=\theta_0^{2/n}\Big\{\frac{\sqrt{2n(n+1)}}{n-1}k\,\mathrm{sech}\Big[k\Big(\theta_0x+\frac{2}{n-1}k\theta_0t+\phi_0\Big)\Big]\Big\}^{2/(n-1)}\,.$$

Case 2 When $(m,n)=(m,m)$, $m\neq1$, $B(m,m)$ becomes

$$u_{tt}-(u^m)_{xx}-(u^m)_{xxxx}=0\,. \tag{31}$$

Under the travelling reduction $u=u(\xi)$, $\xi=k(x-\lambda t)$, we can obtain the exact solution

$$u_1=\begin{cases}\Big\{\dfrac{2m^2\lambda^2}{m+1}\cos^2\Big[\dfrac{m-1}{2m}(x-\lambda t)\Big]\Big\}^{1/(m-1)}, & -\dfrac{\pi}{2}\le\dfrac{m-1}{2m}(x-\lambda t)\le\dfrac{\pi}{2},\\ 0, & \Big|\dfrac{m-1}{2m}(x-\lambda t)\Big|>\dfrac{\pi}{2}\,,\end{cases} \tag{32}$$

where λ is an arbitrary constant, which is compacton solution that is solitary wave with the property that after colliding with other compacton solutions, they re-emerge with the same coherent shape.[18−20]

4 Conclusion and Discussion

In summary, by using both the direct method due to Clarkson and Kruskal and the improved direct method due to Lou *et al.*, we have obtained three types of similarity reductions of the Boussinesq-like equations with nonlinear dispersion. In addition, we firstly obtain bell-shaped solitary wave solutions of $B(1,n)$ equations from the constant-coefficient ordinary differential equation (31) and two direct transformations $Q=V^{2/(n-1)}$, $V=V(Z)=a\,\mathrm{sech}\,(kZ)$. Secondly, we give a compacton solution of $B(m,m)$ equation. The compacton solution is of important physical significance. There exist some open problems which need to be considered further. (i) Do there exist other types of solitary wave solutions of $B(m,n)$ with $m\neq1$? (ii) Does other $B(m,n)$ equation with $m\neq n$ have the compacton solutions? (iii) It is well-known that $B(1,2)$ equation passes the Painlevé test,[7] but $B(1,n)$ $(n\geq3)$ equation is not Painlevé-integrable.[14,15] A natural problem is that whether the other $B(m,n)$ equation is Painlevé-integrable.

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